Bifurcation of an equation arising in porous-medium combustion

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We study the bifurcation of positive solutions of a steady-state problem arising in porous-medium combustion:

\[ \frac{\partial^2 u}{\partial x^2} + \lambda \frac{1 + au}{1 + e^{(1-u)/\epsilon}} = 0 \quad (-1 < x < 1), \]

\[ u(-1) = u(1) = 0. \]

We give explicit criteria for a unique solution for all \( \lambda > 0 \) when \( a = 0 \) and for an S-shaped bifurcation curve when \( a > 0 \). We also prove a conjecture of Norbury & Stuart (1987, IMA J. Appl. Math. 39, 241–57).

1. Introduction

Porous-medium combustion occurs in a number of situations including the burning of cocal, the smouldering of polyurethane, the use of catalytic converters as exhaust filters, and the burning of cigarettes; see [1,5]. Recently, a model for combustion in a porous medium was developed by Norbury & Stuart [5]. Norbury & Stuart developed a three-dimensional model for a chemical process of the type

\[ \text{solid} + O_2 \rightarrow \text{heat} + CO_2 + \text{ash}. \]

Their model represents conservation of mass and energy for both the gas and solid species, while the fluid flow is governed by Darcy's law and the ideal-gas law. Subsequently, in the case of one-space-dimension combustion, they used a number of asymptotic considerations to arrive at a simplified heat-equation model

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda f(u) = \frac{\partial^2 u}{\partial x^2} + \lambda \frac{1 + au}{1 + e^{(1-u)/\epsilon}} \quad (-1 < x < 1), \quad (1.1) \]

with \( u(\pm1, t) = 0 \) and initial conditions.

In (1.1), \( u \) is the dimensional solid temperature, \( \lambda > 0 \) is the Frank–Kamenetskii parameter, and \( f(u) \) is the reaction rate of the chemical reaction, in which \( \epsilon \) is small is the reciprocal activation energy, and \( a \neq 0 \) is a parameter which determines the ratio of the rate of oxygen consumption to that of the solid. Note that combustion of carbonaceous material typically involves a reciprocal activation energy of \( \epsilon \sim 0.1 \); see [4,5].

The steady-state equation associated with (1.1) is the two-point boundary-value
For simplicity of computation in the analysis of problem (1.2), we let
\[ d = \frac{1}{\epsilon} > 0 \]  
(1.3)
and rewrite problem (1.2) as
\[ -\frac{\partial^2 u}{\partial x^2} = \lambda f(u) = \lambda \frac{1 + au}{1 + e^{a(1-u)}}, \]  
(1.4)
\[ u(-1) = u(1) = 0. \]

For problem (1.4), we are interested in positive solutions. More importantly, we are interested in positive solutions, \( u \), with
\[ \|u\|_\infty > 1; \]
see [4, 5]. Let
\[ S = \{(\lambda, u) : \lambda \geq 0 \text{ and } u \text{ is a positive solution of (1.4)}\}. \]  
(1.5)
It can be shown by a time-map formula that the bifurcation curve \( S \) defined above is a smooth continuum satisfying
\[ \lim_{\lambda \to 0} \lambda = 0, \quad \lim_{\lambda \to +\infty} \lambda = \frac{\pi^2}{4a} := \lambda_\infty. \]

Norbury & Stuart [4] studied (1.4) and they showed that

**Proposition** (Theorem 6.2 of [4]) There exists a positive stable solution of problem (1.4) (when viewed as a steady state of (1.1)) for all \( \lambda \), \( d > 0 \), whenever
\[ 0 < \lambda < \lambda_\infty. \]
Through numerical evaluation, Norbury & Stuart [4] claimed that, for any \( \lambda > 0 \), if \( d \) is large enough, two positive numbers \( \lambda, \lambda \) exist, with
\[ \lambda < \lambda_\infty < \lambda, \]
(1.6)
such that the bifurcation curve \( S \) is S-shaped; that is, the bifurcation curve \( S \) has exactly two turning points. Moreover, they claimed that

(i) problem (1.4) has exactly three positive solutions for \( \lambda < \lambda < \lambda_\infty \), exactly two positive solutions for \( \lambda < \lambda < \lambda_\infty < \lambda < \lambda \) and \( \lambda = \lambda, \) exactly one positive solution for \( 0 < \lambda < \lambda \) and \( \lambda = \lambda, \) and no positive solution for \( \lambda > \lambda; \)

