$b$ a constant; and so

$$w = \left(b - \frac{\gamma-1}{2\gamma}\beta r^2\right)^{+\frac{1}{\gamma-1}}, \tag{34}$$

where we took the positive part of the right hand side of (34) to ensure $w \geq 0$. Recalling $v(y) = w(r)$ and (28), we obtain

$$u(x,t) = \frac{1}{t^\alpha}\left(b - \frac{\gamma-1}{2\gamma}\beta\frac{|x|^2}{t^{2\beta}}\right)^{+\frac{1}{\gamma-1}} \quad (x \in \mathbb{R}^n,\ t > 0), \tag{35}$$

where, from (30), (33),

$$\alpha = \frac{n}{n(\gamma-1)+2}, \quad \beta = \frac{1}{n(\gamma-1)+2}. \tag{36}$$

The formulas (35), (36) are *Barenblatt's solution* to the porous medium equation. □

**Remarks.** Observe that Barenblatt's solution has compact support for each time $t > 0$. This is a general feature for (appropriately defined) weak, nonnegative solutions of the porous medium equation with compactly supported initial data. The nonlinear parabolic PDE (27) becomes degenerate wherever $u = 0$, and so the set $\{u > 0\}$ moves with *finite propagation speed*. Consequently the porous medium equation (27) is often regarded as a better model of diffusive spreading than the linear heat equation (which predicts infinite propagation speed). □

## 4.3. TRANSFORM METHODS

In this section we develop some of the theory of Fourier and Laplace transforms, which provides extremely powerful tools for converting certain linear partial differential equations into either algebraic equations or else differential equations involving fewer variables.

### 4.3.1. Fourier transform.

In this section all functions are complex-valued, and $\bar{}$ denotes the complex conjugate.



**a. Definitions and elementary properties.**

**Definition of Fourier transform on $\mathbf{L^1}$.** If $u \in L^1(\mathbb{R}^n)$, we define its *Fourier transform*

$$\hat{u}(y) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot y}u(x)\,dx \quad (y \in \mathbb{R}^n) \tag{1}$$

and its *inverse Fourier transform*

$$\check{u}(y) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot y}u(x)\,dx \quad (y \in \mathbb{R}^n). \tag{2}$$

Since $|e^{\pm ix\cdot y}| = 1$ and $u \in L^1(\mathbb{R}^n)$, these integrals converge for each $y \in \mathbb{R}^n$.

We intend now to extend definitions (1), (2) to functions $u \in L^2(\mathbb{R}^n)$.

**THEOREM 1** (Plancherel's Theorem). *Assume $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{u}, \check{u} \in L^2(\mathbb{R}^n)$ and*

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}. \tag{3}$$

**Proof.** 1. First we note that if $v, w \in L^1(\mathbb{R}^n)$, then $\hat{v}, \hat{w} \in L^\infty(\mathbb{R}^n)$. Also

$$\int_{\mathbb{R}^n} v(x)\hat{w}(x)\,dx = \int_{\mathbb{R}^n} \hat{v}(y)w(y)\,dy, \tag{4}$$

since both expressions equal $\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}v(x)w(y)\,dxdy$. Furthermore, as we will explicitly compute below in Example 1 of §4.3.2,

$$\int_{\mathbb{R}^n} e^{ix\cdot y - t|x|^2}dx = \left(\frac{\pi}{t}\right)^{n/2}e^{-\frac{|y|^2}{4t}} \quad (t > 0).$$

Consequently if $\varepsilon > 0$ and $v_\varepsilon(x) := e^{-\varepsilon|x|^2}$, we have $\hat{v}_\varepsilon(y) = \frac{e^{-\frac{|y|^2}{4\varepsilon}}}{(2\varepsilon)^{n/2}}$. Thus (4) implies for each $\varepsilon > 0$ that

$$\int_{\mathbb{R}^n} \hat{w}(y)e^{-\varepsilon|y|^2}dy = \frac{1}{(2\varepsilon)^{n/2}}\int_{\mathbb{R}^n} w(x)e^{-\frac{|x|^2}{4\varepsilon}}dx. \tag{5}$$

