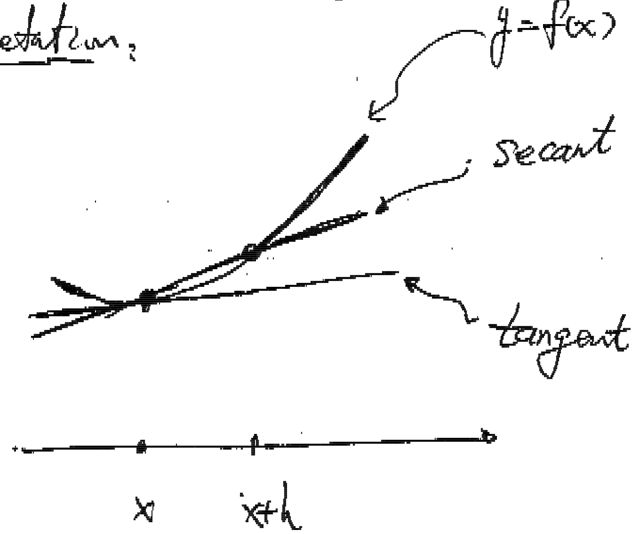
Definition : The derivative of $y = f(x)$ at x :

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Synonyms : y', $\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx} f(x)$, $D_x f$, ---

higher derivatives : $y'' = (y')'$, $y''' = (y'')'$, $y'''' = y^{(4)}$, etc.

Graphic interpretation :



$$\frac{f(x+h) - f(x)}{h} = \text{slope of secant}$$

$$f'(x) = \text{slope of tangent}$$

Examples : (1) $\left(\frac{1}{x}\right)' = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

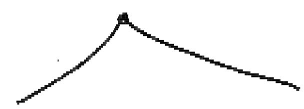
(2) $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$, $x > 0$

(detail : exercise)

Typical examples of "$f(x)$ NOT differentiable at x"

(1) $y = f(x)$ has a corner at $(x, f(x))$

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \neq \lim_{x\to 0^-} \frac{f(x+h)-f(x)}{h}$$



(2) Vertical tangent:

$$f(x) = \begin{cases} \sqrt{x} & , x \geq 0 \\ -\sqrt{-x} & , x \leq 0 \end{cases} \quad \text{at } x = 0$$

(3) f is discontinuous at x

<u>Thm</u>: f is differentiable at c
Then f is continuous at c

<u>pf</u>: $\lim_{x\to c} f(x) - f(c)$

$$= \lim_{x\to c} \left(\frac{f(x)-f(c)}{x-c}\right)(x-c)$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c}(x-c)$$

$$= f'(c) \cdot 0 = 0$$

Differential Rules and derivatives of elementary functions

(1) $\frac{d}{dx}(C) = 0$

(2) $\frac{d}{dx} x^n = n x^{n-1}$ $\quad n = 1, 2, 3, \cdots$

(3) $\frac{d}{dx}(u \pm v) = \frac{d}{dx} u \pm \frac{d}{dx} v$

$\frac{d}{dx}(cu) = c \frac{du}{dx}$

(4) Product Rule: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Pf: use definition, Exercise.

(5) $\frac{d}{dx} x^n$, $n = -m$, $m = 1, 2, 3, \cdots$

$= \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m \frac{d}{dx}(1) - 1 \frac{d}{dx}(x^m)}{(x^m)^2} = -m x^{-m-1} = n x^{n-1}$

$\therefore \frac{d}{dx} x^n = n x^{n-1}$ for $n = 0, \pm 1, \pm 2, \pm 3, \cdots$

(1) $$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{(\sin x \cosh + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h\to 0} \sin x\left(\frac{\cosh - 1}{h}\right) + \lim_{h\to 0}\left(\frac{\cos x \sin h}{h}\right)$$

$$= \lim_{h\to 0} \sin x\left(\frac{-2\sin^2\frac{h}{2}}{h}\right) + \lim_{h\to 0}\left(\frac{\cos x \sin h}{h}\right)$$

$$= \cos x$$

(2) $\frac{d}{dx}\cos x$ can be derived similarly (exercise)

or use the identity

$$\cos x = \sin\left(x+\frac{\pi}{2}\right)$$

$$\therefore \frac{d}{dx}\cos x = \lim_{h\to 0} \frac{\sin\left(x+\frac{\pi}{2}+h\right) - \sin\left(x+\frac{\pi}{2}\right)}{h}$$

$$= \frac{d}{dy}\sin y \Big|_{y = x+\frac{\pi}{2}}$$

$$= \cos\left(x+\frac{\pi}{2}\right) = -\sin x$$

(3) $\frac{d}{dx}\tan x = \sec^2 x$ using quotient rule

(4) $\frac{d}{dx}\cot x = -\frac{d}{dx}\tan\left(x+\frac{\pi}{2}\right) = -\sec^2\left(x+\frac{\pi}{2}\right) = -\csc^2 x$

