Implicit differentiation

When $y = y(x)$ is implicitly defined by a function of 2 variables $F(x, y) = 0$,

(for example $\frac{x^2}{6} + \frac{y^2}{2} = 1$ defines the upper and lower branches of the ellipse as 2 functions $y_\pm = \pm\sqrt{2 - \frac{x^2}{3}}$)

It is possible to compute the derivatives of $y_\pm$ by directly differentiating the expression

$$F(x, y(x)) = 0$$

The underlying mechanism for this operation is:

$$\text{If} \quad f(x) = g(x) \qquad (\text{for all } x)$$
$$\text{then} \quad f'(x) = g'(x).$$

Example:

$$\frac{x^2}{6} + \frac{y^2(x)}{2} = 1$$

Here $f(x) = \frac{x^2}{6} + \frac{y^2(x)}{2}$, $g(x) = 1$

Therefore

$$\begin{array}{ccc} f'(x) & = & g'(x) \\ \| & & \| \\ \frac{x}{3} + y \cdot y' & & 0 \end{array}$$

and $y' = -\frac{x}{3y}$

The advantage of implicit differentiation is that we do not need to know the exact expression of $y = y(x)$, but we can still evaluate $y'(x_0)$, Provided we are given the values of (x_0, y_0) such that $F(x_0, y_0) = 0$.

Explicit: $\boxed{\text{Given } y = y(x),\ x_0,} \Longrightarrow \boxed{\text{evaluate } y'(x_0)}$

Implicit: $\boxed{\text{Given } F(x,y),\ x_0, y_0 \text{ with } F(x_0, y_0) = 0} \Rightarrow \boxed{\text{evaluate } y'(x_0) \text{ in terms of } x_0, y_0}$

In previous example, we can verify the identity

(*) $$\frac{x}{3} + y \cdot y' = 0$$

by subsitituting $y = \pm\sqrt{2 - \frac{x^2}{3}}$ and $y' = \pm \dfrac{-\frac{x}{3}}{\sqrt{2 - \frac{x^2}{3}}}$.

In more complicated cases, it is sometimes impossible (or too much work) to find out the expression $y = y(x)$, Never the less, direct differentiation on $F(x, y(x)) = 0$ still gives an equation for y' (in terms of x and y) and we can easily solve for $y'(x_0)$, once x_0 and $y(x_0)$ is given.

Examples:

$$x + y(x) + \sin(x + y^2(x)) = 0$$

it is not possible to solve $y(x)$ explicitly.

$$\frac{d}{dx} \Rightarrow 1 + y' + \cos(x+y^2)\cdot(1+2y\cdot y') = 0$$

$$\therefore\ y'(x) = \frac{-(1+\cos(x+y^2))}{1+\cos(x+y^2)\cdot 2y}$$

<u>Application:</u> Power Rule for x^r, $r \in \mathbb{Q}$ (rational number)

So far, we have only introduced

$$(x^n)' = n x^{n-1} \qquad n \in \mathbb{Z} \text{ (integers)}$$

For the case of $(x^r)'$, $r \in \mathbb{Q}$. $r = p/q$, $q > 0$

$p, q \in \mathbb{Z}$

we can write $y = x^{p/q}$

and $y^q = x^p$.

View this as a implicit definition for $y(x)$

and proceed with implicit differentiation to

get

$$q y^{q-1} y' = p x^{p-1}$$

$$\text{so} \quad y' = \frac{p x^{p-1}}{q y^{q-1}} = \frac{p}{q} x^{(p/q - 1)} = r x^{r-1}$$