(ii) at the turning points \( (\lambda, u_\lambda) \) and \( (\lambda, u_\lambda) \) of the S-shaped bifurcation curve \( S, \)
\[ \|u_\lambda\|_\infty < 1 < \|u_\lambda\|_\infty \]
(see Fig. 2 and pp. 254–5 of [4]).
In this paper, we prove this claim by Norbury & Stuart [4]. Moreover, we give explicit criteria for a unique solution for all $\lambda > 0$, where $a = 0$, and for an S-shaped bifurcation curve when $a \geq 0$. Our method is based upon detailed analysis of the time-map formula

$$\left[ \lambda (\alpha) \right]^a = 2^{-1} \int_0^\alpha \frac{du}{[F(\alpha) - F(u)]^2} := T(\alpha) \quad (\alpha > 0),$$

(1.7)

where $F(x) = \int_0^x f(u) \, du$ and $\alpha = \|u\|_\infty = u(0)$. Note that when $a = 0$ this time-map formula, $T(\alpha)$, has been applied by Norbury & Stuart [5] to the study of the stable positive solution $u = u(x, d)$ of (1.4) with $\|u\|_\infty > 1$ as $d \to +\infty$ (see the appendix of [5] for details).

2. Main results

The main results are given in Theorem 1 for $a = 0$ and in Theorems 2–4 for $a > 0$ in which we study the shape of the bifurcation curve $S$ of positive solutions for (1.4). For $a \geq 0$ and $d > 0$, we say the bifurcation curve $S$ is a monotone curve if the bifurcation curve $S$ has no turning point. That is, problem (1.4) has at most one positive solution for all $\lambda > 0$; see, for example, Fig. 1(a).

**Theorem 1** (see Fig. 1) Let $a = 0$ and $d > 0$. Then the bifurcation curve $S$ satisfies

$$\lim_{\alpha \to 0} \lambda (\alpha) = 0, \quad (2.1)$$

$$\lim_{\alpha \to +\infty} \lambda (\alpha) / \alpha = 2. \quad (2.2)$$

Moreover,

(i) the bifurcation curve $S$ is a monotone curve for $0 < d \leq 2$;

(ii) a number $d_0 \approx 2.43751$ exists such that for $d > d_0$, the bifurcation curve $S$ is S-shaped. More precisely, two positive numbers $\lambda$ and $\lambda^\ast$ exist, with $\lambda < \lambda^\ast$, such that problem (1.4) has exactly three positive solutions for $\lambda < \lambda < \lambda^\ast$, exactly two positive solutions for $\lambda = \lambda$ and $\lambda = \lambda^\ast$, exactly one positive solution for $0 < \lambda < \lambda$ and $\lambda > \lambda^\ast$. Moreover, at the turning points $(\lambda, u_\lambda)$ and $(\lambda^\ast, u_{\lambda^\ast})$ of the S-shaped bifurcation curve $S$

$$\|u_\lambda\|_\infty < 1 < \|u_{\lambda^\ast}\|_\infty.$$

**Theorem 2** (see Fig. 2) Let $a > 0$ and $d > 0$. Then the bifurcation curve $S$ satisfies

$$\lim_{\alpha \to 0} \lambda (\alpha) = 0, \quad (2.3)$$

$$\lim_{\alpha \to +\infty} \lambda (\alpha) = \lambda_\infty \left( = \frac{\pi^2}{4a} \right). \quad (2.4)$$
Fig. 1. Bifurcation curves for: (a) $a = 0$ and $0 < d < 2$, and (b) $a = 0$ and $d > d_0 \approx 2.43751$.

(i) A number $d_1 \approx 3.05084$ exists which is the unique positive root of equality (3.29) and which is independent of $a$, such that for

$$d > d_1$$

the bifurcation curve $S$ is S-shaped. More precisely, two positive numbers $\lambda$ and $\bar{\lambda}$ exist, with $\lambda < \bar{\lambda}$ and $\lambda < \lambda_\infty$, such that:

**case 1**, if $\bar{\lambda} > \lambda_\infty$ then problem (1.4) has exactly three positive solutions for $\lambda < \lambda < \lambda_\infty$, exactly two positive solutions for $\lambda_\infty < \lambda < \bar{\lambda}$ and $\lambda = \lambda_\infty$, exactly one positive solution for $0 < \lambda < \lambda_\infty$ and $\lambda = \bar{\lambda}$, and no positive solution for $\lambda > \bar{\lambda}$;