2. Now take $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and set $v(x) := \bar{u}(-x)$. Let $w := u * v \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and check (cf. Theorem 2 below) that

$$\hat{w} = (2\pi)^{n/2}\hat{u}\hat{v} \in L^\infty(\mathbb{R}^n).$$

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But

$$\hat{v}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} \tilde{u}(-x)\, dx = \tilde{\hat{u}}(y);$$

and so $\hat{w} = (2\pi)^{n/2}|\hat{u}|^2$.

Now $w$ is continuous and thus

$$\lim_{\varepsilon \to 0} \frac{1}{(2\varepsilon)^{n/2}} \int_{\mathbb{R}^n} w(x) e^{-\frac{|x|^2}{4\varepsilon}}\, dx = (2\pi)^{n/2} w(0),$$

where we employed the lemma from §2.3.1. Since $\hat{w} = (2\pi)^{n/2}|\hat{u}|^2 \geq 0$, we deduce upon sending $\varepsilon \to 0^+$ in (5) that $\hat{w}$ is summable, with

$$\int_{\mathbb{R}^n} \hat{w}(y)\, dy = (2\pi)^{n/2} w(0).$$

Hence

$$\int_{\mathbb{R}^n} |\hat{u}|^2 dy = w(0) = \int_{\mathbb{R}^n} u(x) v(-x)\, dx = \int_{\mathbb{R}^n} |u|^2 dx.$$

The proof for $\check{u}$ is similar. □

**Definition of Fourier transform on $L^2$.** In view of the equality (3) we can define the Fourier transforms of a function $u \in L^2(\mathbb{R}^n)$ as follows. Choose a sequence $\{u_k\}_{k=1}^\infty \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with

$$u_k \to u \quad \text{in } L^2(\mathbb{R}^n).$$

According to (3), $\|\hat{u}_k - \hat{u}_j\|_{L^2(\mathbb{R}^n)} = \|\widehat{u_k - u_j}\|_{L^2(\mathbb{R}^n)} = \|u_k - u_j\|_{L^2(\mathbb{R}^n)}$, and thus $\{\hat{u}_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. This sequence consequently converges to a limit, which we define to be $\hat{u}$:

$$\hat{u}_k \to \hat{u} \quad \text{in } L^2(\mathbb{R}^n).$$

The definition of $\hat{u}$ does not depend upon the choice of approximating sequence $\{\hat{u}_k\}_{k=1}^\infty$. We similarly define $\check{u}$.

Next we record some useful formulas.

**THEOREM 2** (Properties of Fourier transform). *Assume $u, v \in L^2(\mathbb{R}^n)$. Then*

(i) $\int_{\mathbb{R}^n} u\bar{v}dx = \int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}dy$,

(ii) $\widehat{D^\alpha u} = (iy)^\alpha \hat{u}$ *for each multiindex $\alpha$ such that $D^\alpha u \in L^2(\mathbb{R}^n)$,*

(iii) $(u * v)\hat{} = (2\pi)^{n/2}\hat{u}\hat{v}$,

(iv) $u = (\hat{u})\check{}$.



**Proof.** 1. Let $u, v \in L^2(\mathbb{R}^n)$ and $\alpha \in \mathbb{C}$. Then

$$\|u + \alpha v\|^2_{L^2(\mathbb{R}^n)} = \|\hat{u} + \widehat{\alpha v}\|^2_{L^2(\mathbb{R}^n)}.$$

Expanding, we deduce

$$\int_{\mathbb{R}^n} |u|^2 + |\alpha v|^2 + \bar{u}(\alpha v) + u(\overline{\alpha v})\, dx = \int_{\mathbb{R}^n} |\hat{u}|^2 + |\widehat{\alpha v}|^2 + \bar{\hat{u}}(\widehat{\alpha v}) + \hat{u}(\overline{\widehat{\alpha v}})\, dy;$$

and so according to Theorem 1,

$$\int_{\mathbb{R}^n} \alpha\bar{u}v + \bar{\alpha}u\bar{v}\, dx = \int_{\mathbb{R}^n} \alpha\bar{\hat{u}}\hat{v} + \bar{\alpha}\hat{u}\bar{\hat{v}}\, dy.$$

Take $\alpha = 1, i$ and combine the resulting equalities to deduce

$$\int_{\mathbb{R}^n} u\bar{v}\, dx = \int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}\, dy.$$

This proves (i).

