8.8. *Evaluation of certain indeterminate forms*

We are concerned here with the evaluation of the limits of certain expressions.

The basic theorem in this section is the following:

Theorem 8.1. (L'Hôpital). *Let $f(x)$ and $g(x)$ be defined and be differentiable on the set $E - \{p\}$, E open, $p \in E$. Let $g'(x) \neq 0$ for every x in $E - \{p\}$. Let*

$$\lim_{x \to p} (f'(x)/g'(x)) = \lambda, \qquad -\infty \leqslant \lambda \leqslant +\infty.$$

If either

(*a*) *$\lim_{x \to p} f(x) = 0$ and $\lim_{x \to p} g(x) = 0$ or*

(*b*) *$\lim_{x \to p} g(x) = +\infty$,*

then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lambda.$$

Proof. There are three cases:

(1) λ is finite (i.e., it is a real number). Let $\varepsilon > 0$ be given. Assume that condition (*a*) holds. Since

$$\lim_{x \to p} \frac{f'(x)}{g'(x)} = \lambda,$$

there exists $\delta > 0$ such that if $0 < |x - p| < \delta$, then

$$\left| \frac{f'(x)}{g'(x)} - \lambda \right| < \varepsilon \qquad \text{(Fig. 8.10).}$$

Let x_1 and x_2 be two distinct points in the neighborhood $N(p, \delta)$ with $x_2 < x_1 < p$ or $p < x_1 < x_2$ and $g(x_2) \neq 0$.

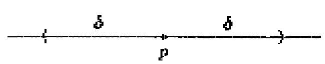

Fig. 8.10

By the generalized law of the mean there exists a point t between x_1 and x_2 such that

$$\frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \frac{f'(t)}{g'(t)}.$$

We shall keep x_2 fixed and let x_1 approach p. Here, one sees that since

$g'(x) \neq 0$ on $E - \{p\}$,

$$g(x_1) \neq g(x_2).$$

Since

$$\lim_{x_1 \to p} f(x_1) = \lim_{x_1 \to p} g(x_1) = 0,$$

$$\frac{f(x_2)}{g(x_2)} = \lim_{x_1 \to p} \frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \lim_{x_1 \to p} \frac{f'(t)}{g'(t)}.$$

Here, since x_2 is fixed, t depends only on x_1.
Since $t \in N(p, \delta)$,

$$\left| \lim_{x_1 \to p} \frac{f'(t)}{g'(t)} - \lambda \right| \leqslant \varepsilon.$$

Thus

$$\left| \frac{f(x_2)}{g(x_2)} - \lambda \right| \leqslant \varepsilon.$$

We have shown that if $0 < |x_2 - p| < \delta$ and $g(x_2) \neq 0$, then

$$\left| \frac{f(x_2)}{g(x_2)} - \lambda \right| \leqslant \varepsilon.$$

Thus

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lambda.$$

Assume now that condition (b) holds. Since

$$\lim_{x \to p} \frac{f'(x)}{g'(x)} = \lambda$$

and

$$\lim_{x \to p} g(x) = +\infty,$$

there exists $\delta > 0$ such that if $0 < |x - p| < \delta$, then

$$\left| \frac{f'(x)}{g'(x)} - \lambda \right| < \frac{\varepsilon}{2}$$

and $g(x) > 0$.

Let x_1 and x_2 be in $N(p, \delta)$ with $x_2 < x_1 < p$ or $p < x_1 < x_2$. We will keep x_2 fixed and let x_1 approach p. There exists $\delta_1 < \delta$, $\delta_1 > 0$ such that if $0 < |x_1 - p| < \delta_1$,

$$g(x_1) > g(x_2) \qquad \text{and} \qquad g(x_1) > 0.$$

Take

$$x_1 \in N(p, \delta_1) - \{p\}.$$

By the generalized law of the mean, there exists a point t between x_1 and x_2 such that