**case 2**, if $\bar{\lambda} = \lambda_\infty$ then problem (1.4) has exactly three positive solutions for $\lambda < \lambda < \lambda_\infty$, exactly two positive solutions for $\lambda = \bar{\lambda}$ and $\lambda = \lambda_\infty$, exactly one positive solution for $0 < \lambda < \lambda_\infty$ and $\lambda = \lambda_\infty$, and no positive solution for $\lambda > \lambda_\infty$;

**case 3**, if $\bar{\lambda} < \lambda_\infty$ then problem (1.4) has exactly three positive solutions for $\lambda < \lambda < \bar{\lambda}$, exactly two positive solutions for $\lambda = \bar{\lambda}$ and $\lambda = \lambda_\infty$, exactly one positive solution for $0 < \lambda < \lambda_\infty$ and $\lambda < \lambda_\infty$, and no positive solution for $\lambda > \lambda_\infty$.

Moreover, at the turning points $(\bar{\lambda}, u_{\bar{\lambda}})$ and $(\lambda_\infty, u_{\lambda_\infty})$ of the S-shaped bifurcation curve $S$,

$$\|u_{\bar{\lambda}}\|_\infty < 1 < \|u_{\lambda_\infty}\|_\infty.$$  

(2.6)
(ii) In addition to (2.5), if
\[ d > d_2 := 2 \log \left( \frac{2 + a}{8a} \right) \pi^2 - 1 \]  
then
\[ \bar{\lambda} > \lambda_\infty; \]
that is, only case 1 occurs (see Fig. 2(a) and figure 5 of [4]).
Note that
\[ \max (d_1, d_2) = \begin{cases} d_1 & \text{if } a > \hat{a}, \\ d_2 & \text{if } a < \hat{a}, \end{cases} \]
for some \( \hat{a} \approx 0.56548 \).

For \( a > 0 \) (compare with \( a = 0 \) in Theorem 1), it is interesting that the bifurcation curve \( S \) is S-shaped if \( d > 0 \) is small. This is equivalent to saying that, for any \( d > 0 \), the bifurcation curve \( S \) is S-shaped if \( a \) is sufficiently large.

**Theorem 3**  
Let \( a > 0 \) and \( d > 0 \). In addition to (2.3) and (2.4), if
\[
\frac{a}{d} > \frac{d}{e^{x^2} + e^{x^2 + d} + e^{x^2 + d}} := A(d)
\]  
then the bifurcation curve \( S \) is S-shaped. More precisely, two positive numbers \( \lambda \) and \( \bar{\lambda} \) exist with \( \lambda < \bar{\lambda} \) and \( \bar{\lambda} < \lambda_{\infty} \) \( (= \pi^2/4a) \) such that:

*case 1*, if \( \lambda > \lambda_{\infty} \) then problem (1.4) has exactly three positive solutions for \( 0 < \lambda < \lambda_{\infty} \), exactly two positive solutions for \( \lambda_{\infty} < \lambda < \bar{\lambda} \) and \( \lambda = \bar{\lambda} \), exactly one positive solution for \( \lambda = \bar{\lambda} \) and no positive solution for \( \lambda > \bar{\lambda} \);

*case 2*, if \( \bar{\lambda} = \lambda_{\infty} \) then problem (1.4) has exactly three positive solutions for \( \bar{\lambda} < \lambda < \lambda_{\infty} \), exactly two positive solutions for \( \lambda = \bar{\lambda} \), exactly one positive solution for \( \lambda = \lambda_{\infty} \) and no positive solution for \( \lambda > \lambda_{\infty} \);

*case 3*, if \( \bar{\lambda} < \lambda_{\infty} \) then problem (1.4) has exactly three positive solutions for \( \lambda < \bar{\lambda} \), exactly two positive solutions for \( \lambda = \bar{\lambda} \) and \( \lambda = \lambda_{\infty} \), exactly one positive solution for \( 0 < \lambda < \lambda_{\infty} \) and \( \bar{\lambda} < \lambda < \lambda_{\infty} \), and no positive solution for \( \lambda = \lambda_{\infty} \).

In addition to (2.2), if
\[
a > \frac{d\pi^2}{8e^{x^2 - 1} - \pi^2 + 8},
\]  
then
\[
\bar{\lambda} > \lambda_{\infty},
\]
that is, only case 1 occurs.

*Note*, for \( a > 0 \), suppose \( d \) is small and that it satisfies (2.8) (contrast with the case when \( d \) is large and satisfies (2.5); see Theorem 2 and Fig. 2); if
\[
0 < d < \frac{2}{1 + a},
\]
then it can be shown that \( \theta'(u) := f(u) - uf''(u) > 0 \) on \((0, 1)\) and hence \( T'(<a) > 0 \) for \( 0 < a \leq 1 \) (see (3.1)-(3.3)). So, at the turning point \((\bar{\lambda}, \|u_{\bar{\lambda}}\|_{\infty})\) of the S-shaped bifurcation curve \( S \),
\[
\|u_{\bar{\lambda}}\|_{\infty} > 1;
\]
compare this with (2.6) in Theorem 2.