2. If $u$ is smooth and has compact support, we calculate

$$\begin{aligned}
\widehat{D^\alpha u}(y) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} D^\alpha u(x)\, dx \\
&= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} D_x^\alpha (e^{-ix\cdot y}) u(x)\, dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} (iy)^\alpha u(x)\, dx = (iy)^\alpha \hat{u}(y).
\end{aligned}$$

By approximation the same formula is true if $D^\alpha u \in L^2(\mathbb{R}^n)$.

3. We compute for $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$ that

$$\begin{aligned}
\widehat{(u * v)}(y) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} \int_{\mathbb{R}^n} u(z) v(x - z)\, dz dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iz\cdot y} u(z) \left( \int_{\mathbb{R}^n} e^{-i(x-z)\cdot y} v(x - z) dx \right) dz \\
&= \int_{\mathbb{R}^n} e^{-iz\cdot y} u(z)\, dz\ \hat{v}(y) = (2\pi)^{n/2} \hat{u}(y)\hat{v}(y).
\end{aligned}$$

4. Fix $z \in \mathbb{R}^n$, $\varepsilon > 0$ and write $v_\varepsilon(x) := e^{iz\cdot x - \varepsilon|x|^2}$. Then

$$\hat{v}_\varepsilon(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot(y-z) - \varepsilon|x|^2} dx = \frac{1}{(2\varepsilon)^{n/2}} e^{-\frac{|y-z|^2}{4\varepsilon}},$$

where we followed computations from the proof of Theorem 1. Utilizing formula (4), we deduce for $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ that

$$\int_{\mathbb{R}^n} \hat{u}(y) e^{ix\cdot y - \varepsilon|y|^2} dy = \frac{1}{(2\varepsilon)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-\frac{|x-z|^2}{4\varepsilon}} dz.$$

The expression on the right converges to $(2\pi)^{n/2} u(z)$ as $\varepsilon \to 0^+$, for each Lebesgue point of $u$. Thus

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(y) e^{ix\cdot y} dy = u(z) \quad \text{for a.e. } z.$$

This proves (iv). □

**b. Applications.**

The Fourier transform is an especially powerful technique for studying linear, constant-coefficient partial differential equations.

**Example 1** (Bessel potentials). We investigate first the PDE

$$-\Delta u + u = f \quad \text{in } \mathbb{R}^n, \tag{6}$$

where $f \in L^2(\mathbb{R}^n)$. To find an explicit formula for $u$, we take the Fourier transform, recalling Theorem 2,(ii) to obtain

$$(1+|y|^2)\hat{u}(y) = \hat{f}(y) \quad (y \in \mathbb{R}^n). \tag{7}$$

The effect of the Fourier transform has been to convert the PDE (6) into the algebraic equation (7), the solution of which is trivial:

$$\hat{u} = \frac{\hat{f}}{1+|y|^2}.$$

Thus

$$u = \left( \frac{\hat{f}}{1+|y|^2} \right)^{\vee}, \tag{8}$$

and so the only real problem is to rewrite the right hand side of (8) into a more explicit form.

