

Brief solutions to selected problems in homework 04

1. Section 10.8: Solutions, common mistakes and corrections:

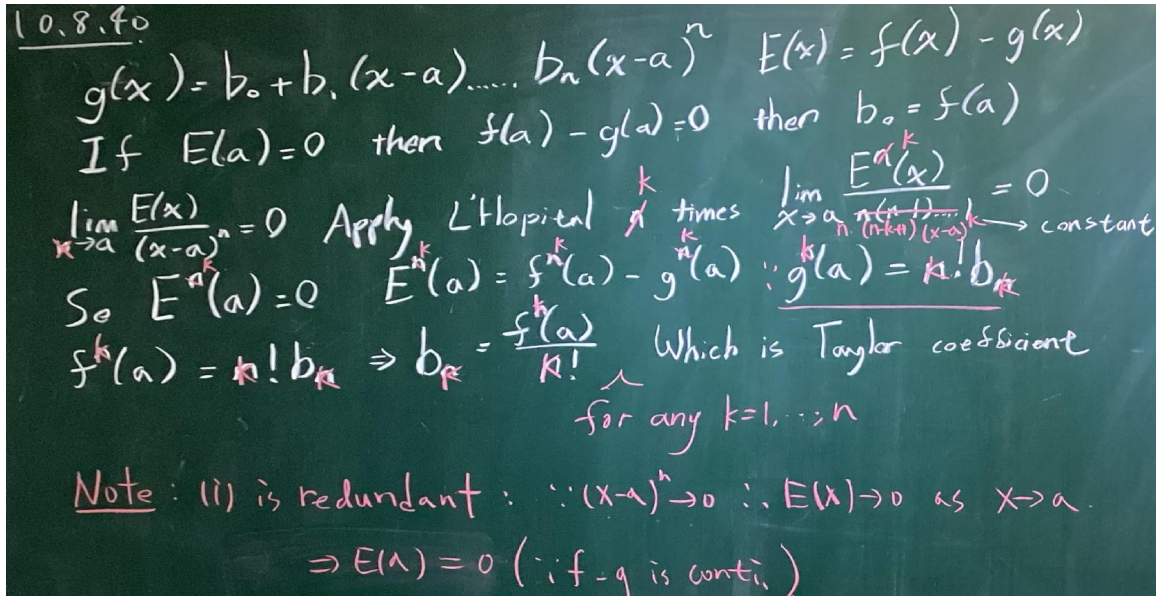


Figure 1: Solution to Section 10.8, problem 40

Remark:

The assumption " $f(x)$ is differentiable on \dots " should be understood as " $f(x)$ is differentiable infinitely many times on \dots ".

Or, at least we need to assume

" $f^{(n)}(x)$ is continuous at $x = a$ (therefore so are $f^{(k)}(x)$, $0 \leq k < n - 1$). "

Note 1:

$\lim_{x \rightarrow a} \frac{F(x)}{G(x)} = 0$ does not imply (from L'Hôpital's Rule) that $\lim_{x \rightarrow a} \frac{F'(x)}{G'(x)} = 0$ unless you

know in advance that $\lim_{x \rightarrow a} \frac{F'(x)}{G'(x)}$ indeed exists (see counter example in Lecture 14 from last semester, 20251021). This is why we need

" $f^{(n)}(x)$ is continuous at $x = a$ (therefore so are $f^{(k)}(x)$, $0 \leq k < n - 1$). "

Note 2:

(i)+(ii)+ " $f(x)$ is differentiable on \dots " does not imply $f''(a)$ exists (counter example: $f(x) = |x|^{2.1} \sin \frac{1}{x}$, $a = 0$).

2.
$$g(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

(i) By def, $g(a) = f(a) \Rightarrow E(a) = 0$

(ii) Note that $g^{(k)}(a) = f^{(k)}(a)$ for $k=1, \dots, n$.

(
$$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\Rightarrow g'(x) = f'(a) + \frac{f''(a)}{1!}(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} \Rightarrow g'(a) = f'(a)$$

$$\Rightarrow \dots \text{ (inductively) }$$

$$\lim_{x \rightarrow a} \frac{f(x) - g(x)}{(x-a)^n} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow a} \frac{f(x) - g'(x)}{n(x-a)^{n-1}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow a} \frac{f'(x) - g''(x)}{n(n-1)(x-a)^{n-2}}$$

$$= \dots = \lim_{x \rightarrow a} \frac{f^{(n)}(x) - g^{(n)}(x)}{n!} = \frac{1}{n!} (f^{(n)}(a) - f^{(n)}(a)) = 0$$

Figure 2: Solution to homework 05, problem 2

Remark: The assumption

" $f^{(n)}(x)$ is continuous at $x = a$ (therefore so are $f^{(k)}(x)$, $0 \leq k < n - 1$."

weaker than the one given in this problem, but is sufficient for this problem.