We obtain the next theorem as an easy corollary to Theorems 2 and 3.
**Theorem 4** (see Fig. 3) Let \( a > 0, \ d > 0 \).

(i) If either (2.5) or (2.8) holds, then the bifurcation curve \( S \) is S-shaped.

(ii) For \( d > 0 \), the bifurcation curve of a positive solution for problem (1.4) is S-shaped if

\[
a > \bar{a} := \max_{d \in [0,d_1]} A(d) \approx 2.10492,
\]

where \( A(d) \) is defined in (2.8).

3. **Proofs of Theorems 1–4**

For Theorem 4, part i is an immediate consequence of Theorems 2 and 3, and part ii follows easily from part i; see Fig. 3. In the following, we shall prove Theorems 1–3 by an analysis of the time map \( T(\alpha) \) in (1.7). First, note that the solutions \( u \) of (1.4) corresponds to \( \|u\|_\omega = \alpha \) and \( T(\alpha) = \lambda^\frac{1}{2} \). Thus,

(A) to show that the bifurcation curve \( S \) is S-shaped on the \((\lambda, \|u\|_\omega)\)-plane is equivalent to showing that the time map \( T(\alpha) \) has exactly two critical points, a local maximum and a local minimum, on \((0, +\infty)\);

(B) to show that the bifurcation curve \( S \) is a monotone curve on the \((\lambda, \|u\|_\omega)\)-plane is equivalent to showing that the time map \( T(\alpha) \) is an increasing function of \( \alpha \) on \((0, +\infty)\).

For \( T(\alpha) \) in (1.7), it is easy to compute that

\[
T'(\alpha) = 2^{-\frac{1}{4}} \int_0^\alpha \frac{\theta'(\alpha) - \theta(u)}{[F'(\alpha) - F(u)]^{\frac{3}{4}}} \frac{du}{\alpha},
\]

(3.1)
where

\[ \theta(x) = 2F(x) - xf(x) \]  
(3.2)

and (3.2) gives

\[ \theta'(x) = f(x) - xf'(x), \]  
(3.3)

\[ \theta''(x) = -xf''(x), \]  
(3.4)

which are useful in our analysis of the time map \( T \). We first study the graph of the nonlinear function \( f(u) \) in (1.4).

**Lemma 5** For \( a \geq 0 \) and \( d > 0 \), \( f(u) = (1 + au)/(1 + e^{d(1-u)}) \in C^2(0, +\infty) \) and it satisfies

1. \( f(u) > 0 \) on \([0, +\infty)\);
2. \( f'(u) > 0 \) on \((0, +\infty)\);
3. \( f(u) \sim 1 + au \) as \( u \to +\infty \);
4. for fixed \( a \geq 0 \), \( f(u) \) has exactly one positive inflection point at some point \( C = C(d) \) such that

\[ f''(u) > 0 \quad \text{on} \quad (0, C), \quad f''(C) = 0, \quad f''(u) < 0 \quad \text{on} \quad (0, +\infty). \]  
(3.5)

More precisely,

\[ C = 1 \quad \text{if} \quad a = 0, \]  
(3.6)

\[ C > 1 \quad \text{if} \quad a > 0, \]  
(3.7)

\[ 1 < C < 2 \quad \text{if} \quad d \geq 2, \]  
(3.8)

\[ \lim_{d \to +\infty} C(d) = 1 \quad \text{if} \quad a > 0. \]  
(3.9)

**Proof.** Parts i and iii are easy. So we will only prove parts ii and iv. For \( f(u) = (1 + au)/(1 + e^{d(1-u)}) \), we compute

\[ f'(u) = \frac{a + (a + d + adu)e^{d(1-u)}}{(1 + e^{d(1-u)})^2} > 0 \quad \text{on} \quad (0, +\infty), \]  
(3.10)

\[ f''(u) = \frac{de^{d(1-u)}(2a - d + 2ae^{d(1-u)} + de^{d(1-u)} - adu + adue^{d(1-u)})}{(1 + e^{d(1-u)})^3}; \]  
(3.11)

