Rigorous analysis and estimates of S-shaped bifurcation curves in a combustion problem with general Arrhenius reaction-rate laws

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We study the bifurcation of solutions of the combustion problem with general Arrhenius reaction-rate laws

\[
\begin{align*}
d^2u &+ \lambda(1 + \epsilon u)^m e^{u/(1+\epsilon u)} = 0, \quad -1 < x < 1, \\
u(-1) &= u(1) = 0.
\end{align*}
\]

For this two point boundary value problem, we give a bifurcation analysis in the physically important ranges 0 < m < 1 and m < 0. It is proved that the bifurcation curve is S-shaped in certain parameter ranges, and a lower bound is given for the Frank–Kamenetskii transition value.

Keywords: bifurcation; S-shaped bifurcation curve; positive solution; time map; Frank–Kamenetskii transition value

1. Introduction

We consider the bifurcation of (positive) solutions of the equation

\[
\frac{d^2u}{dx^2} + \lambda f(u) = \frac{d^2u}{dx^2} + \lambda(1 + \epsilon u)^m e^{u/(1+\epsilon u)} = 0, \quad -1 < x < 1,
\]

with the Dirichlet (Frank–Kamenetskii) boundary conditions

\[
u(-1) = u(1) = 0,
\]

where the bifurcation parameters \(\lambda, \epsilon > 0\) and \(m < 1\). Problem (1.1), (1.2) is the one-dimensional case of an important equation in combustion theory. It can be considered as a special case for an \(n\)-dimensional Dirichlet problem of (1.1), (1.2) for the infinite slab. In equation (1.1), \(\lambda\) is the Frank–Kamenetskii parameter, the reaction term \(f(u) = (1 + \epsilon u)^m e^{u/(1+\epsilon u)}\) is the temperature dependence of the \(m\)th-order reaction rate obeying the general Arrhenius reaction-rate law, \(u\) is the dimensionless temperature, and \(\epsilon = RT_a/E\) is the reciprocal activation energy (see Boddington et al. 1979).

Problem (1.1), (1.2) was studied for \(0 < m < 1\), and particularly, for \(m = 0\) (Arrhenius reaction rate) and \(m = \frac{1}{2}\) (bimolecular reaction rate) by Boddington et al. (1979, 1982, 1983a, b, 1984, 1988, 1989; see also Okoya 1994; Zaturska 1982). Physically, it is also meaningful when \(m\) is negative, particularly, for \(m = -1\) and \(-2\).
Criticality (bifurcation) persists as long as the reciprocal activation energy $\epsilon$ is smaller than a transitional value $\epsilon_{tr}$. At this transitional value $\epsilon_{tr}$, only continuous behavior is possible: ignition phenomena disappears (Boddington et al. 1979). Accurate transitional values for $\epsilon$ have been calculated by numerical quadrature for the infinite slab; i.e. for equation (1.1). We note that in the previous numerical work, especially that by Boddington et al. (1979, 1982, 1983a, b, 1984, 1988, 1989), for $0 \leq m < 1$, they found an S-shaped bifurcation diagram and three solutions for some parameter values. We prove this rigorously in certain parameter ranges of physical interest and give a lower bound for the Frank–Kamenetskii transition value. Similar results are obtained for $m < 0$, especially for $m = -1$ and $-2$.

Let

$$S = \{ (\lambda, \|u\|_{\infty}) : \lambda \geq 0 \text{ and } u \text{ is a solution of (1.1), (1.2)} \}.$$  

We say that the bifurcation curve $S$ is S-shaped if it has exactly two turning points and is a monotone curve if it has no turning point on the $(\lambda, \|u\|_{\infty})$-plane. Boddington et al. (1979, §4) obtained

$$\epsilon_{FK} < \epsilon_{Sem} = \begin{cases} \left(1 - \sqrt{1 - m} \right)^2, & \text{for } m < 1 \text{ and } m \neq 0, \\ \frac{1}{4}, & \text{for } m = 0. \end{cases}$$  

Thus, unless $m < 1$ and $0 < \epsilon < \epsilon_{Sem}$, there is no Frank–Kamenetskii transition. For simplicity, we define

$$\tilde{\epsilon} = \epsilon_{Sem} = \begin{cases} \left(1 - \sqrt{1 - m} \right)^2, & \text{for } m < 1 \text{ and } m \neq 0, \\ \frac{1}{4}, & \text{for } m = 0. \end{cases}$$  

The case $m = 0$ is studied in Wang (1994). The method used in that paper, and in the present work, is to consider the ‘time map’ given by

$$(\lambda(\alpha))^{1/2} = 2^{-1/2} \int_{x_0}^{x_1} \frac{du}{(F(\alpha) - F(u))^{1/2}} \equiv T(\alpha),$$  

where $F(s) = \int_{x_0}^{x} f(x) \, dx$. We note that solutions $u$ of (1.1), (1.2) correspond to $\|u\|_{\infty} = \alpha$ and $T(\alpha) = \sqrt{\lambda}$. This equation is obtained from (1.1), (1.2). Multiplying (1.1) by $u'(x)$ and integrating, we obtain

$$\frac{1}{2} |u'(x)|^2 + \lambda F(u(x)) = \text{const.}$$  

Since we are interested in positive solutions, $u''(x) = -\lambda f(u(x)) < 0$ for $-1 < x < 1$ which implies that $u'(x) < 0$ for $0 < x < 1$ by (1.2). Thus $\|u\|_{\infty} = u(0)$. If we let $\alpha = u(0)$ and substitute $x = 0$ in (1.6), then we have

$$u'(x) = -\sqrt{2\lambda F(\alpha) - F(u)}, \quad 0 < x < 1.$$  

Integrating (1.7) in $[0, 1]$, we obtain (1.5).

**Theorem 1.1.** (Wang, 1994, theorem 1). For $m = 0$, in addition to (1.8) and (1.9), the bifurcation curve $S$ is S-shaped for $\epsilon$ in the interval

$$0 < \epsilon < 0.2224.$$
S-shaped bifurcation curves in a combustion problem

Figure 1. Bifurcation diagrams of $T(\alpha)$ for (1.1), (1.2) for $m < 1$. $\alpha = \|u\|_\infty$, $T(\alpha) = \sqrt{\lambda}$.

A lower bound for $\epsilon_{FK}^{tr}$ is then obtained, that is,

$$0.2224 \leq \epsilon_{FK}^{tr} (< 0.25), \quad \text{for } m = 0.$$

We note that, for $m = 0$: (i) numerically, it was found that $\epsilon_{FK}^{tr} \approx 0.24578$, see Boddington et al. (1979, p. 441); (ii) typically, for most materials $\epsilon$ is in the range of 0.01 to 0.1, but larger values are possible (Burnell et al. 1989, p. 149). Moreover, reactions with $\epsilon$ close to 0.2 have been reported (see Zaturska 1982, p. 380 and Boddington et al. 1970, p. 415).

By (1.5), to show the bifurcation curve $S$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane is equivalent to show that the time map $T(\alpha)$ has exactly two critical points, a local maximum and a local minimum, on $(0, \infty)$.

It is easy to show the following.

