Informal Definition of Limit (of functions' value)

If $f(x) \longrightarrow L$ when $x \to c$

we say $\lim_{x\to c} f(x) = L$

Remark: The definition of $\lim_{x\to c} f(x)$ is not related to "$f(c)$".

Examples: (1) $f(x) = x$

(2) $f(x) = k$ (constant function)

(3) $f(x) =$ polynomials and rational functions

Properties of limits

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exists (and finite)

Then

- $\lim_{x\to c}\left(f(x) \pm g(x)\right) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
- $\lim_{x\to c}\left(f(x)\cdot g(x)\right) = \left(\lim_{x\to c} f(x)\right)\cdot\left(\lim_{x\to c} g(x)\right)$
- $\lim_{x\to c}\left(k f(x)\right) = k \lim_{x\to c} f(x)$
- $\lim_{x\to c}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, Provided $\lim_{x\to c} g(x) \neq 0$

Corollary: (1), (2) $\Rightarrow$ (3)

Examples of "$\lim_{x \to c} f(x)$ does not exist"

(1) $f(x) = \lfloor x \rfloor$ = integer part of x

i.e. $f(x) = n$ iff $n \le x < n+1$
(if and only if)
integer

(2) ~~$f(x) = \{$~~

"$f(x) = \sin(\frac{1}{x})$, $x \neq 0$ ($f(0)$ is not relevant)

In (1): we have $\lim_{x \to n^-} f(x) = n-1$

while $\lim_{x \to n^+} f(x) = n$

$\lim_{x \to n^-} \neq \lim_{x \to n^+}$ so $\lim_{x \to n}$ does not exist.

In (2) $f(\frac{1}{n\pi}) = 0$ $n = 1, 2, 3, \ldots$

while $f(\frac{1}{(2n \pm \frac{1}{2})\pi}) = \pm 1$,

values of $f(x)$ oscillates between $[-1, 1]$
as $x \to 0$.

Example (3):

$\lim_{x\to 0} \frac{1}{x^2} = ?$

We call it ∞ instead of non-existent.

But $\lim_{x\to 0} \frac{1}{x}$ does not exist, (why?)

Limits involving Infinity

$x\to\infty$ means x gets larger and larger

$x\to -\infty$ means x gets smaller and smaller
(taking account the sign)
~~i.e. large and negative~~
(i.e. negative with large absolute value)

Examples

- $\lim_{x\to\infty} \frac{1}{x} = 0$ ("0^+", to be more specific)
- $\lim_{x\to 0^+} \frac{1}{x} = \infty$
- $\lim_{x\to 0^-} \frac{1}{x} = -\infty$
- $\lim_{x\to -\infty} \frac{1}{x} = 0^-$
- Properties of limits on Page 2-1 remains valid if $x\to c$ is replaced by $x\to\infty$ or $x\to -\infty$

Limits of rational functions as $x \to \pm\infty$

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0} = \frac{\pm\infty}{\pm\infty} = ?$$

Trick: Find $p = \max(m,n)$ and devide by x^p on ~~top and~~ numerator and denominator

Conclusion:
- $n < m$, $\Rightarrow$ $\lim_{x\to\pm\infty} \frac{(\ )}{(\ )} = 0$
- $n = m$, $\lim_{x\to\pm\infty} \frac{(\ )}{(\ )} = \frac{a_n}{b_n}$
- $n > m$, $\lim_{x\to\pm\infty} \frac{(\ )}{(\ )} = \infty$ or $-\infty$

Definition of $\lim_{x \to c} f(x) = L$
Using ε, δ

How to specify "as $x \to c$" and "$f(x) \to L$" ?

and " "$x \to c$" implies "$f(x) \to L$" "

or " $f(x) \to L$ as $x \to c$ " ?

Ans: Definition of $\lim_{x \to c} f(x) = L$

Ans: Given $\varepsilon > 0$, we can always find a $\delta > 0$ such that

"$0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$"

Remark: (1) This definition is "operational", to verify whether $\lim_{x \to c} f(x) = L$, one needs to develop a strategy to find a $\delta > 0$ for any given $\varepsilon > 0$

(2) We do not care whether $f(c) = L$,

therefore we use "$0 < |x - c| < \delta$"

instead of "$|x - c| < \delta$"

. If we use "$|x - c| < \delta$", the definition ~~we~~ will impose "$\lim_{x \to c} f(x) = L$" AND "$f(c) = L$"

Example : Verify $\lim_{x \to 6} x^2 = 6^2$

Step 1. Given any $\varepsilon > 0$, find an interval around 6 on which $|x^2 - 6^2| < \varepsilon$, that is $\sqrt{6^2 - \varepsilon} < x < \sqrt{6^2 + \varepsilon}$

(we may assume ε small so that $6^2 - \varepsilon > 0$)

Step 2 : Try to fit an interval of the form

$(6-\delta, 6) \cup (6, 6+\delta)$ $\{$ that is, $0 < |x-6| < \delta \}$

inside $(\sqrt{6^2-\varepsilon}, \sqrt{6^2+\varepsilon})$

For this purpose, it suffices to take

$\delta = \min\{6 - \sqrt{6^2-\varepsilon}, \sqrt{6^2+\varepsilon} - 6\}$

(or any smaller positive number)

Step 3 Check indeed that

$0 < |x-6| < \delta$ implies $|x^2 - 6^2| < \varepsilon$

Detail left as exercise.