$$\frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \frac{f'(t)}{g'(t)},$$

$$\frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} \cdot \frac{g(x_1) - g(x_2)}{g(x_1)} = \frac{f'(t)}{g'(t)} \left[1 - \frac{g(x_2)}{g(x_1)}\right],$$

$$\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_1)} = \frac{f'(t)}{g'(t)} \left[1 - \frac{g(x_2)}{g(x_1)}\right],$$

$$\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_1)} - \lambda = \frac{f'(t)}{g'(t)} - \lambda - \frac{f'(t)}{g'(t)}\frac{g(x_2)}{g(x_1)},$$

$$\left|\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_1)} - \lambda\right| \leqslant \left|\frac{f'(t)}{g'(t)} - \lambda\right| + \left|\frac{f'(t)}{g'(t)}\right|\left|\frac{g(x_2)}{g(x_1)}\right|.$$

Now $f'(t)/g'(t) = \lambda + \mu$, where $|\mu| < \varepsilon/2$.

$$\left|\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_1)} - \lambda\right| \leqslant \left|\frac{f'(t)}{g'(t)} - \lambda\right| + \left(|\lambda| + \frac{\varepsilon}{2}\right)\left(\left|\frac{g(x_2)}{g(x_1)}\right|\right).$$

There exists $\delta_2 < \delta_1$, $\delta_2 > 0$ such that if x_1 is in $N(p, \delta_2) - \{p\}$,

$$\left|\frac{g(x_2)}{g(x_1)}\right| < \frac{\varepsilon/2}{|\lambda| + (\varepsilon/2)}.$$

Take x_1 in $N(p, \delta_2) - \{p\}$. Then

$$\left|\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_1)} - \lambda\right| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus if $0 < |x_1 - p| < \delta_2$,

$$\left|\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_1)} - \lambda\right| < \varepsilon \qquad \left|\frac{f(x_1)}{g(x_1)} - \lambda\right| < \varepsilon + \left|\frac{f(x_2)}{g(x_1)}\right|.$$

There exists $\delta_3 < \delta_2$, $\delta_3 > 0$ such that if $|x_1 - p| < \delta_3$, then

$$\left|\frac{f(x_2)}{g(x_1)}\right| < \varepsilon.$$

Thus if $|x_1 - p| < \delta_3$, then

$$\left|\frac{f(x_1)}{g(x_1)} - \lambda\right| < 2\varepsilon,$$

i.e.,

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lambda.$$

(2) $\lambda = +\infty$. Assume condition (a) to hold. Let $M > 0$ be given. Since

$$\lim_{x \to p} \frac{f'(x)}{g'(x)} = +\infty,$$

there exists $\delta > 0$ such that if $0 < |x - p| < \delta$, then

$$\frac{f'(x)}{g'(x)} > M.$$

Let x_1 and x_2 be two distinct points in $N(p, \delta)$, with $g(x_2) \neq 0$ and $x_2 < x_1 < p$ or $p < x_1 < x_2$. By the generalized law of the mean, there exists a point t between x_1 and x_2 such that

$$\frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \frac{f'(t)}{g'(t)}.$$

We will keep x_2 fixed and let x_1 approach p. Since

$$\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0,$$

we have

$$\frac{f(x_2)}{g(x_2)} = \lim_{x_1 \to p} \frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \lim_{x_1 \to p} \frac{f'(t)}{g'(t)}.$$

Since $t \in N(p, \delta)$,

$$\lim_{x_1 \to p} \frac{f'(t)}{g'(t)} \geqslant M.$$

Thus $f(x_2)/g(x_2) \geqslant M$. Thus

$$\lim_{x\to p}\frac{f(x)}{g(x)} = +\infty.$$

Assume now that condition (b) holds. Let $M > 0$ be given. Since

$$\lim_{x\to p}\frac{f'(x)}{g'(x)} = +\infty \quad \text{and} \quad \lim_{x\to p} g(x) = +\infty,$$