10.8.5

$$\frac{1}{x} = \frac{1}{x-2+2} = \frac{1}{2(1 + \frac{x-2}{2})} \quad r = \frac{x-2}{2}$$

$$f(x) = \frac{1}{x} = \frac{1}{2} \left(1 - \frac{x-2}{2} + \left(\frac{x-2}{2}\right)^2 - \left(\frac{x-2}{2}\right)^3 + \dots \right)$$

$$= \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \dots \quad (R=2 > 0)$$

by thm A $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{x \cdot 2} x^n = \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \dots$$

$$\left(\Rightarrow \frac{f^{(n)}(2)}{n!} = a_n \cdot f(x) = \frac{1}{x} \right)$$

Figure 3: Solution to Section 10.8, problem 5, method 2

10.8.35
(Last Week)

$$(fg)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}$$

$$f(x) = \cos x \ln(1+x) + \sin x \frac{1}{1+x} \Rightarrow f'(0) = 0$$

$$f''(x) = (-\sin x) \ln(1+x) + 2\cos x \frac{1}{1+x} + \sin x \frac{-1}{(1+x)^2} \Rightarrow f''(0) = 2$$

$$f^{(3)}(x) = \sin x \cdot \ln(1+x) + 3(-\sin x) \frac{1}{1+x} + 3(\cos x) \frac{-1}{(1+x)^2} + \sin x \left(\frac{2}{(1+x)^3} \right)$$

$$\Rightarrow f^{(3)}(0) = -3$$

$$f^{(4)}(x) = \sin x \cdot \ln(1+x) + 4(-\cos x) \frac{1}{1+x} + 6(\sin x) \left(\frac{-1}{(1+x)^2} \right)$$

$$+ 4\cos x \left(\frac{2}{(1+x)^3} \right) + \sin x \left(\frac{-6}{(1+x)^4} \right) \Rightarrow f^{(4)}(0) = 4$$

$$\Rightarrow T_{f,0}(x) = 0 + 0 + \frac{2}{2!} x^2 + \frac{-3}{3!} x^3 + \frac{4}{4!} x^4 - \dots$$

$$= x^2 - \frac{1}{2} x^3 + \frac{1}{6} x^4 - \dots$$

Figure 4: Solution to Section 10.8, problem 35, method 1

sec 10.8
35

$$f(x) = (\sin x)(\ln(1+x))$$

$$(\sin x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad)A$$

conv. on \mathbb{R}

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad)B$$

conv. on $(-1,1)$

$$(\sin x) \ln(1+x) = AB$$

$$= \sum_{n=0}^{\infty} C_n x^n$$

$$\Rightarrow T_{f(x),0} = AB = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

Thm A:
conv. on $(-1,1)$

Figure 5: Solution to Section 10.8, problem 35, method 2

10.8.41

$$1^{\circ} \frac{d}{dx} \ln(\cos x) = \frac{-\sin x}{\cos x} = -\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)}{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)} = -x - \frac{x^3}{3} - \frac{2}{15}x^5 - \dots$$

$$\Rightarrow \int_0^x \frac{-\sin t}{\cos t} dt = \int_0^x \left(-t - \frac{t^3}{3} - \frac{2}{15}t^5 - \dots\right) dt$$

$$\Rightarrow \ln(\cos x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 + \dots$$

$\therefore f(x)$ has a power series representation, $\therefore T_{f,0}(x) = f(x)$