**Proposition 1.2.** (See figure 1). For $m < 1$ and $\epsilon > 0$, the bifurcation curve $S$ satisfies

$$\lim_{\alpha \to 0} \lambda(\alpha) = 0,$$

$$\lim_{\alpha \to \infty} \lambda(\alpha) = \infty.$$

The next corollary follows immediately from proposition 1.2 and (1.3).

**Corollary 1.3.** (See figure 1). For $m < 1$, in addition to (1.8) and (1.9), the bifurcation curve $S$ is a monotone curve if $\epsilon \geq \tilde{\epsilon}$.

The main results of this paper are as follows.

**Theorem 1.4.** (See figure 1). For $0 < m < 1$, in addition to (1.8) and (1.9): (i) the bifurcation curve, $S$, is S-shaped if

$$0 < \epsilon < \max\left(\frac{1}{5}, \frac{1}{2} \tilde{\epsilon}\right) = \begin{cases} \frac{1}{5}, & \text{if } 0 < m \leq \frac{1}{2}(2\sqrt{10} - 5) \approx 0.66228 \\ \frac{1}{2} \tilde{\epsilon}, & \text{if } \frac{1}{2}(2\sqrt{10} - 5) < m < 1; \end{cases}$$

(ii) in particular, for $m = \frac{1}{2}$, the bifurcation curve, $S$, is S-shaped for $\epsilon$ in an

Theorem 1.5. (See figure 1.) For $m < 0$, in addition to (1.8) and (1.9):  
(i) the bifurcation curve $S$ has at least two turning points if  
$$0 < \epsilon < \frac{1}{2} \tilde{\epsilon};$$  
(ii) in particular, for $m = -1$, the bifurcation curve, $S$, is S-shaped if  
$$0 < \epsilon < \frac{1}{2} \tilde{\epsilon} = \frac{1}{2}(3 - 2\sqrt{2}) \approx 0.08579;$$  
(iii) for $m = -2$, the bifurcation curve, $S$, is S-shaped if  
$$0 < \epsilon < \frac{1}{2} \tilde{\epsilon} = \frac{1}{4}(2 - \sqrt{3}) \approx 0.06699. $$

Theorem 1.4 and corollary 1.3 imply that  
$$\max\left(\frac{1}{15}, \frac{1}{2} \tilde{\epsilon}\right) \leq \epsilon_{tr}^{FK} < \tilde{\epsilon}, \quad \text{for } 0 < m < 1.$$  
In particular,  
$$0.30028 \leq \epsilon_{tr}^{FK} < \tilde{\epsilon} = 6 - 4\sqrt{2} \approx 0.34315, \quad \text{for } m = \frac{1}{2}.$$  
Numerically, for $m = \frac{1}{2}$, it was found that $\epsilon_{tr}^{FK} \approx 0.33692$ (see Boddington et al. 1979, p. 441).

Theorem 1.5 and corollary 1.3 imply that  
$$\frac{1}{2} \tilde{\epsilon} \leq \epsilon_{tr}^{FK} < \tilde{\epsilon}, \quad \text{for } m < 0.$$  
In particular,  
$$(0.08579 \approx) \frac{1}{2}(3 - 2\sqrt{2}) \leq \epsilon_{tr}^{FK} < 3 - 2\sqrt{2} \approx 0.17157, \quad \text{for } m = -1,$$  
$$(0.06699 \approx) \frac{1}{4}(2 - \sqrt{3}) \leq \epsilon_{tr}^{FK} < \frac{1}{2}(2 - \sqrt{3}) \approx 0.13397, \quad \text{for } m = -2.$$  
In §3, we prove theorem 1.4. In §4, we prove theorem 1.5.

2. Preliminary results

We first remark that many computations in this paper have been checked by using the symbolic manipulator MATHEMATICA.

For $T(\alpha)$ in (1.5), it is easy to compute that  
$$T'(\alpha) = 2^{-3/2} \int_{0}^{\alpha} \frac{\theta(\alpha) - \theta(u)}{(F(\alpha) - F(u))^{3/2}} \frac{du}{\alpha},$$  
where  
$$\theta(u) = 2F(u) - uf(u).$$  
(2.2) gives  
$$\theta'(u) = f(u) - uf'(u)$$  
$$= [1 - (1 - 2\epsilon + \epsilon m)u + \epsilon^2(1 - m)u^2][(1 + \epsilon u)^{m-2}e^{u/(1+\epsilon u)},$$  
$$\theta''(u) = -uf''(u) = u[(1 - 2\epsilon + 2\epsilon m - \epsilon^2 m + \epsilon^2 m^2) - 2\epsilon^2(1 - m + \epsilon m - \epsilon m^2)u$$  
$$- \epsilon^4 m(1 - m)u^2](1 + \epsilon u)^{m-4}e^{u/(1+\epsilon u)},$$  
$$\theta'''(u) = -uf'''(u) = \ldots$$

which are useful in our analysis of the time map \( T \). For \( m < 1 \), if \( 0 < \epsilon < \tilde{\epsilon} \), it can be easily shown that the function \( \theta'(u) = f(u) - uf''(u) \) in (2.3) has exactly two zeros at \( A \) and \( B \) with \( 0 < A < B \),

\[
A \equiv \frac{1 - 2\epsilon + \epsilon m - \sqrt{1 - 4\epsilon + 2\epsilon m + \epsilon^2 m^2}}{2\epsilon^2(1 - m)}, \tag{2.5}
\]

\[
B \equiv \frac{1 - 2\epsilon + \epsilon m + \sqrt{1 - 4\epsilon + 2\epsilon m + \epsilon^2 m^2}}{2\epsilon^2(1 - m)}. \tag{2.6}
\]

To study the bifurcation curve \( S \), we first study the graph of the nonlinearity \( f(u) \) in (1.1) for \( 0 < m < 1 \) and \( m < 0 \) in next two lemmata for which we omit the proofs.

**Lemma 2.1.** For \( 0 < m < 1 \) and \( \epsilon > 0 \), \( f(u) = (1 + \epsilon u)^m e^{u/(1+\epsilon u)} \in C[0, \infty) \cap C^2(0, \infty) \) and satisfies:

(i) \( f(u) > 0 \) on \( [0, \infty) \);

(ii) \( f(u) \sim (1 + \epsilon u)^m \) as \( u \to \infty \);

(iii) if

\[
0 < \epsilon < \epsilon_c \equiv \frac{-1 + \sqrt{1 - m + m}}{(1 - m)m} (> 0), \tag{2.7}
\]

then, on \( (0, \infty) \), \( f(u) \) has exactly one inflection point at \( C \),

\[
C \equiv \frac{-1 + \sqrt{1 - m + m - \epsilon m + \epsilon m^2}}{\epsilon^2(1 - m)m} (> 0) \tag{2.8}
\]

such that

\[
f''(u) > 0, \text{ on } (0, C), \quad f''(u) < 0, \text{ on } (C, \infty). \tag{2.9}
\]

**Lemma 2.2.** For \( m < 0 \) and \( \epsilon > 0 \), \( f(u) = (1 + \epsilon u)^m e^{u/(1+\epsilon u)} \in C[0, \infty) \cap C^2(0, \infty) \) and satisfies:

(i) \( f(u) > 0 \) on \( [0, \infty) \);

(ii) \( f(u) \sim (1 + \epsilon u)^m \) as \( u \to \infty \);

(iii) if

\[
0 < \epsilon < \epsilon_c \equiv \frac{-1 + \sqrt{1 - m + m}}{(1 - m)m} (> 0), \tag{2.10}
\]

then, on \( (0, \infty) \), \( f(u) \) has exactly two inflection points at \( C \) and \( D \) with \( 0 < C < D \),

\[
D \equiv \frac{-1 - \sqrt{1 - m + m - \epsilon m + \epsilon m^2}}{\epsilon^2(1 - m)m}, \tag{2.11}
\]

\[
C \equiv \frac{-1 + \sqrt{1 - m + m - \epsilon m + \epsilon m^2}}{\epsilon^2(1 - m)m}, \tag{2.12}
\]

such that

\[
f''(u) > 0, \text{ on } (0, C); \quad f''(u) < 0, \text{ on } (C, D); \quad f''(u) > 0, \text{ on } (D, \infty). \tag{2.13}
\]

**3. Proof of theorem 1.4**

The proof of theorem 1.4 is based upon the next lemma which is a slight variation of lemma 3 of Wang et al. (1994).