there exists $\delta > 0$ such that if $0 < |x - p| < \delta$, then

$$\frac{f'(x)}{g'(x)} > 4M$$

and $g(x) > 0$. Let x_1 and x_2 be two points in $N(p, \delta) - \{p\}$ with $x_2 < x_1 < p$ or $p < x_1 < x_2$. We will keep x_2 fixed and let x_1 approach p. There exists $\delta_1 < \delta, \delta_1 > 0$ such that if $|x_1 - p| < \delta_1$,

$$g(x_1) > g(x_2).$$

Take $x_1 \in N(p, \delta_1) - \{p\}$. By the generalized law of the mean, there exists a point t between x_1 and x_2 such that

$$\frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \frac{f'(t)}{g'(t)}.$$

Since $t \in N(p, \delta) - \{p\}$, $f'(t)/g'(t) \geqslant 2M$.

$$\frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)}\,\frac{g(x_1) - g(x_2)}{g(x_1)} = \frac{f'(t)}{g'(t)}\left[1 - \frac{g(x_2)}{g(x_1)}\right],$$

$$\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_1)} = \frac{f'(t)}{g'(t)}\left[1 - \frac{g(x_2)}{g(x_1)}\right]$$

There exists $\delta_2 < \delta_1, \delta_2 > 0$, such that if $0 < |x_1 - p| < \delta_2$, then

$$1 - \frac{g(x_2)}{g(x_1)} > \frac{1}{2}.$$

Thus if $|x_1 - p| < \delta_2$, then

$$\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_1)} > 4M \cdot \tfrac{1}{2} = 2M.$$

$$\frac{f(x_1)}{g(x_1)} > 2M + \frac{f(x_2)}{g(x_1)}.$$

Now there exists $\delta_3 < \delta_2, \delta_3 > 0$, such that if $|x_1 - p| < \delta_3$, then

$$\frac{f(x_2)}{g(x_1)} < M.$$

Thus if $|x_1 - p| < \delta_3$, then

$$\frac{f(x_1)}{g(x_1)} > 2M - M.$$

We have shown that if $|x_1 - p| < \delta_3$, then $f(x_1)/g(x_1) > M$, i.e.,

$$\lim_{x\to p}\frac{f(x)}{g(x)} = +\infty.$$

(3) $\lambda = -\infty$. This reduces to case (2) after replacing $f(x)$ by $-f(x)$.

This completes the proof of the theorem.

In Theorem 8.1 p was supposed to be some real number. We wish to extend the theorem to the case when $p = +\infty$ or $-\infty$.

One recalls that the statement

$$\lim_{x\to\infty} f(x) = \lambda, \qquad \lambda \text{ finite},$$

means that for each $\varepsilon > 0$ there exists $M > 0$ such that if $x > M$, then $|f(x) - \lambda| < \varepsilon$.

Let us define the concept of a neighborhood of ∞ thus:

$$N(\infty, M) = \{\text{all real numbers } r \text{ such that } r > M\}.$$

We may now phrase the definition of the above limit statement thus: For each $\varepsilon > 0$ there exists an $M > 0$ such that if x is in $N(\infty, M)$, then $|f(x) - \lambda| < \varepsilon$.

If $\lambda = \infty$, the above limit statement may be defined thus: For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if x is in $N(\infty, \delta)$ then $f(x)$ is in $N(\infty, \varepsilon)$.

Similarly the statement $\lim_{x\to-\infty} f(x) = \lambda$, λ finite, means that for each $\varepsilon > 0$ there exists $N > 0$ such that if $x < -N$, then $|f(x) - \lambda| < \varepsilon$. By defining the neighborhood $N(-\infty, N)$ to mean the set {all real numbers r such that $r < -N$}, we may phrase the definition of the above limit statement thus:

For each $\varepsilon > 0$ there exists $N > 0$ such that if x is in $N(-\infty, N)$, then $|f(x) - \lambda| < \varepsilon$.

