

Brief solutions to selected problems in homework 04

1. Section 10.6: Homework 04, problem 01:

$$\int_{\pi}^{\infty} \frac{\sin t}{t} dt = \lim_{b \rightarrow \infty} \int_{\pi}^b \frac{\sin t}{t} dt$$
 Int by part $u = \frac{1}{t}$ $dv = \sin t dt$

$$\lim_{b \rightarrow \infty} \left. \frac{-\cos t}{t} \right|_{\pi}^b - \int_{\pi}^b \frac{\cos t}{t^2} dt$$

$$\lim_{b \rightarrow \infty} \frac{-\cos b}{b} \rightarrow 0$$
 Squeeze Theorem

$$\left| \frac{\cos t}{t^2} \right| \leq \frac{1}{t^2}$$
 By p-series $\int_{\pi}^{\infty} \frac{1}{t^2} dt$ conv. Theorem
 (direct comparison of integrals) $\Rightarrow \int_{\pi}^{\infty} \frac{\cos t}{t^2} dt$ conv.

$$\Rightarrow \int_{\pi}^{\infty} \frac{\sin t}{t} dt$$
 conv.

$$\int_{\pi}^{\infty} \left| \frac{\sin t}{t} \right| dt$$

$$= \lim_{b \rightarrow \infty} \int_{\pi}^b \left| \frac{\sin t}{t} \right| dt$$
 floor function
 For a given $b \in \mathbb{R}$, let $N = \lfloor \frac{b}{\pi} \rfloor$
 $\Rightarrow N$ is the largest integer such that $N\pi \leq b$
 $\therefore \int_{\pi}^b = \int_{\pi}^{N\pi} + \int_{N\pi}^b$ & $b \rightarrow \infty \Rightarrow N \rightarrow \infty$
 \therefore It suffices to show $\lim_{N \rightarrow \infty} \int_{\pi}^{N\pi} \left| \frac{\sin t}{t} \right| dt$ div.

$$\int_{\pi}^{N\pi} \left| \frac{\sin t}{t} \right| dt = \sum_{k=1}^{N-1} \int_{k\pi}^{(k+1)\pi} \left| \frac{\sin t}{t} \right| dt$$

$$\left(\frac{1}{k\pi} \leq \frac{1}{(k+1)\pi} \right) \geq \sum_{k=1}^{N-1} \int_{k\pi}^{(k+1)\pi} \left| \frac{\sin t}{t} \right| dt$$

$$\left(\int_{k\pi}^{(k+1)\pi} |\sin t| dt = \left| -\cos t \right|_{k\pi}^{(k+1)\pi} = 2 \right)$$

$$= \frac{2}{\pi} \sum_{k=1}^{N-1} \frac{1}{k+1}$$
 Since $\sum_{k=1}^{\infty} \frac{1}{k+1}$ div. $\Rightarrow \lim_{N \rightarrow \infty} \int_{\pi}^{N\pi} \left| \frac{\sin t}{t} \right| dt$ div.

Figure 1: Solution to Homework 04, problem 1

2. Section 10.7: Solutions, common mistakes and corrections:

10.7.15

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+3}} \quad \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{x^n} \right| =$$

$$= \lim_{n \rightarrow \infty} |x| \left(\frac{n+3}{n+4} \right)^{\frac{1}{2}} = |x|$$

at $x=1$ $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}} = \infty$ $\therefore \sum_{n=0}^{\infty} \frac{1}{n} = \infty$ *div.*
by limit comp.

at $x=-1$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}} < \infty$ by Leibniz test

$\Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+3}}$ conv in $(-1, 1)$ $R=1$

abs. conv. $(-1, 1)$ ~~condi conv.~~
 $x=-1$

Figure 2: Solution to Section 10.7, problem 15

$$\sum_{n=0}^{\infty} \frac{\sqrt{n} x^n}{3^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1} x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n} x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1} x}{n \cdot 3} \right|$$

$$= \left| \frac{x}{3} \right|$$

$$\left| \frac{x}{3} \right| < 1 \Rightarrow |x| < 3 \Rightarrow R=3$$

$x=3$ $x=-3$

$$\sum_{n=0}^{\infty} \sqrt{n} \text{ div.} \quad \sum_{n=0}^{\infty} -\sqrt{n} \text{ div.}$$