Lemma 3.1. Consider

\[
\begin{cases}
-u''(x) = \lambda \tilde{f}(u(x)), & -1 < x < 1, \\
u(-1) = u(1) = 0,
\end{cases}
\quad (3.1)
\]

where \( \lambda > 0 \) and \( \tilde{f} \in C[0, \infty) \cap C^2(0, \infty) \) satisfying:

(H1) \( \tilde{f}(u) > 0 \) on \( [0, \infty) \);
(H2) \( \tilde{f}(u) \sim (1 + \epsilon u)^m \) for some \( m < 1 \) and \( m \neq 0 \) and \( \epsilon > 0 \) as \( u \to \infty \);
(H3) there exists a number \( C > 0 \) such that

\[
\tilde{f}''(u) > 0, \quad \text{on } (0, C), \quad \tilde{f}''(C) < 0, \quad \text{on } (C, \infty);
\]

(H4) \( \tilde{\theta}(C) \equiv 2 \tilde{F}(C) - C \tilde{f}(C) < 0 \), where \( \tilde{F}(u) \equiv \int_0^u \tilde{f}(s) \, ds \).

Then, in addition to (1.8) and (1.9), the bifurcation curve of positive solutions for (3.1) is S-shaped on the \( (\lambda, \| u \|_\infty) \)-plane.

Outline of the proof of lemma 3.1. First, (1.8) and (1.9) follow by (H1) and (H2). Secondly, it is known that to show that the bifurcation curve of positive solutions for (3.1) is S-shaped on the \( (\lambda, \| u \|_\infty) \)-plane is equivalent to show that the time map \( T(\alpha) \) defined in (1.5) has exactly two critical points, a local maximum and a local minimum, on \( (0, \infty) \). The latter is shown as follows.

Step 1. By (H1)–(H3) and (2.4), it is easy to see that \( \tilde{\theta}(0) = 0, \quad \tilde{\theta}'(0) = \tilde{f}(0) > 0, \quad \lim_{u \to \infty} \tilde{\theta}'(u) = \infty, \) and \( \tilde{\theta} \) has a unique inflection point at \( C \) on \( (0, \infty) \) such that

\[
\tilde{\theta}''(u) > 0, \quad \text{on } (0, C), \quad \tilde{\theta}''(C) < 0, \quad \text{on } (C, \infty). \quad (3.2)
\]

Moreover, by (H4), \( \tilde{\theta}(C) < 0 \). Thus the graph of \( \tilde{\theta} \) is typically depicted in figure 2.

Let \( A \) be the first positive critical point of \( \tilde{\theta} \), \( B \) the first positive zero of \( \tilde{\theta} \), \( D \) the second positive critical point of \( \tilde{\theta} \), and \( E \) the point on \( (D, \infty) \) such that \( \tilde{\theta}(E) = \tilde{\theta}(A) \). Then by (2.1),

\[
T'(\alpha) > 0, \quad \text{for } \alpha \in (0, A], \quad T'(\alpha) < 0, \quad \text{for } \alpha \in [B, D], \quad T'(\alpha) > 0, \quad \text{for } \alpha \in [E, \infty).
\]

So, the time map \( T \) has at least one critical point on \( (A, B) \), and at least one critical point on \( (D, E) \).

Step 2. A result of Laetsch (1970, theorem 3.2) implies that \( T \) has at most one critical point on \( (0, C) \), and hence \( T \) has exactly one critical point, a relative maximum on \( (A, B) \).

Step 3. An estimate of Smoller & Wasserman (1981, p. 275) implies that
\[ T''(\alpha) + \frac{2}{\alpha} T'(\alpha) > 0, \quad \text{for } \alpha \in (D, E). \]
This shows that \( T \) has exactly one critical point, a relative minimum, on \((D, E)\).

Step 4. By above, the time map \( T(\alpha) \) has exactly two critical points, a local maximum and a local minimum, on \((0, \infty)\).

(a) Proof of part (i) of theorem 1.4

To prove part (i) of theorem 1.4, by lemmata 3.1 and 2.1, it suffices to show the next key lemma 3.2, which relies next on lemmata 3.3–3.6. The proof of lemma 3.2 is given at the end of this section.

First, it is easy to show that
\[ \hat{\epsilon} < \epsilon_c, \quad \text{for } m < 1 \text{ and } m \neq 0. \] (3.3)
In particular,
\[ \max\left(\frac{1}{5}, \frac{1}{2}\hat{\epsilon}\right) < \hat{\epsilon} < \epsilon_c, \quad \text{for } 0 < m < 1. \] (3.4)
For fixed \( m \) with \( 0 < m < 1 \) and \( 0 < \epsilon < \hat{\epsilon} \), we recall \( A \) and \( C \) in (2.5) and (2.8),
\[ A \equiv \frac{1 - 2\epsilon + cm - \sqrt{1 - 4\epsilon + 2cm + \epsilon^2m^2}}{2\epsilon^2(1 - m)} \quad (> 0), \]
\[ C \equiv \frac{-1 + \sqrt{1 - m + m - cm + \epsilon^2m^2}}{\epsilon^2(1 - m)m} \equiv C(\epsilon) \quad (> 0), \]
and let
\[ \theta_u \equiv f(u) - uf'(u) = [1 - (1 - 2\epsilon + cm)u + \epsilon^2(1 - m)u^2](1 + \epsilon u)^{m-2}e^{u/(1+\epsilon u)}, \]
see (2.3).

Lemma 3.2. Let \( 0 < m < 1 \). If \( 0 < \epsilon < \max\left(\frac{1}{5}, \frac{1}{2}\hat{\epsilon}\right) \) then \( \theta(C) < 0 \).

The next lemma follows by straightforward but tedious algebra. We omit the proof.