2 $^{\circ}$ (a) linearization: 0 #

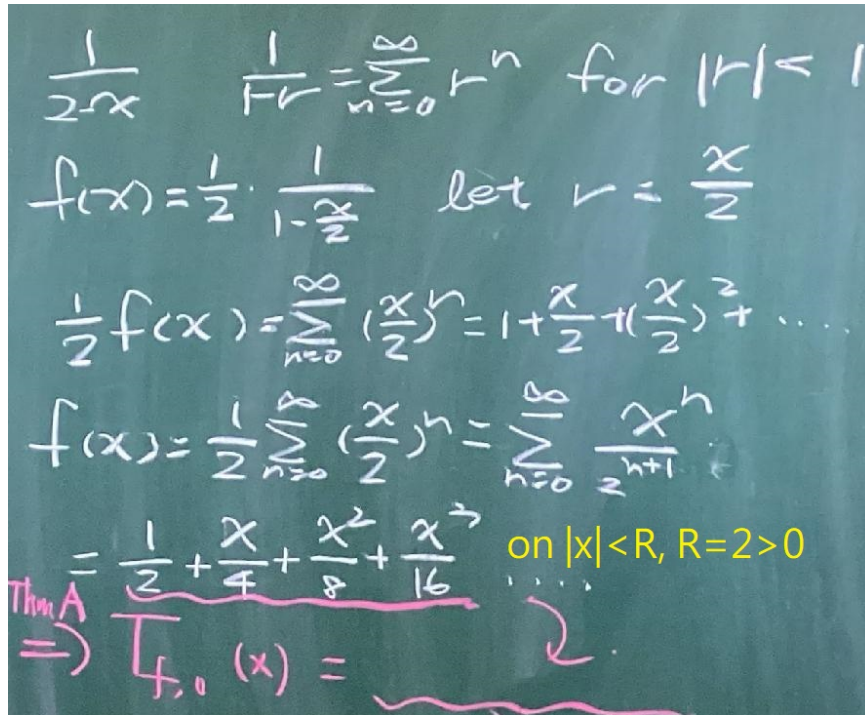
(b) quadratic: $-\frac{1}{2}x^2$ #

$= -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6$ #

Figure 6: Solution to Section 10.8, problem 41

Method 2: $\ln(\cos x)$ is obviously infinitely many times differentiable near $x = 0$. Since we don't need $T_{\ln(\cos x), 0}$ in this problem, it will be easier to find linear and quadratic approximation by differentiating $\ln(\cos x)$ twice at $x = 0$.

2. Section 10.9: Solutions, common mistakes and corrections:



$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \text{ for } |r| < 1$$

$$f(x) = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} \text{ let } r = \frac{x}{2}$$

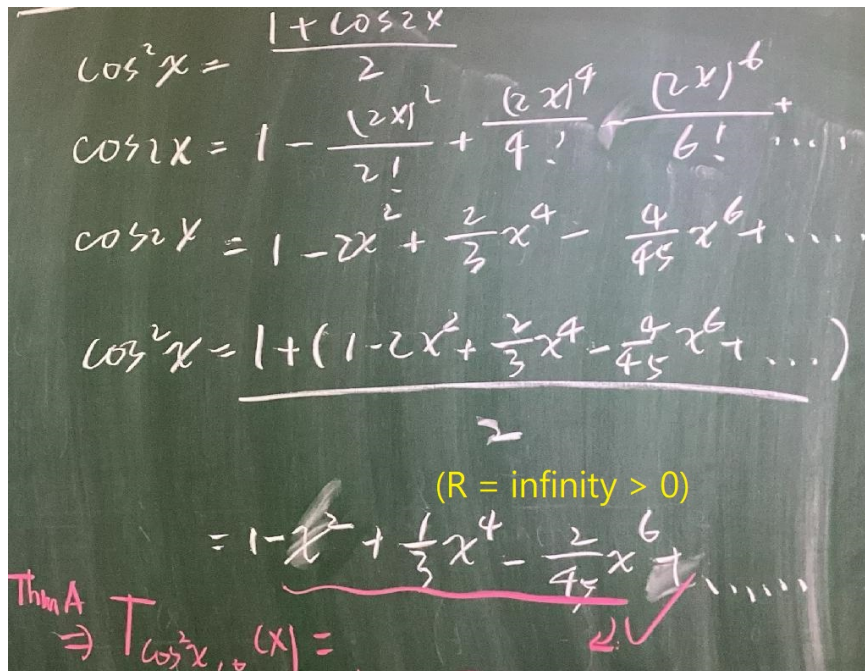
$$\frac{1}{2} f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \dots$$

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \dots \text{ on } |x| < R, R=2 > 0$$

Thm A $\Rightarrow T_{f,0}(x) = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \dots$

Figure 7: Solution to Section 10.9, problem 10



$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$$

$$\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots$$

$$\cos^2 x = \frac{1 + (1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots)}{2}$$

