Newton's Method:

How to solve a nonlinear algebraic equation: $f(x)=0$ numerically and efficiently?

We have learned:

Bisection method: (use Intermediate value Theorem to find a root)

f: continuous

$f(a)\,f(b) < 0 \implies f$ has a root on (a, b)

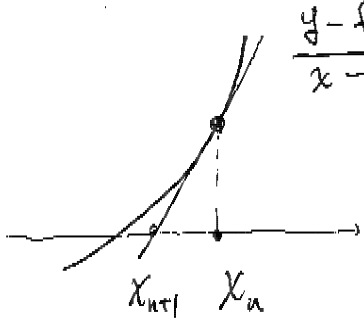
Check whether $f(\frac{a+b}{2}) > 0$ or < 0

and shrink the interval to either $(a, \frac{a+b}{2})$ or $(\frac{a+b}{2}, b)$

Then continue on.

New method: Newton's method.

x_n: n th approximate root



$$\frac{y-f(x_n)}{x-x_n} = f'(x_n)$$

x_{n+1}: solve $\begin{cases} \dfrac{y-f(x_n)}{x-x_n} = f'(x_n) \\ y = 0 \end{cases}$

to get $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Example: $\sqrt{2}$ is a root of $f(x) = x^2 - 2 = 0$

Since $f(1)f(2) < 0$

we know the root lies on the interval $(1,2)$

We start with $x_0 = \frac{3}{2}$

with Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$$

The first few appoximation and errors are

	x_n	$x_n - \sqrt{2}$
$n=0$	1.5	8.58×10^{-2}
$n=1$	1.416666	2.45×10^{-3}
$n=2$	1.414215	2.12×10^{-6}
$n=3$	1.41421356	1.59×10^{-12}

This behavior is typical for Newton's method: Prop: $|x_{n+1} - \sqrt{2}| \simeq c\,|x_n - \sqrt{2}|^2$

Pf: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x = x - \frac{f(x)}{f'(x)}$ $(\because f(x) = 0)$

$x_{n+1} - x = x_n - x - (g(x_n) - g(x))$ where $g(x) = \frac{f(x)}{f'(x)}$

ξ between x_n and x

~~$\xi \in (x_n, x)$~~

$$= (x_n - x) - g'(x)(x_n - x) - \frac{g''(\xi)}{2}(x_n - x)^2 \quad \left(\begin{array}{l}\text{Linear approximation}\\ \text{and error term}\end{array}\right)$$

$$= -\frac{g''(\xi)}{2}(x_n - x)^2 \quad (\because g'(x) = 1, \text{ exercise})$$