$\sum_{n=0}^{\infty} \frac{\sqrt{n} x^n}{3^n}$ conv. absly at $x=3, -3$

Figure 3: Solution to Section 10.7, problem 19

10.7.23

$\therefore R=1$

1^o $a_n = (1 + \frac{1}{n})^n$, $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} (1 + \frac{1}{n}) = 1$

for $x=1$, $\sum_{n=1}^{\infty} (1 + \frac{1}{n})^n$ $\therefore \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e \neq 0$

$\therefore \sum_{n=1}^{\infty} (1 + \frac{1}{n})^n$ div

for $x=-1$, $\sum_{n=1}^{\infty} (-1)^n \cdot (1 + \frac{1}{n})^n$

for the same reason, it div

\therefore interval of con: $(-1, 1)$ #

2^o $\therefore R=1$ \therefore for $\forall x$ satisfies $|x| < 1$, it abs con

no conditionally con #

Figure 4: Solution to Section 10.7, problem 23

$\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$, $U_n = \frac{x^n}{n(\ln n)^2}$

Ratio Test - $\lim_{n \rightarrow \infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} \cdot n(\ln n)^2}{(n+1)[\ln(n+1)]^2 \cdot x^n} \right|$

$= |x| \lim_{n \rightarrow \infty} \frac{n}{n+1} \left(\lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} \right)^2$

$= |x| \Rightarrow \sum U_n$ conv. abs. for $|x| < 1 \Rightarrow -1 < x < 1$

$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln 2}$

$\exists \sum U_n$ conv. $\frac{1}{n(\ln n)^2}$

$x=1$ $\Rightarrow \sum U_n$ conv. $\Rightarrow \sum \frac{(-1)^n}{n(\ln n)^2}$ conv. ✓

Ⓐ $R=1, -1 \leq x \leq 1$

Ⓑ $-1 \leq x \leq 1$

con

Ⓒ No X ✓

Figure 5: Solution to Section 10.7, problem 29

$$\lim_{n \rightarrow \infty} \left| \left(\frac{n}{n+1} \right)^n x^n \right|$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \cdot |x|$$

$$\left(\frac{n}{n+1} \right)^n = \left(\frac{1}{1+\frac{1}{n}} \right)^n \rightarrow e^{-1}$$

$$\lim = \frac{|x|}{e}$$

$$\frac{|x|}{e} < 1 \Rightarrow |x| < e$$

$$R = e$$

Figure 6: Solution to Section 10.7, problem 40

3. Section 10.7: Homework 05, problem 03:

$\phi(x) = \sum_{n=0}^{\infty} (x-2)^n$
 $R=1$
 $\sum_{n=0}^{\infty} (x-2)^n$
 只要 $R=1$ 就包含在 $|x-2| < 1$ $(1, 5)$ conv

(a) $\sum_{n=0}^{\infty} (x-2)^n$ $|x-2| < 1$ $|x| < 3$ (c) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ $|x-2| < 1$ conv

$x=1 = \sum (-1)^n$ div
 $x=3 = \sum 1$ div

(d) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$ ✓

(b) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} (-1)^n = \sum_{n=1}^{\infty} \frac{(2-x)^n}{n}$ ✓
 $x=3 = \sum \frac{(-1)^n}{n}$ conv
 $x=1 = \sum \frac{1}{n}$ div

Figure 7: Solution to Homework 05, problem 3

4. Section 10.7: Homework 05, problem 04:

$$\begin{aligned}
 & \left(1 - \frac{x^2}{2} + \frac{17}{24}x^4 + \dots \right) \left(1 - x^2 + x^4 - x^6 + x^8 - \dots \right) \\
 & \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) \left(1 - x^2 + x^4 - x^6 + x^8 - \dots \right) \\
 & \hline
 & -\frac{x^2}{2} + \frac{23x^4}{24} - \frac{719x^6}{720} \\
 & -\frac{x^2}{2} + \frac{x^2}{4} - \frac{x^6}{48} \\
 & \hline
 & \frac{17x^4}{24} - \frac{714x^6}{720}
 \end{aligned}$$