Lemma 3.3. For fixed \( m \) with \( 0 < m < 1 \), if \( 0 < \epsilon < \max\left(\frac{1}{5}, \frac{1}{2}\hat{\epsilon}\right) \), then at \( u = C(\epsilon) \), the function \( \theta_u \) as a function of \( \epsilon \) satisfies:

(i) \[ \frac{\partial \theta_u}{\partial \epsilon} = \frac{\partial \theta_u(\epsilon)}{\partial \epsilon} > 0; \]

(ii) \[ 2 + \theta_u(\epsilon) < 0. \]

For \( m < 1 \) and \( m \neq 0 \), by (3.3), for \( 0 < \epsilon < \hat{\epsilon} \left( < \epsilon_c \right) \), we obtain
\[ C - 2A = \frac{m(-1 + \sqrt{1 - m + cm}) + m^2\sqrt{1 - 4\epsilon + 2cm + \epsilon^2m^2}}{\epsilon^2(1 - m)m^2}. \]
Since \( m < 1 \) and \( m \neq 0 \), \( \epsilon^2(1 - m)m^2 > 0 \). Thus \( C - 2A > 0 \) if
\[ K \equiv [m^2\sqrt{1 - 4\epsilon + 2cm + \epsilon^2m^2}] - [m(-1 + \sqrt{1 - m + cm})]^2 \\
\quad = m^4(m^2 - 1)(\epsilon - \hat{\epsilon})(\epsilon - \tilde{\epsilon}) \quad (\text{if } m \neq -1) \\
\quad > 0, \quad \text{(3.5)} \]

where we let, if \( m \neq -1 \),

\[
\dot{\epsilon} = \frac{-m(1 - \sqrt{1 - m - 2m + m^2}) + \sqrt{(-4 + 4\sqrt{1 - m + 5m})(1 - m)m^3}}{m^2(m^2 - 1)}, \quad (3.6)
\]

\[
\ddot{\epsilon} = \frac{-m(1 - \sqrt{1 - m - 2m + m^2}) - \sqrt{(-4 + 4\sqrt{1 - m + 5m})(1 - m)m^3}}{m^2(m^2 - 1)}. \quad (3.7)
\]

(Note: the above analysis for \( C > 2A \) holds for \( m < 0 \) if (3.5) is satisfied. It is used in § 4b later.) For \( 0 < m < 1 \), it is easy to see that

\[
m^2(m^2 - 1) < 0,
\]

\[-m(1 - \sqrt{1 - m - 2m + m^2}) = -m[(1 - m)^2 - \sqrt{1 - m}] > 0,
\]

\[-(4 + 4\sqrt{1 - m + 5m})(1 - m)m^3 = \{4[\sqrt{1 - m} - (1 - m)] + m\}(1 - m)m^3 > 0.
\]

So, for \( 0 < m < 1 \),

\[
\dot{\epsilon} < 0. \quad (3.8)
\]

Moreover, further manipulations enable us to prove the next lemma for which we omit the proof. We note that this lemma can be checked by simply using Mathematica.

**Lemma 3.4.** If \( 0 < m < 1 \) then \( \dot{\epsilon} > \max \left( \frac{1}{3}, \frac{1}{2}\bar{\epsilon} \right) > 0 \).

Lemma 11 together with (3.5) and (3.8) implies the next lemma immediately.

**Lemma 3.5.** Let \( 0 < m < 1 \). If \( 0 < \epsilon < \bar{\epsilon} \) then \( C > 2A \).

**Lemma 3.6.** Let \( 0 < m < 1 \). If \( 0 < \epsilon < \bar{\epsilon} \) then

(i) \( \theta''(0) < 0 \);

(ii) \( \theta''(C) > 0 \);

(iii) \( \theta''(u) \) changes sign exactly once on \( (0, C) \).

**Proof of lemma 3.6.** For \( 0 < m < 1 \), we first recall (2.7) that

\[
\epsilon_c \equiv \frac{-1 + m + \sqrt{1 - m}}{m(1 - m)} \quad (> 0),
\]

and let

\[
\epsilon_c \equiv \frac{-1 + m - \sqrt{1 - m}}{m(1 - m)} \quad (< 0).
\]

By (2.4), it can be computed that

\[
\theta''(u) = e^{u/(1+\epsilon u)}(1 + \epsilon u)^{-\delta}[-(1 + 2\epsilon - 2\epsilon m + \epsilon^2 m^2 - \epsilon^2 m^2) + (1 - 4\epsilon + 3\epsilon m - 3\epsilon^2 m - 2\epsilon^3 m + 3\epsilon^2 m^2 + \epsilon^3 m^2 + \epsilon^3 m^3)u + \epsilon^2(5 - 6\epsilon - 3m + 12m - 6m^2 + 3\epsilon m^2 - 3\epsilon^2 m^3)u^2 - \epsilon^4(4 - 7m + 2\epsilon m + 3m^2 - 5\epsilon m^2 + 5\epsilon m^3)u^3 - \epsilon^6m(1 - m)^2u^4] \equiv e^{u/(1+\epsilon u)}(1 + \epsilon u)^{-\delta}Q(u). \quad (3.9)
\]

Thus, on \([0, \infty), \theta''(u) > 0 \text{ if and only if } Q(u) > 0 \). Now for \( 0 < m < 1 \) and \( 0 < \epsilon < \bar{\epsilon} \), it can be computed that

(i) $Q(0) = -1 + 2(1 - m)\epsilon + m(1 - m)\epsilon^2$

$$= m(1 - m)\left(\epsilon - \frac{-1 + m - \sqrt{1 - m}}{m(1 - m)}\right)\left(\epsilon - \frac{-1 + m + \sqrt{1 - m}}{m(1 - m)}\right)$$

$$= m(1 - m)(\epsilon - \bar{\epsilon}_c)(\epsilon - \epsilon_c) < 0,$$

since $\bar{\epsilon}_c < 0$ and $0 < \bar{\epsilon} < \epsilon_c$. So $\theta'''(0) < 0$.

(ii) $Q(C) = \frac{2}{\epsilon^2(1 - m)m^3}[4 - 4\sqrt{1 - m} - 3m + \sqrt{1 - mm}]$

$$+ m(2 - 2\sqrt{1 - m} - 2m + \sqrt{1 - mm})\epsilon] > 0,$$

if

$$0 < \epsilon < \epsilon_2 \equiv \frac{4 - 4\sqrt{1 - m} - 3m + \sqrt{1 - mm}}{m(-2 + 2\sqrt{1 - m} + 2m - \sqrt{1 - mm})} \quad (> 0). \quad (3.10)$$

We note that $\epsilon_2 > 0$ since, for $0 < m < 1$, it is easy to show that

$$4 - 4\sqrt{1 - m} - 3m + \sqrt{1 - mm} > 0,$$

and

$$m(-2 + 2\sqrt{1 - m} + 2m - \sqrt{1 - mm}) = \sqrt{1 - m}(1 + \sqrt{1 - m})^2 > 0. \quad (3.11)$$

Now

$$\epsilon_2 - \bar{\epsilon} = \frac{8 - 8\sqrt{1 - m} - 8m + 4\sqrt{1 - mm} + m^2}{m^2(-2 + 2\sqrt{1 - m} + 2m - \sqrt{1 - mm})}$$

$$\equiv \frac{N_6(m)}{m^2(-2 + 2\sqrt{1 - m} + 2m - \sqrt{1 - mm})}, \quad (3.12)$$

where $N_6(m) = 8 - 8\sqrt{1 - m} - 8m + 4\sqrt{1 - mm} + m^2$ satisfies the following conditions.

(i) $N_6(0) = 0$.

(ii) $N_6'(m) = 2(1 - \sqrt{1 - m})^3/\sqrt{1 - m} > 0$ for $0 < m < 1$.