(R = infinity > 0)

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots$$

Thm A $\Rightarrow T_{\cos^2 x, 0}(x) = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots$

Figure 8: Solution to Section 10.9, problem 17

$$\frac{X^2}{1-2X} = X^2 \left(\frac{1}{1-2X} \right)$$

$$= X^2 (1 + (2X) + (2X)^2 + (2X)^3 + \dots)$$

$$= X^2 + 2X^3 + 4X^4 + 8X^5 + \dots, \quad |x| < 0.5$$

ThmA $R = 0.5 > 0$ $|x| < 0.5$

$\Rightarrow T_{f,0}(x) =$

Figure 9: Solution to Section 10.9, problem 19

$$\sin X = X - \frac{X^3}{3!} + \frac{X^5}{5!} - \frac{X^7}{7!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{\sin X} = 1 + \left(X - \frac{X^3}{3!} + \frac{X^5}{5!} - \dots \right) + \frac{1}{2} \left(X - \frac{X^3}{3!} + \frac{X^5}{5!} - \dots \right)^2 + \frac{1}{6} \left(X - \frac{X^3}{3!} + \frac{X^5}{5!} - \dots \right)^3 + \dots$$

$$= 1 + X + \frac{1}{2}X^2 - \frac{1}{8}X^4 + \dots \quad R = \text{infinity} > 0$$

$\Rightarrow T_{e^{\sin X}, 0}(x) =$

ThmA

Figure 10: Solution to Section 10.9, problem 33

$\underline{10.9, 50}$
 P Accurate to n decimal $\Rightarrow |P - \pi| < 10^{-n}$
 Let $\varepsilon = P - \pi$ (error)
 $(P + \sin P) - \pi = (\pi + \varepsilon) + \sin(\pi + \varepsilon) - \pi$
 $= \varepsilon + \sin(\pi + \varepsilon) = \varepsilon - \sin \varepsilon$
 We know $\sin \varepsilon = \varepsilon - \frac{\varepsilon^3}{6} + \frac{\varepsilon^5}{120} \dots$
 so $\varepsilon - \sin \varepsilon = \frac{\varepsilon^3}{6} - \frac{\varepsilon^5}{120} + \dots$ Since $|\varepsilon| < 10^{-n}$
 and series alternating and decreasing
 It is upper bounded by $\frac{\varepsilon^3}{6}$ Alter. Series Estimation Thm
 $\left| \frac{\varepsilon^3}{6} \right| < \frac{(10^{-n})^3}{6} = \frac{10^{-3n}}{6} < 10^{-3n}$

Figure 13: Solution to Section 10.9, problem 50

\Rightarrow 因 Power Series 在 $(-R, R)$ 可
 逐項微分
 $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n \cdot X^{n-k}$
 ($n=0$ or $n=k$ are both OK)
 $\Rightarrow f^{(k)}(0) = k! a_k$
 $\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$
 $\Rightarrow \sum_{n=0}^{\infty} a_n X^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} X^n$ #

Figure 14: Solution to Section 10.9, problem 51

3. Section 10.10: Solutions, common mistakes and corrections:

(19) 估計 $\int_0^{0.1} \frac{\sin x}{x} dx$ (error) $< 10^{-8}$

(5) $\frac{1}{2}$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

交錯級數 (alternating series)

(5) $\frac{1}{2}$

$$\Rightarrow \left| \frac{\sin x}{x} - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right) \right| < \left| -\frac{x^6}{7!} \right|$$

$$\Rightarrow \left| \int_0^{0.1} \frac{\sin x}{x} dx - \int_0^{0.1} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right) dx \right| < \left| -\int_0^{0.1} \frac{x^6}{7!} dx \right|$$

$$< \left| \int_0^{0.1} \frac{(0.1)^6}{7!} dx \right| = (0.1 - 0) \cdot \frac{(0.1)^6}{7!} = \frac{(0.1)^7}{7!} < 10^{-8}$$