(5) $\frac{d}{dx}\sec x = \sec x \tan x$ (quotient rule)

(6) $\frac{d}{dx}\csc x = -\csc x \cot x$

Derivative of composite functions

$$\frac{d}{dx} f(g(x)) = \frac{d}{dy} f(y)\Big|_{y=g(x)} \cdot \frac{d}{dx} g(x)$$

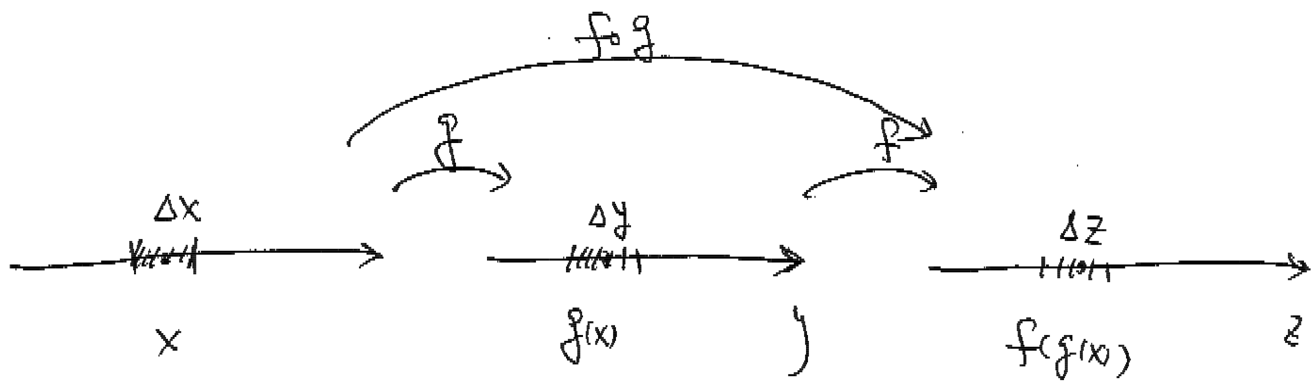
Interpretation of chain rule in terms of "rate of change"

$$y = g(x)$$

$$\frac{d}{dx} g(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

ie. $\therefore$ $g'(x) \cong \dfrac{\text{change in } y}{\text{change in } x}$ near x.

$f'(g(x)) \cong \dfrac{\text{change in } z}{\text{change in } y}$ near $g(x)$.



$$\therefore \frac{d}{dx} f(g(x)) \cong \frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x} = \frac{d}{dy} f(y)\Big|_{y=g(x)} \cdot \frac{d}{dx} g(x)$$

Symbolically: $\boxed{\dfrac{df}{dx} = \dfrac{df}{\cancel{dg}} \dfrac{\cancel{dg}}{dx}}$

Examples

(1) $$\frac{d}{dx}\tan\left(\frac{1}{x}\right) = \frac{d}{dy}\tan y\Big|_{y=\frac{1}{x}} \cdot \frac{d}{dx}\frac{1}{x}$$

$$= -\frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)$$

(2) $$\frac{d}{dx}\left(\frac{2x+1}{x+1}\right)^2$$

$$= \frac{d}{dy}(y^2)\Big|_{y=\frac{2x+1}{x+1}} \quad \frac{d}{dx}\left(\frac{2x+1}{x+1}\right)$$

$$\frac{2x+1}{x+1} = 2 - \frac{1}{x+1} \qquad \therefore \quad \frac{d}{dx}\left(\frac{2x+1}{x+1}\right) = -\frac{d}{d(x+1)}\left(\frac{1}{x+1}\right)\cdot\frac{d(x+1)}{dx}$$

and $$\frac{d}{dx}\left(\frac{2x+1}{x+1}\right)^2 = 2\left(\frac{2x+1}{x+1}\right)\cdot\frac{1}{(x+1)^2}$$

(3) $$\frac{d}{dx}\sin(\cos x)$$

$$= \frac{d}{dy}\sin y\Big|_{y=\cos x} \cdot \frac{d}{dx}\cos x$$

$$= -\cos(\cos x)\cdot\sin x$$