So $N_6(m) > 0$ and hence $0 < \bar{\epsilon} < \epsilon_2$ by (3.11) and (3.12). It follows that, if $0 < \epsilon < \bar{\epsilon}$, then $Q(C) > 0$ and hence $\theta'''(C) > 0$.

(iii) By (i) and (ii), $Q(0) < 0$ and $Q(C) > 0$. In addition, it can be computed that

$$Q\left(-\frac{1}{\epsilon}\right) = \frac{1}{\epsilon} > 0.$$

For $0 < m < 1$, the function $Q(u)$ in (3.9) is a quadratic polynomial with negative leading coefficient $-\epsilon^5m(1 - m)^2$. So $Q(u)$ changes sign exactly once on $(0, C)$. Hence $\theta'''(u)$ changes sign exactly once on $(0, C)$.

The proof of lemma 3.6 is now complete.

We are now in a position to prove lemma 3.2 by previous lemmata.

**Proof of lemma 3.2.** By previous lemmata 2.1 and 3.3–3.6, lemma 3.2 follows by applying the same argument used in the proof of lemma 2.1 of Wang et al. (1994). The detail proof of lemma 3.2 is omitted.
Throughout this section, we let $m = \frac{1}{2}$. To prove part (ii) of theorem 1.4 for $m = \frac{1}{2}$, as a special case in §3a for $0 < m < 1$, by lemmata 3.1 and 2.1, it suffices to show that the following is true.

**Lemma 3.7.** Let $m = \frac{1}{2}$. If $0 < \epsilon < 0.30028$ then $\theta(C) < 0$.

We recall some results obtained in §3a as in (3.13)-(3.19) for $m = \frac{1}{2}$.

\[
\begin{align*}
\epsilon &= 6 - 4\sqrt{2} \approx 0.34315, \\
\epsilon_c &= 2\sqrt{2} - 2 \approx 0.82843, \\
A &= \frac{2 - 3\epsilon - \sqrt{4 - 12\epsilon + \epsilon^2}}{2\epsilon^2} > 0, \quad \text{if } 0 < \epsilon < \bar{\epsilon}, \\
C &= \frac{2\sqrt{2} - 2 - \epsilon}{\epsilon^2} > 0, \quad \text{if } 0 < \epsilon < \epsilon_c,
\end{align*}
\]

\[
\begin{align*}
\theta'(u) &= f(u) - uf'(u) = \frac{1}{2}[2 + (-2 + 3\epsilon)u + \epsilon^2u^2](1 + \epsilon u)^{-3/2}e^{u/(1+\epsilon u)}, \\
\theta''(u) &= -uf''(u) = \frac{1}{2}[(-4 + 4\epsilon + \epsilon^2) + \theta(-4 + 8\epsilon + 3\epsilon^2)u \\
&+ \epsilon^3(4 + 3\epsilon)u^2 + \epsilon^5u^3](1 + \epsilon u)^{-9/2}e^{u/(1+\epsilon u)}, \\
\theta'''(u) &= e^{u/(1+\epsilon u)}(1 + \epsilon u)^{-11/2}[\frac{1}{8}[-8 + 8\epsilon + 2\epsilon^2] + (-8 + 20\epsilon + 6\epsilon^2 + 5\epsilon^3)u \\
&+ 2\epsilon(28 - 12\epsilon + 3\epsilon^2)u^2 - \epsilon(10 + \epsilon)u^3 - 6u^4]
\end{align*}
\]

see (1.4), (2.7), (2.5), (2.8), (2.3), (2.4), (3.9). We also recall lemmata 3.6 for $m = \frac{1}{2}$ as in the next lemma.

**Lemma 3.8.** For $m = \frac{1}{2}$, if $0 < \epsilon < \bar{\epsilon} = 6 - 4\sqrt{2} \approx 0.34315$ then:
(i) $\theta'''(0) < 0$;
(ii) $\theta'''(C) > 0$;
(iii) $\theta'''(u)$ changes sign exactly once on $(0, C)$.

Moreover, we show the following.

**Lemma 3.9.** For $m = \frac{1}{2}$:
(i) $\theta'''(A) < 0$ if $0 < \epsilon < 7 - 3\sqrt{5} \approx 0.29180$;
(ii) $\theta'''(A) > 0$ if $7 - 3\sqrt{5} < \epsilon < \bar{\epsilon} = 6 - 4\sqrt{2} \approx 0.34315$.

Proof of lemma 3.9. Let $m = \frac{1}{2}$. By (3.19), on $[0, \infty)$, $\theta'''(u) > 0$ if and only if $Q(u) > 0$. We compute that

\[
Q(A) = \frac{256(-4 + 14\epsilon - \epsilon^2)}{(32 - 84\epsilon + 20\epsilon^2 - \epsilon^3) + (16 - 14\epsilon + \epsilon^2)\sqrt{4 - 12\epsilon + \epsilon^2}}
\]

\[
\begin{align*}
&< 0, \quad \text{if } 0 < \epsilon < 7 - 3\sqrt{5} \approx 0.29180, \\
&> 0, \quad \text{if } 7 - 3\sqrt{5} < \epsilon < 6 - 4\sqrt{2} < 7 + 3\sqrt{5},
\end{align*}
\]

in which: (i) $7 - 3\sqrt{5}$ and $7 + 3\sqrt{5}$ are two zeros of the quadratic polynomial $-4 + 14\epsilon - \epsilon^2$; and (ii) $32 - 84\epsilon + 20\epsilon^2 - \epsilon^3 > 0$ and $(16 - 14\epsilon + \epsilon^2)\sqrt{4 - 12\epsilon + \epsilon^2} > 0$ for $0 < \epsilon < \bar{\epsilon} = 6 - 4\sqrt{2} \approx 0.34316$. Thus lemma 3.9 follows immediately. ■
S-shaped bifurcation curves in a combustion problem

S-shaped bifurcation curves in a combustion problem

Figure 3. Graphs of $\hat{\epsilon}$ and $\bar{\epsilon}$.

Figure 4. The graph of $y = \theta'(x)$ on $(0, C)$ for $0 < \epsilon \leq 7 - 3\sqrt{5}$; $U = (A, 0)$, $P = (C, 0)$, $Q = (C, \theta'(C))$, $V = (C, \theta'(N))$.

For $A$ and $C$ in (3.15) and (3.16), it is easy to obtain the next lemma for which we omit the proof.

**Lemma 3.10.** Let $m = \frac{1}{2}$. If

$$0 < \epsilon < \frac{1}{9}(2(1 - 2\sqrt{2} + \sqrt{8\sqrt{2} - 6})) \approx 0.31781,$$

then

$$C - 2A = \frac{-4 + 2\sqrt{2} + 2\epsilon + \sqrt{4 - 12\epsilon + \epsilon^2}}{\epsilon^2} > 0.$$

We are now in a position to prove lemma 3.7.

**Proof of lemma 3.7.** The proof of lemma 3.7 is similar to that of lemma 3 of Wang (1994). For the convenience of discussion, we divide the interval $0 < \epsilon < 0.30028$ into two subintervals: (I) $0 < \epsilon \leq 7 - 3\sqrt{5}$ ($\approx 0.29180$); and (II) $7 - 3\sqrt{5} < \epsilon < 0.30028$.