For $\lambda = \infty$, the above limit statement may be defined thus: For each $\varepsilon > 0$ there exists $N > 0$ such that if x is in $N(-\infty, N)$, then $f(x)$ is in $N(\infty, \varepsilon)$.

The statement

$$\lim_{x \to -\infty} f(x) = -\infty$$

may be defined thus: For each $\varepsilon > 0$ there exists $\delta > 0$ such that if x is in $N(-\infty, \delta)$, then $f(x)$ is in $N(-\infty, \varepsilon)$.

The statement

$$x \in E - \{+\infty\}, \qquad +\infty \in E$$

is to be taken to mean that E has no upper bound and that x is any real number in E. Similarly, the statement

$$x \in E - \{-\infty\}, \qquad -\infty \in E,$$

is to be taken to mean that E has no lower bound and that x is any real number in E.

With the above interpretations one may now state L'Hôpital's theorem in broadest form.

Theorem *Let $f(x)$ and $g(x)$ be defined and be differentiable on the set $E - \{p\}$, E open and $p \in E$, p being a real number, or $+\infty$, or $-\infty$. Let $g'(x) \neq 0$ for all x in $E - \{p\}$. Let*

$$\lim_{x \to p} \frac{f'(x)}{g'(x)} = \lambda, \qquad -\infty \leqslant \lambda \leqslant +\infty.$$

If either

(*a*) $\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0,$

or

(*b*) $\lim_{x \to p} g(x) = \pm\infty,$

then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lambda.$$

Proof. With the above interpretations, the proof of Theorem 8.1 holds for this broader theorem with no essential change.

We will now discuss the evaluation of certain indeterminate forms. The basic problem is the evaluation of $\lim_{x \to p} f(x)/g(x)$ when either

(1) $\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0$, or

(2) $\lim_{x \to p} g(x) = \pm\infty$, $-\infty \leqslant p \leqslant +\infty$; $g'(x) \neq 0$ on some neighborhood of p, it being known that $f(x)$ and $g(x)$ are differentiable on this neighborhood.

The theorem of L'Hôpital immediately permits us to replace $\lim_{x \to p} f(x)/g(x)$ by its equal $\lim_{x \to p} f'(x)/g'(x)$. If the function $f'(x)/g'(x)$ is seen to satisfy the condition of the L'Hôpital theorem, then, of course, one may in turn replace

$$\lim_{x \to p} \frac{f'(x)}{g'(x)}$$

by

$$\lim_{x \to p} \frac{f''(x)}{g''(x)}$$

The process may be repeated until one arrives at a form $f^{(n)}(x)/g^{(n)}(x)$ for which $\lim_{x \to p}(f^{(n)}(x)/g^{(n)}(x))$ is obtainable directly.

Since direct substitution in $f(x)/g(x)$ results, say, in 0/0, one may say that, in this case, $f(x)/g(x)$ assumes the indeterminate form 0/0. When

$$\lim_{x \to p} g(x) = \infty \qquad \text{and} \qquad \lim_{x \to p} f(x) = \infty,$$

one says that $f(x)/g(x)$ assumes the indeterminate form ∞/∞.

We wish to point out that if

$$\lim_{x \to p} f(x) = A, \quad 0 \neq A \neq \infty, \quad \text{and} \quad \lim_{x \to p} g(x) = B, \quad 0 \neq B \neq \infty,$$

$\lim_{x \to p} f(x)/g(x)$ is, in general, distinct from $\lim_{x \to p} f'(x)/g'(x)$.

EXAMPLE 1. Evaluate $\lim_{x \to 1} (x^2 + x - 2)/(x - 1)$.

Solution. Since x is not permitted to assume the value 1, one sees that

$$\lim_{x \to 1} \frac{(x - 1)(x + 2)}{x - 1} = \lim_{x \to 1} (x + 2) = 3.$$

L'Hôpital's rule gives

$$\lim_{x \to 1} \frac{2x + 1}{1} = \frac{3}{1} = 3.$$