The Sandwich Theorem.

<u>Suppose</u> $g(x) \le f(x) \le h(x)$

for $x \in (c-\delta, c) \cup (c, c+\delta)$, $\delta > 0$

<u>and</u> $\lim_{x\to c} = \lim_{x\to c} h(x) = L$

Then $\lim_{x\to c} f(x) = L$

Examples

(1) $\sin\theta < \theta < \tan\theta \quad \theta \sim 0$

$\Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

(2) $\lim_{x\to 0} x \sin\frac{1}{x} = \lim_{x\to 0} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 0$

Continuity

Continuity at a point

$f : [a,b] \longrightarrow \mathbb{R}$.

If $c \in (a,b)$, $f(x)$ is continuous at c iff

$$\lim_{x\to c} f(x) = f(c)$$

$$\left(\begin{array}{l} f(x) \text{ is continuous at } a \text{ iff} \\ \qquad \lim_{x\to a^+} f(x) = f(a) \\ f(x) \text{ is continuous at } b \text{ iff} \\ \qquad \lim_{x\to b^-} f(x) = f(b) \end{array} \right)$$

Remark: $\lim_{x\to c} f(x) = f(c)$ means

(1) f is defined on c

(2) $\lim_{x\to c} f(x)$ exists

(3) they are equal.

~~Typical cases of failure of~~

Typical cases of non-continuity

(1). f is not defined on c

$f(x) = \sin\frac{1}{x}$ $\quad x \neq 0$

$f(0)$ not defined.

(2) $\lim_{x \to c} f(x)$ does not exist.

- $f(x) = \sin\frac{1}{x}$, $\quad c = 0$
- $f(x) = [x]$, $\quad x = n$.

(3) $\lim_{x \to c} f(x)$ exists, but $\neq f(c)$

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 1 & x = 0 \end{cases}$$

It is possible to Redefine $f(c)$ in case (3) to make $f(x)$ continuous at c

$f(c) \stackrel{\text{def}}{=}$?

Prop: f and g are continues at c

then so is $f \pm g$, $f \cdot g$, $K \cdot g$ and f/g (If $g(c) \neq 0$)

Corollary: continuity of the constant function $f(x) = c$ and identity function $f(x) = x$

Implies the continuity of ~~polynomials~~ and rational functions (if denominator $\neq 0$)

Prop: (Continuity of Composition of functions)

Suppose $f(x)$ is continuous at $x = c$

$g(y)$ is continuous at $y = f(c)$

Then $g \circ f$ is continuous at $x = c$

Pf: Left as an exercise for practice

Examples (1) $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

$\Rightarrow f(x)$ is discontinuous everywhere

(2) $\sqrt{x \cos x}$ is continuous at $x = \frac{\pi}{3}$

from the two Propositions above.

Defining continuity in terms of ε and δ.

Given $\varepsilon>0$, $\exists\ \delta>0$ (δ may depend on ε) such that

"$|x-c|<\delta \implies |f(x)-f(c)|<\varepsilon$"

Remark: $|x-c|<\delta$ can be replaced by $0<|x-c|<\delta$ here, the results are the same.

Q: Why bother to define continuity using ε, δ?

Remark: A not so obvious fact (Beyond this course)

$f_1(x), f_2(x), \cdots f_n(x), \cdots$ defined on (a,b)

Suppose each $f_n(x)$ is continuous on (a,b) and If $\lim_{N\to\infty} \sum_{i=1}^{N} f_i(x) = g(x)$ exists for each $x \in (a,b)$, then $g(x)$ may or may not be continuous!

Examples (Also beyond this course!)

(1) $\sum_{i=1}^{n} f_i(x) = x^n$ on $[0,1]$

(2) $\sum_{i=1}^{n} f_i(x) = \tan^{-1}(nx)$ on $\mathbb{R}$.

Examples: Prove $f+g$ is continuous at c if both f and g are.

(detail: see 2.5)

Q: Where does the argument fail if one tries to repeat the proof on

$$f_1(x)+f_2(x)+\cdots+f_n(x)+\cdots=g(x)$$

mentioned on page 2-11?

Application of continuity

Intermediate value Theorem: If $f(x)$ is cont on $[a,b]$ $\Longrightarrow$ f takes any value between $f(a)$ and $f(b)$

Remark: [illegible]

Homework: find an example (with sketching)

Corollary: f and g are continuous on $[a,b]$

$f(a)<g(a)$, $f(b)>g(b)$,

Then $\exists\ c\in(a,b)$ such that $f(c)=g(c)$

Example: Does $\cos x - x = 0$ have a root?

Ans: Since $|\cos x| \le 1$

We have $\cos x - x > 0$ on $x = -2$

< 0 on $x = 2$

$\therefore \exists \; x_0 \in (-2, 2)$ st $\cos x_0 - x_0 = 0$

Q: How to locate x_0 better?

Ans. Bisection, see Section 2.4 for detail.