**Subinterval I.** For $0 < \epsilon \leq 7 - 3\sqrt{5}$, by lemmata 3.8 and 3.9, there exists a number $M$ with $0 < A \leq M < C$ such that the function $\theta'$ satisfies (see figure 4):

$$\begin{align*}
(\theta')''(x) &< 0, \quad \text{for } x \in (0, M); \\
(\theta')'''(x) &> 0, \quad \text{for } x \in (M, C);
\end{align*}$$

$$\theta'(0) = 1, \quad \theta'(A) = 0, \quad \theta'(C) < 0, \quad (\theta')'(x) < 0, \quad \text{for } x \in (0, C).$$

(3.20)

Let $U = (A, 0)$, $P = (C, 0)$, $Q = (C, \theta'(C))$ (see figure 4). Then the tangent line of $y = \theta'(x)$ at $U = (A, 0)$ will intersect the line through the points $P$ and $Q$ at some point $V = (C, \theta'(N))$. There are two subcases to be considered.

**Subcase A.** $\theta'(C) \leq \theta'(N)$. In this subcase, by lemma 3.10, $C > 2A > 0$ and by

(3.20) the convexity of \(\theta'(x)\) on \((0, C)\), it is easy to see that
\[
0 < \int_0^A \theta'(t) \, dt < -\int_A^C \theta'(t) \, dt < -\int_A^C \theta'(t) \, dt,
\]
(3.21)
(see figure 4a). Thus in this subcase (A)
\[
\theta(C) = \int_0^C \theta'(t) \, dt = \int_0^A \theta'(t) \, dt + \int_A^C \theta'(t) \, dt < 0.
\]
(3.22)

**Subcase B.** \(\theta'(C) > \theta'(N)\). In this subcase we obtain
\[
\int_0^A \theta'(t) \, dt < \text{area}(\Delta OUW),
\]
(3.23)
\[
-\int_A^C \theta'(t) \, dt > \text{area}(\Delta UPQ),
\]
(3.24)
where \(O\) is the origin and \(W\) is the intersection point of the tangent line of \(y = \theta'(x)\) at the point \(U = (A, 0)\) with the positive \(y\)-axis (see figure 4b). We compute that
\[
\text{area}(\Delta OUW) = -\frac{1}{2} A^2 \theta''(A)
\]
\[
= -\left[ \exp \left( \frac{2 - 3\epsilon - \sqrt{4 - 12\epsilon + \epsilon^2}}{\epsilon(2 - \epsilon - \sqrt{4 - 12\epsilon + \epsilon^2})} \right) \right. \\
\times \left. \frac{(2 - 3\epsilon - \sqrt{4 - 12\epsilon + \epsilon^2})^3}{4\sqrt{2\epsilon^5/2}(2 - \epsilon - \sqrt{4 - 12\epsilon + \epsilon^2})^{9/2}} \right] \\
\times \left[ 16 - 52\epsilon + 16\epsilon^2 - \epsilon^3 - (8 - 10\epsilon + \epsilon^2)\sqrt{4 - 12\epsilon + \epsilon^2} \right],
\]
(3.25)
\[
\text{area}(\Delta UPQ) = \frac{1}{2}(C - A)(-\theta'(C))
\]
\[
= \left[ \exp \left( \frac{2 - 2\sqrt{2} + \epsilon}{\epsilon(2 - 2\sqrt{2})} \right) \right. \\
\times \left. \frac{[20 - 14\sqrt{2} - (2 - 2\sqrt{2})\epsilon](-6 + 4\sqrt{2} + \epsilon + \sqrt{4 - 12\epsilon + \epsilon^2})}{8\sqrt{2}(\sqrt{2} - 1)^{5/2}\epsilon^{3/2}} \right],
\]
(3.26)

Let
\[
G_1(\epsilon) \equiv \text{area}(\Delta OUW) - \text{area}(\Delta UPQ) = (3.25) - (3.26).
\]
(3.27)
It can be easily shown that \(G_1(\epsilon) < 0\) for \(0 < \epsilon < 7 - 3\sqrt{5} \approx 0.29180\). Hence, in this subcase (B), by (3.23) and (3.24), we obtain
\[
\theta(C) = \int_0^C \theta'(t) \, dt = \int_0^A \theta'(t) \, dt + \int_A^C \theta'(t) \, dt < \text{area}(\Delta OUW) - \text{area}(\Delta UPQ) = G_1(\epsilon) < 0, \quad \text{for } 0 < \epsilon < 7 - 3\sqrt{5}.
\]
(3.28)

**Subinterval II.** For \(7 - 3\sqrt{5} < \epsilon < 0.30028\), similarly, by lemmata 3.8 and 3.9, there exists a number \(M\) with \(0 < M < A\) such that the function \(\theta'\) satisfies (3.20) (see figure 5).

Thus it is easy to see that
\[
\int_0^A \theta'(t) \, dt < A = \frac{2 - 3\epsilon - \sqrt{4 - 12\epsilon + \epsilon^2}}{2\epsilon^2},
\]
(3.29)
Figure 5. The graph of $y = \theta'(x)$ on $(0, C)$ for $7 - 3\sqrt{5} < \epsilon < 0.30028$; $U = (A, 0)$, $P = (C, 0)$, $Q = (C, \theta'(C))$.

\[-\int_{A}^{C} \theta'(t) \, dt > \text{area}(\Delta U P Q) = \frac{1}{2}(C - A)(-\theta'(C)) = \left[ \exp \left( \frac{2 - 2\sqrt{\epsilon}}{\epsilon(2 - 2\sqrt{\epsilon})} \right) \right] \times \frac{[20 - 14\sqrt{2} - (2 - \sqrt{2})\epsilon][-6 + 4\sqrt{2} + \epsilon + \sqrt{4 - 12\epsilon + \epsilon^2}]}{8\sqrt{2}(\sqrt{2} - 1)^{5/2} \epsilon^{5/2}}.\] (3.30)

Let

$$G_2(\epsilon) \equiv A - \text{area}(\Delta U P Q).$$ (3.31)

$G_2(\epsilon)$ defines a unique $\epsilon^* \approx 0.300281$ such that

$$G_2(\epsilon) < 0, \quad \text{for } 0 < \epsilon < \epsilon^*, \quad G_2(\epsilon^*) = 0,$$

$$G_2(\epsilon) > 0, \quad \text{for } \epsilon^* < \epsilon < 0.34315.$$ (3.32)

So by (3.29)–(3.32),

$$\theta(C) = \int_{0}^{C} \theta'(t) \, dt = \int_{0}^{A} \theta'(t) \, dt + \int_{A}^{C} \theta'(t) \, dt < A - \text{area}(\Delta U P Q) = G_2(\epsilon) < 0, \quad \text{for } 7 - 3\sqrt{5} < \epsilon < 0.30028 < \epsilon^*.\] (3.33)

Combining (3.28) and (3.33), for $m = \frac{1}{2}$, we obtain $\theta(C) < 0$ for $0 < \epsilon < 0.30028$.