Invoking Theorem 2,(iii), we see

$$u = \frac{f * B}{(2\pi)^{n/2}}, \tag{9}$$



where

$$\hat{B} = \frac{1}{1+|y|^2}. \tag{10}$$

We solve for $B$ as follows. Since $\frac{1}{a} = \int_0^\infty e^{-sa} ds$ for each $a > 0$, we have $\frac{1}{1+|y|^2} = \int_0^\infty e^{-t(1+|y|^2)} dt$. Thus

$$B = \left( \frac{1}{1+|y|^2} \right)^{\vee} = \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{-t} \left( \int_{\mathbb{R}^n} e^{ix\cdot y - t|y|^2} dy \right) dt. \tag{11}$$

Now if $a, b \in \mathbb{R}$, $b > 0$, and we set $z = b^{1/2}x - \frac{a}{2b^{1/2}} i$, we find

$$\int_{-\infty}^{\infty} e^{iax - bx^2} dx = \frac{e^{-a^2/4b}}{b^{1/2}} \int_\Gamma e^{-z^2} dz,$$

$\Gamma$ denoting the contour $\left\{ \mathrm{Im}(z) = -\frac{a}{2b^{1/2}} \right\}$ in the complex plane. Deforming $\Gamma$ into the real axis, we compute $\int_\Gamma e^{-z^2} dz = \int_{-\infty}^{\infty} e^{-x^2} dx = \pi^{1/2}$; and hence

$$\int_{-\infty}^{\infty} e^{iax - bx^2} dx = e^{-a^2/4b} \left( \frac{\pi}{b} \right)^{1/2}. \tag{12}$$

Thus

$$\int_{\mathbb{R}^n} e^{ix\cdot y - t|y|^2} dy = \prod_{j=1}^n \int_{-\infty}^{\infty} e^{ix_j y_j - ty_j^2} dy_j = \left( \frac{\pi}{t} \right)^{n/2} e^{-\frac{|x|^2}{4t}} \tag{13}$$

by (12). Consequently, we conclude from (11), (13) that

$$B(x) = \frac{1}{2^{n/2}} \int_0^\infty \frac{e^{-t - \frac{|x|^2}{4t}}}{t^{n/2}} dt \quad (x \in \mathbb{R}^n). \tag{14}$$

$B$ is called a *Bessel potential*. Employing (9), we derive then the formula

$$u(x) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-t - \frac{|x-y|^2}{4t}}}{t^{n/2}} f(y)\, dy dt \quad (x \in \mathbb{R}^n) \tag{15}$$

for the solution of (6). □

**Example 2** (Fundamental solution of heat equation). Consider again the initial-value problem for the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{16}$$

We establish a new method for solving (16) by computing $\hat{u}$, the Fourier transform of $u$ in *the spatial variables $x$ only*. Thus

$$\begin{cases} \hat{u}_t + |y|^2 \hat{u} = 0 & \text{for } t > 0 \\ \hat{u} = \hat{g} & \text{for } t = 0; \end{cases}$$

whence

$$\hat{u} = e^{-t|y|^2} \hat{g}.$$

Consequently $u = \left(e^{-t|y|^2}\hat{g}\right)^\vee$, and therefore

$$u = \frac{g * F}{(2\pi)^{n/2}}, \tag{17}$$

where $\hat{F} = e^{-t|y|^2}$. But then

$$F = \left(e^{-t|y|^2}\right)^\vee = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y - t|y|^2}\, dy = \frac{1}{(2t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

by (13). Invoking (17), we compute

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \quad (x \in \mathbb{R}^n,\ t > 0), \tag{18}$$

in agreement with §2.3.1. The Fourier transform has provided us with a new derivation of the fundamental solution of the heat equation. □

**Example 3** (Fundamental solution of Schrödinger's equation). Let us next look at the initial-value problem for Schrödinger's equation

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{19}$$

Here $u$ and $g$ are complex-valued.

If we formally replace $t$ by $it$ on the right hand side of (18), we obtain the formula

$$u(x,t) = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} g(y)\, dy \quad (x \in \mathbb{R}^n,\ t > 0), \tag{20}$$



where we interpret $i^{\frac{1}{2}}$ as $e^{\frac{i\pi}{4}}$. This expression clearly makes sense for all times $t > 0$, provided $g \in L^1(\mathbb{R}^n)$. Furthermore if $|y|^2 g \in L^1(\mathbb{R}^n)$, we can check by a direct calculation that $u$ solves $iu_t + \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$. (We will not discuss here the sense in which $u(\cdot,t) \to g$ as $t \to 0^+$, but see §4.5.3 below and Problem 5.)