(3.10) is part ii. For part iv, let

\[ H(u) = 2a - d + 2ae^{d(1-u)} + de^{d(1-u)} - adu + adue^{d(1-u)}. \]  
(3.12)
It is easy to see that finding the inflection point $C$ for $f(u)$ is equivalent to finding the zero point of $H(u)$. Now

$$H(0) = 2a + 2ae^d + d(e^d - 1) > 0,$$

$$H(1) = 4a \begin{cases} = 0 & \text{if } a = 0, \\ > 0 & \text{if } a > 0, \end{cases}$$

$$H(2) = 2a - d - 2ad + (2a + d + 2ad)e^{-d}$$

$$= 2a - d - 2ad + \frac{1}{2}(2a + d + 2ad)$$

(since $e^{-d} < \frac{1}{2}$ if $d < 2$)

$$= -\frac{d}{2} + (\frac{1}{2} - \frac{1}{2}d)a$$

if $d > 2$

$$< 0 \text{ if } d < 2,$$

$$H'(u) = -(ad + d^2 + ad^2)ue^{-d} (1 - u) - ad < 0 \text{ on } (0, +\infty).$$

Now (3.13)–(3.15) prove (3.5)–(3.8), since in (3.11), for $d > 0$,

$$de^{d(1-u)} > 0 \text{ and } (1 + e^{d(1-u)})^3 > 0.$$

Next, we prove (3.9). First,

$$H(1 + 1/d) = e^{-1}[3 + a + a + d(1 + a)(1 - e)]$$

$$< 0 \text{ if } d \text{ is large enough;}$$

(3.17)

secondly, (3.14) and (3.17) prove that (3.9) holds. Thus part iv holds. □

The next lemma is the key lemma to Theorems 1–4, in which we study the bifurcation curve of positive solutions for problem (1.4); this is equivalent to studying $\lambda(\alpha) (= [T(\alpha)]^2)$ in (1.7).

**Lemma 6** For a given $a \geq 0$,

$$\lim_{\alpha \to 0} \lambda(\alpha) = 0,$$

$$\lim_{\alpha \to +\infty} \frac{\lambda(\alpha)}{\alpha} = 2 \text{ if } a = 0,$$

$$\lim_{\alpha \to +\infty} \lambda(\alpha) = \frac{\pi^2}{4\alpha} \text{ if } a > 0.$$  

Moreover, for $d > 0$, let $C = C(d)$ be the unique positive inflection point for $f(u)$ in (1.4).

(i) If

$$\theta'(C) = 0,$$
then the bifurcation curve $S$ is a monotone curve (see Fig. 1(a));

(ii) if

$$\theta(C) < 0,$$  \hspace{1cm} (3.22)

then $T'(C) < 0$ and the bifurcation curve $S$ is S-shaped (see Fig. 1(b) for $a = 0$ and Fig. 2 for $a > 0$).

**Proof.** Since $f(0) = 1/(1 + e^a) > 0$, (3.18) follows from the well-known result that

$$\lim_{a \to 0} [\lambda(a)]^{1/2} = \lim_{a \to 0} T(\alpha) = 0;$$

see theorem 2.9 of [3].

Now (3.19) is the result of a theorem in [2]; and (3.20) is a well-known result (theorem 2.7 in [3]). By (3.2)-(3.5), it is easy to see that

$$\theta(0) = 0, \quad \theta'(0) = f(0) > 0,$$

$$\theta''(x) > 0 \quad \text{on} \quad (0, C), \quad \theta''(C) = 0, \quad \theta'(x) < 0 \quad \text{on} \quad (C, +\infty).$$

Thus:

(i) If $\theta'(C) \geq 0$, $\theta(x)$ is a strictly increasing function on $(0, +\infty)$, which implies that $T(\alpha)$ is a strictly increasing function of $\alpha$ on $(0, +\infty)$, by (3.1). So the bifurcation curve $S$ is a monotone curve.

(ii) If $\theta(C) < 0$, then $T'(C) < 0$ by (3.1). Moreover, since $f(x) \sim 1 + ax$ as $x \to +\infty$, $\lim_{x \to +\infty} \theta(x) = +\infty$. Then the same argument used to prove theorem 3 in [6] can be applied to show that the corresponding time map $T(\alpha)$ to $f(u)$ has exactly two critical points, one local maximum and one local minimum, on $(0, +\infty)$. That is, by (1.7), the bifurcation curve $S$ is S-shaped. \(\square\)

The proof of part ii of Lemma 6 showing that the bifurcation curve is S-shaped is also of general interest for nonlinearities. We write it as the next theorem.