The proof of lemma 3.7 is now complete. \hfill \qed

4. Proof of theorem 1.5

(a) Proof of part (i) of theorem 1.5

For fixed $m < 0$ and $0 < \epsilon < \epsilon_c$, we recall $C$ in (2.8),

$$C (= C(\epsilon)) \equiv \frac{-1 + \sqrt{1 - m} + m - \epsilon m + \epsilon m^2}{\epsilon^2(1 - m)m} > 0.$$ The function $\theta(u)$ in (2.2) satisfies:

(i) $\theta(0) = 0, \theta'(0) = 1$;

(ii) $\theta''(u) < 0$ for $0 < u < C$;

by (2.3), (2.4) and (2.13). In addition, suppose that,

$$\theta(C) < 0,$$
then it is easy to see that $T'(C) < 0$ by (2.1). It is well known that $\lim_{\alpha \to 0} T(\alpha) = 0$ and $\lim_{\alpha \to \infty} T(\alpha) = \infty$. Thus, the time map $T(\alpha)$ has at least two critical points, a local maximum and a local minimum, on $(0, \infty)$. Hence the bifurcation curve $S$ has at least two turning points and part (i) of theorem 1.5 follows. We then show the following.

**Lemma 4.1.** Let $m < 0$. If $0 < \epsilon < \frac{1}{2} \tilde{\epsilon}$ then $\theta(C) < 0$.

Similar to lemma 3.2, lemma 4.1 follows from lemmata 2.2 and 4.2–4.5 by applying the same argument used in lemma 6 of Wang et al. (1994). The detailed proof of lemma 4.1 is omitted.

Let
\[
\theta_u \equiv f(u) - uf'(u) = [1 + (-1 + 2\epsilon - cm)u + \epsilon^2(1 - m)u^2](1 + \epsilon u)^{m-2}\epsilon u/(1+\epsilon u),
\]
see (2.3).

Similar to lemma 3.3 for $0 < m < 1$, the next lemma follows by straightforward but tedious algebra. We omit the proof.

**Lemma 4.2.** For fixed $m < 0$, if $0 < \epsilon < \frac{1}{2} \tilde{\epsilon} < \epsilon_c$, then at $u = C(\epsilon)$, the function $\theta_u$ satisfies:

(i) $\partial \theta_u / \partial \epsilon = \partial \theta_u(\epsilon) / \partial \epsilon > 0$;

(ii) $2 + \theta_u(\epsilon) < 0$.

It was shown in §3a that, for $m < 0$ and $0 < \epsilon < \tilde{\epsilon} (< \epsilon_c)$, \[C - 2A = \frac{m(-1 + \sqrt{1 - m + \epsilon m}) + m^2\sqrt{1 - 4\epsilon + 2cm + \epsilon^2 m^2}}{\epsilon^2(1 - m)m^2} > 0,\] if
\[K \equiv (m^2\sqrt{1 - 4\epsilon + 2cm + \epsilon^2 m^2})^2 - [m(-1 + \sqrt{1 - m + \epsilon m})]^2
= m^4(m^2 - 1)(\epsilon - \tilde{\epsilon})(\epsilon - \hat{\epsilon}) \quad \text{(if } m \neq -1)\]
\[> 0, \quad \text{(4.1)}\]
where, if $m \neq -1$,
\[\hat{\epsilon} \equiv \frac{-m(1 - \sqrt{1 - m - 2m + m^2}) + \sqrt{(-4 + 4\sqrt{1 - m + 5m})(1 - m)m^3}}{m^2(m^2 - 1)},\]
\[\tilde{\epsilon} \equiv \frac{-m(1 - \sqrt{1 - m - 2m + m^2}) - \sqrt{(-4 + 4\sqrt{1 - m + 5m})(1 - m)m^3}}{m^2(m^2 - 1)},\]
see (3.5)–(3.7).

Moreover, similar to lemma 3.4 for $0 < m < 1$, further manipulations enable us to prove the next lemma for which we omit the proof. We note that this lemma can be checked by simply using MATHEMATICA.

**Lemma 4.3.** (See figure 3).

(i) If $-1 < m < 0$ then $\hat{\epsilon} < 0 < \frac{1}{2} \tilde{\epsilon} < \tilde{\epsilon}.$
(ii) If $m < -1$ then $0 < \frac{1}{2} \tilde{\epsilon} < \hat{\epsilon} < \tilde{\epsilon}.$

Lemma 4.3 together with (4.2) implies the next lemma.

**Lemma 4.4.** Let $m < 0$. If $0 < \epsilon < \frac{1}{2} \tilde{\epsilon}$ then $C > 2A$. 

Lemma 3.6, it can be shown that, for $m < 0$, the function $\theta(u)$ changes sign exactly once on $(0, C)$.

Proof. Recall (3.9),

$$\theta''(u) = e^{u/(1+\epsilon u)}(1+\epsilon u)^{m-6}(-1+6\epsilon-2\epsilon m+\epsilon^2 m - \epsilon^2 m^2) + (1+\epsilon u)^{m-6}(-1+6\epsilon-2\epsilon m+\epsilon^2 m - \epsilon^2 m^2) + \cdots$$

Thus, on $[0, \infty)$, $\theta''(u) > 0$ if and only if $Q(u) > 0$. Similar to that of proof of lemma 3.6, it can be shown that, for $m < 0$, if $0 < \epsilon < \tilde{\epsilon}$ then

$$Q(0) < 0, \quad Q(C) > 0, \quad Q(D) < 0.$$  \hfill (4.4)

Thus (i)–(iii) follows.

For $m < 0$, the function $Q(u)$ in (4.3) is a quadratic polynomial with positive leading coefficient $-m(1-m)^2$. So by (4.4), $Q(u)$ changes sign exactly once on $(0, C)$. Hence $\theta''(u)$ changes sign exactly once on $(0, C)$.

The proof of lemma 4.5 is now complete. \hfill \Box

Lemma 4.6. For fixed $m < 0$, if $0 < \epsilon < \frac{1}{2}\tilde{\epsilon}$ then $\theta(u)$ satisfies:

(i) $\theta(0) = 0$, $\theta'(0) = 1$, $\theta(A) = 0$;
(ii) $\theta''(u) < 0$ on $(0, C)$, $\theta''(u) > 0$ on $(C, D)$, $\theta''(u) < 0$ on $(D, \infty)$;
(iii) $\theta(C) < 0$, $\theta(A) > 0$;
(iv) $\lim_{u \to -\infty} \theta'(u) = 0$;
(v) $\theta'(u) > 0$ on $[D, \infty)$, $\theta'(C) < 0$;
(vi) $\lim_{u \to -\infty} \theta(u) = \infty$ if $-1 \leq m < 0$ and $\lim_{u \to -\infty} \theta(u) < \infty$ and exists if $m < -1$.  

Proof. (i) $\theta(0) = 0$ and $\theta'(0) = 1$ by (2.2) and (2.3) at $u = 0$, and $\theta'(A) = 0$ since $A$ is the first zero of $\theta'(u)$ on $(0, \infty)$.
(ii) It follows from (2.4) and (2.13).
(iii) $\theta(C) < 0$ by (4.4) and hence $\theta(A) > 0$ by (i) and (ii) since $0 < A < C$.
(iv) For $m < 0$, since
$$\theta'(u) = [1 - (1 - 2\epsilon + em)u + \epsilon^2(1 - m)u^2](1 + \epsilon u)^{m-2}e^{u/(1+\epsilon u)},$$
it is easy to see that $\lim_{u \to \infty} \theta'(u) = 0$.
(v) $\theta'(u) > 0$ on $[D, \infty)$ since for $m < 0$, $\theta'(u) > 0$ if $u$ is large enough and by (ii) $\theta'(u)$ is strictly decreasing on $[D, \infty)$. Moreover, $\theta'(C) < 0$ by (i)–(iii).
(vi) It holds since $\theta'(u) \sim (1 + \epsilon u)^m$ as $u \to \infty$.