Let us next rewrite formula (20) as

$$u(x,t) = \frac{e^{\frac{i|x|^2}{4t}}}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-ix\cdot y}{2t}} e^{\frac{i|y|^2}{4t}} g(y)\, dy.$$

Since $\left|e^{\frac{i|x|^2}{4t}} e^{\frac{i|y|^2}{4t}}\right| = 1$, we can check as in Theorem 1 that if $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} \quad (t > 0). \tag{21}$$

Hence the mapping $g \mapsto u(\cdot,t)$ preserves the $L^2$-norm. Therefore we can extend formula (20) to functions $g \in L^2(\mathbb{R}^n)$, in the same way that we extended the definition of Fourier transform. □

**Remark.** We call

$$\Psi(x,t) := \frac{1}{(4\pi i t)^{n/2}} e^{\frac{i|x|^2}{4t}} \quad (x \in \mathbb{R}^n,\ t \neq 0) \tag{22}$$

the *fundamental solution* of Schrödinger's equation. Note that formula (20), $u = g * \Psi$, makes sense for all times $t \neq 0$, even $t < 0$. Thus we in fact have solved

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^n \times (-\infty,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{23}$$

In particular, Schrödinger's equation is *reversible in time*, whereas the heat equation is not (in spite of Theorem 11 in §2.3.4). □

**Example 4** (Wave equation). We next analyze the initial-value problem for the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g,\ u_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{24}$$

where for simplicity we suppose the initial velocity to be zero. Take as before $\hat{u}$ to be the Fourier transform of $u$ in the variable $x \in \mathbb{R}^n$. Then

$$\begin{cases} \hat{u}_{tt} + |y|^2 \hat{u} = 0 & \text{for } t > 0 \\ \hat{u} = \hat{g},\ \hat{u}_t = 0 & \text{for } t = 0. \end{cases} \tag{25}$$

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This is an ODE for each fixed $y \in \mathbb{R}^n$. We look for a solution having the form $\hat{u} = \beta e^{t\gamma}$ ($\beta, \gamma \in \mathbb{C}$). Plugging into (25) gives $\gamma^2 + |y|^2 = 0$ and so $\gamma = \pm i|y|$. Remembering the initial conditions from (25), we deduce

$$\hat{u} = \frac{\hat{g}}{2}(e^{it|y|} + e^{-it|y|}).$$

Inverting, we find

$$u(x,t) = \left[\frac{\hat{g}}{2}(e^{it|y|} + e^{-it|y|})\right]^\vee;$$

and consequently

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\hat{g}(y)}{2}(e^{i(x\cdot y + t|y|)} + e^{i(x\cdot y - t|y|)})\, dy \tag{26}$$

for $x \in \mathbb{R}^n$, $t \geq 0$. We will further analyze this formula in certain asymptotic limits later, in §4.5.3. See also Example 1(ii) in §4.2.1. □

**Example 5** (Telegraph equation). The initial-value problem for the one-dimensional *telegraph equation* is

$$\begin{cases} u_{tt} + 2du_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{27}$$

for $d > 0$, the term "$2du_t$" representing a physical damping of wave propagation. As before

$$\begin{cases} \hat{u}_{tt} + 2d\hat{u}_t + |y|^2\hat{u} = 0 & \text{for } t > 0 \\ \hat{u} = \hat{g},\ \hat{u}_t = \hat{h} & \text{for } t = 0. \end{cases} \tag{28}$$