**Theorem 7** (cf. theorem 3 of [6]) Consider

$$-u''(x) = \lambda f(u(x)) \quad (-1 < x < 1),$$

where $\lambda > 0$ and $f$ is defined on $[0, +\infty)$ satisfying

(H1) $f \in C^2([0, +\infty), f(u) > 0$ on $[0, +\infty)$, and $f''(u) > 0$ on $[0, +\infty)$;

(H2) there exists a number $C > 0$ such that

$$f''(u) > 0 \quad \text{on} \quad (0, C), \quad f''(C) = 0, \quad f''(u) < 0 \quad \text{on} \quad (C, +\infty);$$

(H3) $f(u) \sim 1 + au$ for some $a \geq 0$ as $u \to +\infty$;

(H4) $\tilde{T}(C) := 2\tilde{F}(C) - C\tilde{f}(C) < 0$ (where $\tilde{F}(u) := \int_0^u \tilde{f}(s) \, ds$).

Then, in addition to (3.18)-(3.20), the bifurcation curve $S$ is S-shaped.

We are now in a position to prove Theorems 1-3.
Proof of Theorem 1  Now (2.1) and (2.2) follow from Lemma 6. Moreover, by (3.6), for \( a = 0 \), we have \( C = 1 \).

(i) It can be shown that
\[
\theta'(C) = \theta'(1) = \frac{1}{2}(2 - d) \geq 0 \quad \text{if } 0 < d \leq 2.
\]
Thus, by Lemma 6, the bifurcation curve \( S \) is a monotone curve for \( 0 < d \leq 2 \).

(ii) It can be shown that
\[
\theta(C) = \theta(1) = 2F(1) - f(1) = \frac{[3d + 4 \log 2 - 4 \log (1 + e^d)]/2d}{< 0 \quad \text{if } d > d_0,}
\]
where \( d_0 \approx 2.43751 \) is the unique positive root of
\[
3d + 4 \log 2 - 4 \log (1 + e^d) = 0.
\]
Thus, by Lemma 6, the bifurcation curve \( S \) is \( S \)-shaped if \( d > d_0 \approx 2.43751 \). Moreover, \( T'(C) = T'(1) < 0 \) by Lemma 6. By the graph of \( T(\alpha) \) on \((0, +\infty)\) and by (1.7), at the turning points \((\lambda, u_A)\) and \((\lambda, u_B)\) of the \( S \)-shaped bifurcation curve \( S \),
\[
\|u_A\|_\infty < 1 < \|u_B\|_\infty;
\]
see Fig. 1(a). \( \square \)

Proof of Theorem 2. Now (2.3) and (2.4) follow from Lemma 6. Moreover, by (3.7), for \( a > 0 \), we have \( C > 1 \). In the following, we show that
\[
\theta(C(d)) < 0
\]
if \( d > d_1 \) for some \( d_1 \approx 3.05084 \), which is the unique positive root of (3.29). By (3.3),
\[
\theta'(0) = f(0) = \frac{1}{1 + e^d} > 0.
\]
By (3.4) and (3.5),
\[
\theta''(u) = -uf''(u) < 0 \quad \text{on } (0, C). \tag{3.24}
\]
Thus the function \( \theta'(u) \) is a strictly decreasing function on \((0, C)\). To show \( \theta(C(d)) < 0 \) for \( d > d_1 \), we compute and find that
\[
\theta'(\frac{1}{2}) = \frac{4 + 4e^{d_1} - 2de^{d_1} - ade^{d_1}}{4(1 + e^{d_1})^2} < 0 \iff \frac{4e^{-d_1^2} + 4 - 2d}{d} < a, \tag{3.25}
\]
in which, for any \( a > 0 \), \( (4e^{-d_1^2} + 4 - 2d)/d < a \) for \( d > d_1 \) for some \( d_1 \approx 2.55693 \), which is the unique positive root of
\[
4e^{-d_1^2} + 4 - 2d = 0.
\]
Thus, for $d > d_3 \approx 2.556\,93$,

$$\theta'(C) < \theta'(1) < \theta'(\frac{1}{2}) < 0. \tag{3.26}$$

Since $\theta'(u)$ is a strictly decreasing function on $(0, C)$ and $\theta'(u) < 0$ on $(1, C)$,

$$\theta(C) = \int_0^C \theta'(u) \, du$$

$$< \int_0^1 \theta'(u) \, du \quad (= \theta(1))$$

$$= \int_0^1 \theta'(u) \, du + \int_\frac{1}{2}^1 \theta'(u) \, du$$

$$< \int_0^1 \theta'(0) \, du + \int_\frac{1}{2}^1 \theta'(\frac{1}{2}) \, du$$

$$= \frac{1}{2} [\theta'(0) + \theta'(\frac{1}{2})]$$

$$= \frac{1}{2} \left( \frac{1}{1 + e^d} + \frac{4 + 4e^{d/2} - 2de^{d/2} - ade^{d/2}}{4(1 + e^{d/2})^2} \right). \tag{3.27}$$

