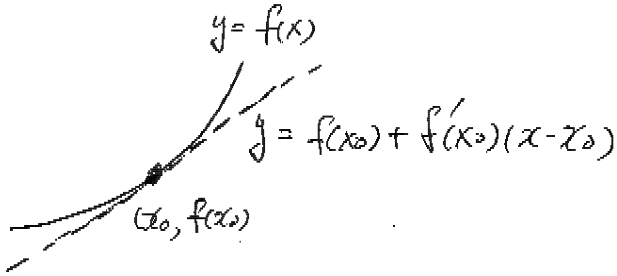
Linearization and differentials

Basic idea:



Tangent lines are good approximations of the original function near the tangent point

$$f(x) \cong f(x_0) + f'(x_0)(x - x_0)$$

when $|x - x_0|$ is small.

The "linear" function (A straight line)

$$y = L(x) := f(x_0) + f'(x_0)(x - x_0)$$

is called the linearization of f at x_0

It can be used to approximate $f(x)$ near x_0

Examples

- $(1+x)^k \cong 1 + kx \qquad |x| << 1.$
- $\sqrt{1+x} = (1+x)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}x, \qquad |x| \ll 1$
- $\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \cong 1 - \frac{x^2}{2}, \qquad |x| \ll 1$

Example

(a) $\sqrt{4.01} = 2\sqrt{1.0025}$

$\simeq 2(1 + \frac{1}{2}0.0025)$

$= 2.0025$

(b) $\dfrac{\sqrt{4.02}}{2+\sqrt{9.02}}$:

$\sqrt{4.02} \simeq 2.005$

$\sqrt{9.02} \simeq 3\sqrt{1.002} \sim 3.003$

$\dfrac{\sqrt{4.02}}{2+\sqrt{9.02}} \simeq \dfrac{2.005}{5.003} = \dfrac{2}{5}\dfrac{(1.0025)}{(1.0006)}$

$\dfrac{1+x}{1+y}$ $\quad |x| \ll 1, |y| \ll 1$

$\simeq (1+x)(1-y) \simeq 1+x-y$

$\therefore$ Ans $\sim \frac{2}{5}(1.0019) = 0.400076$

Example : Volume of a ball

$V(r) = \frac{4}{3}\pi r^3$

$V(r+\Delta r) \cong V(r) + V'(r)\Delta r$

On the other hand, Let $A(r)$ be the area of a sphere with radius r

then $V(r+\Delta r) - V(r) \sim A(r)\cdot\Delta r$ ((Base Area)$\cdot$ height)

$\Rightarrow A(r) = V'(r) = 4\pi r^2$

Error of linear approximation

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \varepsilon \cdot (x-x_0)$$

with $\lim_{x \to x_0} \varepsilon = 0$

Typically, $$\left|\varepsilon \cdot (x-x_0)\right| \le \left(\frac{1}{2} \max_{\substack{\xi \text{ between} \\ x_0 \text{ and } x}} \left| f''(\xi) \right| \right) \cdot (x-x_0)^2$$

This is easily seen if $f(x)$ happens to be a quadratic polynomial

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + C(x-x_0)^2$$

$$\Rightarrow \quad \frac{d^2}{dx^2} (\text{both sides}) \Big|_{x = x_0}$$

$$\Rightarrow \quad C = \frac{1}{2} f''(x_0) \longleftrightarrow (\text{a special case})$$

The general statement

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2}(x-x_0)^2$$

for some ξ between ~~x_0 and~~ x_0 and x

can be obtained using Mean Value Theorem, see Chapter 4.

<u>Example</u>: $\sqrt{4.01} \cong 2.0025$, ~~what~~ How large is the error?

<u>Ans</u> Consider $f(x) = \sqrt{x}$, $x_0 = 4$

$$\Rightarrow f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \qquad f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$$

$$\therefore \text{Error} \le \left(\frac{1}{2} \max_{\xi \in (4, 4.01)} \frac{1}{4} \xi^{-\frac{3}{2}} \right) \cdot (0.01)^2 \simeq \frac{1}{8} \cdot 4^{-\frac{3}{2}} \cdot (0.01)^2 = \frac{1}{64} \times 10^{-}$$