We again seek a solution of the form $\hat{u} = \beta e^{t\gamma}$ ($\beta, \gamma \in \mathbb{C}$). Plugging into (28), we deduce $\gamma^2 + 2d\gamma + |y|^2 = 0$; whence $\gamma = -d \pm (d^2 - |y|^2)^{1/2}$. Consequently

$$\hat{u}(y,t) = \begin{cases} e^{-dt}(\beta_1(y)e^{\gamma(y)t} + \beta_2(y)e^{-\gamma(y)t}) & \text{if } |y| \leq d \\ e^{-dt}(\beta_1(y)e^{i\delta(y)t} + \beta_2(y)e^{-i\delta(y)t}) & \text{if } |y| \geq d \end{cases}$$

for $\gamma(y) := (d^2 - |y|^2)^{1/2}$ ($|y| \leq d$), $\delta(y) := (|y|^2 - d^2)^{1/2}$ ($|y| \geq d$), where $\beta_1(y)$ and $\beta_2(y)$ are selected so that

$$\hat{g}(y) = \beta_1(y) + \beta_2(y)$$

and

$$\hat{h}(y) = \begin{cases} \beta_1(y)(\gamma(y) - d) + \beta_2(y)(-\gamma(y) - d) & \text{if } |y| \leq d \\ \beta_1(y)(i\delta(y) - d) + \beta_2(y)(-i\delta(y) - d) & \text{if } |y| \geq d. \end{cases}$$

We thereby obtain the representation formula:

$$u(x,t) = \frac{e^{-dt}}{(2\pi)^{n/2}} \int_{\{|y| \leq d\}} \beta_1(y)e^{ix\cdot y + \gamma(y)t} + \beta_2(y)e^{ix\cdot y - \gamma(y)t}\, dy$$
$$+ \frac{e^{-dt}}{(2\pi)^{n/2}} \int_{\{|y| \geq d\}} \beta_1(y)e^{i(x\cdot y + \delta(y)t)} + \beta_2(y)e^{i(x\cdot y - \delta(y)t)}\, dy.$$

Notice the terms $e^{-dt}$, which correspond to damping as $t \to \infty$. □



### 4.3.2. Laplace transform.

Remember that we write $R_+ = (0,\infty)$.

**DEFINITION.** *If $u \in L^1(\mathbb{R}_+)$, we define its* Laplace transform *to be*

$$u^\#(s) := \int_0^\infty e^{-st}u(t)\, dt \quad (s \geq 0). \tag{29}$$

Whereas the Fourier transform is most appropriate for functions defined on all of $\mathbb{R}$ (or $\mathbb{R}^n$), the Laplace transform is useful for functions defined only on $\mathbb{R}_+$. In practice this means that for a partial differential equation involving time, it may be useful to perform a Laplace transform in $t$, holding the space variables $x$ fixed. (This is the opposite of the technique from Examples 2–5 above.)

**Example 1** (Resolvents and Laplace transform). Consider again the heat equation

$$\begin{cases} v_t - \Delta v = 0 & \text{in } U \times (0,\infty) \\ v = f & \text{on } U \times \{t = 0\}, \end{cases} \tag{30}$$

and perform a Laplace transform with respect to time:

$$v^\#(x,s) = \int_0^\infty e^{-st}v(x,t)\, dt \quad (s > 0).$$

What PDE does $v^\#$ satisfy? We compute

$$\Delta v^\#(x,s) = \int_0^\infty e^{-st}\Delta v(x,t)\, dt = \int_0^\infty e^{-st}v_t(x,t)\, dt$$
$$= s\int_0^\infty e^{-st}v(x,t)\, dt + e^{-st}v|_{t=0}^{t=\infty} = sv^\#(x,s) - f(x).$$

Think now of $s > 0$ being fixed, and write $u(x) := v^\#(x,s)$. Then

$$-\Delta u + su = f \quad \text{in } U. \tag{31}$$

Thus the solution of the resolvent equation (31) with right hand side $f$ is the Laplace transform of the solution of the heat equation (30) with initial data $f$. (If $U = \mathbb{R}^n$ and $s = 1$, we could now represent $v$ in terms of the fundamental solution, to rederive formula (15).) □

The connection between the resolvent equation and the Laplace transform will be made clearer by the discussion in §7.4 of semigroup theory.