In (3.27),

$$\frac{1}{1 + e^d} + \frac{4 + 4e^{d/2} - 2de^{d/2} - ade^{d/2}}{4(1 + e^{d/2})^2} < 0$$

if and only if

$$\frac{2(4 + 6e^{d/2} - de^{d/2} + 4e^d + 2e^{3d/2} - de^{3d/2})}{de^{d/2}(1 + e^d)} < a. \tag{3.28}$$

Now (3.28) holds for any $a > 0$ if

$$d > d_1,$$

where $d_1 \approx 3.050\,84$ is the unique positive root of

$$4 + 6e^{d/2} - de^{d/2} + 4e^d + 2e^{3d/2} - de^{3d/2} = 0. \tag{3.29}$$

We conclude that if $d > d_1 \approx 3.050\,84$, then

$$\theta(C(d)) < \theta(1) < 0, \tag{3.30}$$

and hence Lemma 6 implies that $T'(C) < 0$ (C > 1) and that the bifurcation curve S is S-shaped. Similarly, as before, by the graph of $T(\alpha)$ on $(0, +\infty)$ and by (1.7), at the turning points $(\lambda, u_\lambda)$ and $(\lambda, u_\lambda)$ of the S-shaped bifurcation curve S

$$\|u_\lambda\|_\infty < 1 < \|u_\lambda\|_\infty.$$
In the above, we know that $C > 1$ and that the time map $T(\alpha)$ has exactly one critical point, a local maximum, say at $\alpha = \widetilde{\alpha} < C$ on $(0, C)$. Then, by (1.7),
\[
\bar{\lambda} = [T(\widetilde{\alpha})]^2 > [T(\frac{1}{2})]^2 = \left(2^{-1} \int_0^{\frac{1}{2}} [F(\frac{1}{2}) - F(u)]^{-1} du \right)^2 = \left(2^{-1} \int_0^{\frac{1}{2}} [f(\frac{1}{2})(\frac{1}{2} - u)]^{-1} du \right)^2
\]
for some $b$, with $0 < u < b < \frac{1}{2}$, by the mean value theorem.

Since $f'(u) > 0$ on $(0, +\infty)$,
\[
\bar{\lambda} > \left(2^{-1} \int_0^{\frac{1}{2}} [f(\frac{1}{2})(\frac{1}{2} - u)]^{-1} du \right)^2 = \left[ f(\frac{1}{4}) \right]^{-1} \left(2^{-1} \int_0^{\frac{1}{2}} (\frac{1}{2} - u)^{-1} du \right)^2 = \left[ f(\frac{1}{4}) \right]^{-1} = \frac{1 + e^{\frac{d}{2}}}{1 + \frac{1}{2}a}.
\]

Thus
\[
\bar{\lambda} > \lambda_{\infty}
\]
if
\[
\frac{1 + e^{\frac{d}{2}}}{1 + \frac{1}{2}a} > \frac{\pi^2}{4a},
\]
which is equivalent to (2.7), that is, to
\[
d > 2 \log \left[ \frac{(2 + a)}{8a} \right] \pi^2 - 1.
\]
That is, in addition to (2.5), if (2.7) holds then $\bar{\lambda} > \lambda_{\infty}$.  \(\Box\)

**Proof of Theorem 3** Now (2.3) and (2.4) follow from Lemma 6. Similarly, as before, we next show that $\theta(C(d)) < 0$ if (2.8) holds. First, suppose (2.8) holds. It is then easy to compute, and verify, that for $d < 0$

(i) \[
a > \frac{d(2e^2 + 2e^{2d} + 2e^{1+d} + e^{2+d})}{e^{1+d} + e^{2+d}} > de^{1-d},
\]
which implies
\[
\theta'(1/d) = \frac{de^{1-d} - a}{d(1 + e^{d-1})^2} < 0; \quad (3.31)
\]
(ii) 

\[ a > \frac{d(2e^2 + e^{2d} + 2e^{1+d} + e^{2+d})}{e^{1+d} + e^{2+d}} > \frac{1}{4}d(e^{2d} - 1), \]

which implies

\[ f''(2/d) = \frac{4de^{2+2d}[a - \frac{1}{4}d(e^{2d} - 1)]}{(e^2 + e^d)^3} > 0, \]

and hence

\[ \frac{1}{d} < \frac{2}{d} < c, \quad (3.32) \]

by (3.5).