Lemma 4.6 and a slight modification of lemma 3.1 imply the next lemma immediately.

**Lemma 4.7.** For $m < 0$ and $0 < \epsilon < \frac{1}{2} \hat{\epsilon}$, the bifurcation curve $S$ is S-shaped if
$$\theta(D) > \theta(A) \quad (\epsilon > 0).$$

By some numerical simulations, it seems that (4.5) holds if and only if negative $m$ is sufficiently close to 0. ((4.5) seems to hold for $-20 < m < 0$.) Although we are not able to prove (4.5) for general negative $m$ close to 0, we are able to prove lemma 4.7 for individual negative $m$ close to 0. In particular, we next prove two important values for $m = -1$ and $-2$.

Let $m < 0$ be fixed and $\epsilon > 0$. For $A$, $B$, $C$ and $D$ given in (2.5), (2.6), (2.12) and (2.11), we define the function $Z_m(\epsilon)$ and compute that
$$Z_m(\epsilon) \equiv \theta'(C) + \theta'(D - B + A)$$
$$= \left\{ \begin{array}{l}
\exp \left( \frac{-1 + \sqrt{1 - m + m - \epsilon m + \epsilon^2 m^2}}{\epsilon(1 - m)m} \right) \left( \frac{-1 + \sqrt{1 - m + m}}{\epsilon(1 - m)m} \right)^{m-2} \\
\times \left( \frac{2 - 2\sqrt{1 - m - 2m + \sqrt{1 - mm} + \epsilon \sqrt{1 - mm}}}{\epsilon^2(1-m)m^2} \right)
\end{array} \right\}
$$
$$+ \left\{ \begin{array}{l}
\exp \left( \frac{1 + \sqrt{1 - m - m + \epsilon m - \epsilon^2 m^2} + m \sqrt{1 - 4\epsilon + 2m(1 + \epsilon^2 m^2)}}{\epsilon(1 - m)m} \right) \\
\times \left( \frac{-1 - \sqrt{1 - m - m - \epsilon m + \epsilon^2 m^2}}{\epsilon(1 - m)m} \right)^{m-2} \left( \frac{1}{\epsilon^2(1-m)m^2} \right) \\
\times \left( 2 + 2\sqrt{1 - m - m} - \sqrt{1 - mm + m^2} - 4\epsilon m^2 - \epsilon \sqrt{1 - mm + 2m^3} + \epsilon^2 m^4 + m(2 + 2\sqrt{1 - m - m - \epsilon m^2}) \sqrt{1 - 4\epsilon + 2m + \epsilon^2 m^2} \right)
\end{array} \right\}. $$

By lemma 4.6,
$$\theta(D) - \theta(A) = \int_A^D \theta'(u) \, du = \int_A^B \theta'(u) \, du + \int_B^D \theta'(u) \, du$$
$$> (B - A)\theta'(C) + (B - A)\theta'(D - B + A)$$
$$= (B - A)[\theta'(C) + \theta'(D - B + A)] = (B - A)Z_m(\epsilon),$$
(see figure 6). Thus the inequality (4.5) $\theta(D) > \theta(A)$ holds if $Z_m(\epsilon) > 0$. It can be
shown that, for $m = -1$ and $-2$

$$Z_{-1}(\epsilon) = \left\{ \begin{array}{l} \exp \left( \frac{-2 + \sqrt{2} + 2\epsilon}{(-2 + \sqrt{2})\epsilon} \right) \left( \frac{-4\epsilon(4 - 3\sqrt{2} + \sqrt{2}\epsilon)}{(-2 + \sqrt{2})^3} \right) \\ + \left\{ \exp \left( \frac{2 + \sqrt{2} - 2\epsilon - \sqrt{1 - 6\epsilon + \epsilon^2}}{\epsilon(2 + \sqrt{2} - \sqrt{1 - 6\epsilon + \epsilon^2})} \right) \\ \times \left( \frac{4\epsilon[5 + 3\sqrt{2} - 6\epsilon - \sqrt{2}\epsilon + \epsilon^2 - (3 + 2\sqrt{2} - \epsilon)\sqrt{1 - 6\epsilon + \epsilon^2}]}{(2 + \sqrt{2} - \sqrt{1 - 6\epsilon + \epsilon^2})^3} \right) \right\} > 0, \quad \text{for} \quad 0 < \epsilon < \frac{1}{2}\tilde{\epsilon} = \frac{1}{2}(3 - 2\sqrt{2}). \quad (4.6) \\
\end{array} \right.$$ 

$$Z_{-2}(\epsilon) = \left\{ \begin{array}{l} \exp \left( \frac{-3 + \sqrt{3} + 6\epsilon}{(-3 + \sqrt{3})\epsilon} \right) \times \left( \frac{108\epsilon^2(6 - 4\sqrt{3} + 4\sqrt{3}\epsilon)}{(-3 + \sqrt{3})^4} \right) \\ + \left\{ \exp \left( \frac{3 + \sqrt{3} - 6\epsilon - 2\sqrt{1 - 8\epsilon + 4\epsilon^2}}{\epsilon(3 + \sqrt{3} - 2\sqrt{1 - 8\epsilon + 4\epsilon^2})} \right) \\ \times \left( \frac{216\epsilon^2[5 + 2\sqrt{3} - 16\epsilon - 2\sqrt{3}\epsilon + 8\epsilon^2 - (4 + 2\sqrt{3} - 4\epsilon)\sqrt{1 - 8\epsilon + 4\epsilon^2}]}{(3 + \sqrt{3} - 2\sqrt{1 - 8\epsilon + 4\epsilon^2})^4} \right) \right\} > 0, \quad \text{for} \quad 0 < \epsilon < \frac{1}{2}\tilde{\epsilon} = \frac{1}{2}(2 - \sqrt{3}). \quad (4.7) \\
\end{array} \right.$$ 

By (4.6), for $m = -1$, (4.5) holds and hence by lemma 4.7 the bifurcation curve $S$ is S-shaped for $0 < \epsilon < \frac{1}{2}\tilde{\epsilon} = \frac{1}{2}(3 - 2\sqrt{2})$. By (4.7), for $m = -2$, (4.5) holds and hence by lemma 4.7 the bifurcation curve $S$ is S-shaped for $0 < \epsilon < \frac{1}{2}\tilde{\epsilon} = \frac{1}{2}(2 - \sqrt{3})$. This completes the proof of parts (ii) and (iii) of theorem 1.5.

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References


Boddington, T., Feng, C.-G. & Gray, P. 1984 Thermal explosion and the disappearance of criticality in systems with distributed temperatures. II. An asymptotic analysis of criticality at the extremes of Biot number ((Bi) → 0, (Bi) → ∞) for generalized reaction-rate laws. Proc. R. Soc. Lond. A 390, 301–322.