(5) $\frac{1}{2}$

$$\Rightarrow \int_0^{0.1} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right) dx \approx \int_0^{0.1} \frac{\sin x}{x} dx$$

$$\left| \frac{\sin x}{x} - \sum_{n=0}^N (-1)^n \frac{x^{2n}}{(2n+1)!} \right| < \left| (-1)^{N+1} \frac{x^{2(N+1)}}{(2N+3)!} \right|$$

$$\Rightarrow \left| \int_0^{0.1} \frac{\sin x}{x} dx - \int_0^{0.1} \sum_{n=0}^N (-1)^n \frac{x^{2n}}{(2n+1)!} dx \right| < \left| \int_0^{0.1} (-1)^{N+1} \frac{x^{2N+1}}{(2N+3)!} dx \right|$$

$$= \left| \int_0^{0.1} \frac{x^{2N+1}}{(2N+3)!} dx \right| = \left| \frac{x^{2N+2}}{(2N+2)(2N+3)!} \right|_0^{0.1} = \frac{(0.1)^{2N+2}}{(2N+2)(2N+3)!}$$

$\Rightarrow \frac{(0.1)^{2N+2}}{(2N+2)(2N+3)!} < 10^{-8} \Rightarrow N \geq 2$

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Figure 15: Solution to Section 10.10, problem 19

10.10.31

$$\lim_{t \rightarrow 0} \frac{1 - \cos t - \frac{t^2}{2}}{t^4}$$

2nd solution

$$= \lim_{t \rightarrow 0} \frac{1 - \frac{t^2}{2} - \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right)}{t^4}$$

$$= \lim_{t \rightarrow 0} \frac{-\frac{1}{24}t^4 + 0(t^6)}{t^4} = -\frac{1}{24}$$

(0/0, LH)

$$\lim_{t \rightarrow 0} \frac{\sin t - 2t}{4t^3}$$

(0/0, LH)

$$\lim_{t \rightarrow 0} \frac{\cos t - 2}{12t^2}$$

(0/0, LH)

$$\lim_{t \rightarrow 0} \frac{-\sin t}{24t} \left(\lim_{t \rightarrow 0} \frac{\sin t}{t}, \text{squeeze} \right)$$

$$= -\frac{1}{24}$$

Figure 16: Solution to Section 10.10, problem 31

$$\lim_{x \rightarrow \infty} x^2 \left(e^{-\frac{1}{x^2}} - 1 \right)$$

$$= x^2 \left(-1 + 1 - \frac{1}{x^4} + \frac{1}{2x^6} - \frac{1}{6x^8} + \dots \right)$$

$$\neq -1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \dots$$

$$= \lim_{t \rightarrow 0^+} \frac{1}{t^2} \left(e^{-t^2} - 1 \right)$$

$$= \lim_{t \rightarrow 0} \frac{1}{t^2} \left(-t^2 + \frac{t^4}{2!} \right)$$

$$= -1$$

Figure 17: Solution to Section 10.10, problem 35

$$\ln(1+x^2) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x \rightarrow 0} \frac{x^2 - \frac{1}{2}x^4 + \dots}{\frac{x^2}{2!} - \frac{x^4}{4!} + \dots}$$

$$= 2$$

Figure 18: Solution to Section 10.10, problem 37

$$-1 + 2x - 3x^2 + \dots = \sum_{k=0}^{\infty} (-1)^{k+1} (k+1) x^k$$

$$= \frac{d}{dx} \sum_{k=0}^{\infty} (-1)^{k+1} x^{k+1}$$

$$= \frac{d}{dx} \left(\frac{1}{1+x} - 1 \right)$$

$$= -\frac{1}{(1+x)^2} \quad (\text{when } |x| < 1)$$

Figure 19: Solution to Section 10.10, problem 51

$$\begin{aligned}
 [\tan^{-1} t]_x^\infty &= \frac{\pi}{2} - \tan^{-1} x \\
 &= \int_x^\infty \frac{1}{1+t^2} dt \\
 &= \int_x^\infty \left(\frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \dots \right) dt \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \dots \right) \Big|_x^\infty \\
 &= \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \dots
 \end{aligned}$$

$\tan^{-1}(x) = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots ; x > 1$

Figure 20: Solution to Section 10.10, problem 66

$$\begin{aligned}
 [\tan^{-1}(x)]_{-\infty}^x &= \frac{\pi}{2} + \tan^{-1}(x) \\
 &= \int_{-\infty}^x \frac{1}{1+t^2} dt \\
 &= -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots ; x < -1, \\
 \Rightarrow \text{When } x < -1, \tan^{-1}(x) &= \frac{-\pi}{2} - \frac{1}{x} + \dots
 \end{aligned}$$

Figure 21: Solution to Section 10.10, problem 66, continued