Now, by (3.31) and (3.32), and by the fact that the function \( \theta'(u) \) is strictly decreasing on \( (0, C) \) (since \( \theta''(u) = -uf''(u) < 0 \) on \( (0, C) \)), we have

\[ \theta(C) = \int_0^C \theta'(u) \, du \]

\[ < \int_0^{2/d} \theta'(u) \, du \]

\[ = \int_0^{1/d} \theta'(u) \, du + \int_{1/d}^{2/d} \theta'(u) \, du \]

\[ < \int_0^{1/d} \theta'(0) \, du + \int_{1/d}^{2/d} \theta'(1/d) \, du \]

\[ = \frac{1}{d} [\theta'(0) + \theta'(1/d)] \]

\[ = \frac{1}{d} \left( \frac{1}{1 + e^d} + \frac{de^2 - a e^{d+1}}{d(e + e^d)^2} \right) \]

\[ = \frac{d(2e^2 - e^{2d} + 2e^{1+d} + e^{2+d}) - a(e^{1+d} + e^{1+2d})}{d^2(1 + e^d)(e + e^d)^2} \]

\[ < 0, \]

if

\[ a > \frac{d(2e^2 + e^{2d} + 2e^{1+d} + e^{2+d})}{e^{1+d} + e^{1+2d}}, \]

which is (2.8). Thus, if (2.8) holds, by Lemma 6, the bifurcation curve \( S \) is S-shaped.

In addition to (2.8), suppose (2.9) holds, we next show

\[ \bar{\lambda} > \lambda_\infty \quad (= \pi^2/4a). \]
In the above, we know that $C > 2/d$ and that the time map $T(\alpha)$ has exactly one critical point, a local maximum, say at $\alpha = \bar{\alpha} < C$, on $(0, C)$. Then by (1.7),

$$\bar{\lambda} = [T(\bar{\alpha})]^2 > [T(1/d)]^2 = \left(2^{-1} \int_0^{1/d} [F(1/d) - F(u)]^{-1} \, du \right)^2 = \left(2^{-1} \int_0^{1/d} [f(b)(1/d - u)]^{-1} \, du \right)^2$$

for some $b$, with $0 < u < b < 1/d$, by the mean value theorem; since $f'(u) > 0$ on $(0, +\infty)$

$$\bar{\lambda} > \left(2^{-1} \int_0^{1/d} [f(1/d)(1/d - u)]^{-1} \, du \right)^2 = [f(1/d)]^{-1} \left(2^{-1} \int_0^{1/d} (1/d - u)^{-1} \, du \right)^2 = [f(1/d)]^{-1} \frac{2}{d} = \frac{2(1 + e^{d-1})}{a + d}.$$ 

Thus

$$\bar{\lambda} > \lambda_\infty$$

if

$$\frac{2(1 + e^{d-1})}{a + d} > \frac{\pi^2}{4a},$$

which is equivalent to (2.9), that is, to

$$a > \frac{d\pi^2}{8e^{d-1} - \pi^2 + 8}.$$ 

That is, in addition to (2.8), if (2.9) holds then $\bar{\lambda} > \lambda_\infty$. □

4. A conjecture

We finish by giving a conjecture for the bifurcation diagram for problem (1.4). By Theorems 1–4, and by some numerical evaluation, we advance the conjecture that, for any $a \geq 0$ and for $d > 0$, problem (1.4) has at most three positive solutions for each $\lambda > 0$. Moreover, on defining the sets

$$F = \{(a, d) : a > 0, d > 0\}, \quad F_0 = \{(a, b) : a \geq 0, d > 0\},$$

there exists a simple connected compact set $M \subset F_0$ with $M \cap F \neq \emptyset$, which contains the set $\{(0, d) : 0 < d \leq 2\}$ such that
(i) the bifurcation curve \( S \) is S-shaped if \((a, d) \in F_0 \setminus M\),

(ii) the bifurcation curve \( S \) is a monotone curve if \((a, d) \in M\);

see Fig. 3.

\textit{Note}, numerical evaluation shows that for \( a = 1 \), the bifurcation curves \( S \) are S-shaped for all \( d > 0 \), and for \( a = 0.01 \) the bifurcation curves \( S \) are monotonic for \( 0.01 < d < 1 \).

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\textbf{References}