Figure 8: Solution to Homework 05, problem 4

5. Section 10.7: Solutions, common mistakes and corrections:

$$\begin{aligned}
 g(x) &= \frac{3}{x-2} = \frac{3}{3 - (-(x-5))} = \frac{1}{1 - \left(-\frac{(x-5)}{3}\right)} \\
 r &= -\left(\frac{x-5}{3}\right) \quad a_n = 1 \\
 \sum_{n=0}^{\infty} \left(\frac{x-5}{-3}\right)^n &= \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n (x-5)^n \\
 \Rightarrow \text{conv when } \left|\frac{x-5}{3}\right| &< 1 \\
 \Rightarrow 2 < x < 8
 \end{aligned}$$

Figure 9: Solution to Section 10.7, problem 51

10.7.55

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

(a) $\cos x = \frac{d}{dx}(\sin x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$

(b) all x converge $\#$

$$\sin 2x = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots$$

$$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \dots$$

all x converge $\#$

(c) $2 \sin x \cos x$

$$x^1 \quad 2(x \cdot 1) = 2x$$

$$x^3 \quad 2\left[x \cdot \left(-\frac{x^2}{2!}\right) + \left(-\frac{x^3}{3!}\right) \cdot 1\right] = -\frac{4}{3}x^3$$

$$x^5 \quad 2\left[x \cdot \left(\frac{x^4}{4!}\right) + \left(-\frac{x^3}{3!}\right) \cdot \left(-\frac{x^2}{2!}\right) + \left(\frac{x^5}{5!}\right) \cdot 1\right]$$

$$\sin 2x = 2 \sin x \cos x = \frac{4}{15}x^5 + \dots$$

Figure 10: Solution to Section 10.7, problem 55

a. $\ln|\sec x| = \int \tan x \, dx$

$$\int \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835}\right)$$

$$= \frac{C+x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14175}$$

$C=0$ since $\ln|\sec 0|=0$
converges for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

b. $\sec^2 x = \frac{d}{dx} \tan x$

$$\frac{d}{dx} \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835}\right)$$

$$= 1 + x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \frac{62}{95}x^8$$

converge on $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$C_1 \sec^2 x = 1 + x^2 + \frac{2}{3}x^4 + \dots$$

Figure 11: Solution to Section 10.7, problem 57

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\Rightarrow \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} n x^n \Rightarrow \left(\frac{x}{(1-x)^2}\right)' = \frac{1+x}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n$$

$$\Rightarrow \frac{(1+x)x}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n \quad \text{take } x = \frac{1}{2}$$

$$\Rightarrow \frac{(1+\frac{1}{2})\frac{1}{2}}{(1-\frac{1}{2})^3} = 6 = \sum_{n=0}^{\infty} \frac{n^2}{2^n} \quad \checkmark$$

Figure 12: Solution to Section 10.7, problem 60

6. Section 10.7: Homework 05, problem 06:

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\int \frac{1}{1+x} dx = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\int \frac{1}{1-x} dx = -\ln(1-x) = -\left(-x + \frac{x^2}{2} - \frac{x^3}{3} + \dots\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

(c=0 since ln(1+0)=0)
(c=0 since ln(1-0)=0)

Figure 13: Solution to Homework 05, problem 6

Remark: A better alternative is to use the definite integral version of Theorem 22 (Term by Term Integration). See page 5 and page 7 of Lecture 08 (v02).

7. Section 10.8: Solutions, common mistakes and corrections:

10.8

5. $f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$

① $a=2$

$1+x+x^2+\dots = \frac{1}{1-x}$ if $|x| < 1$

$P_0(x) = \frac{1}{2}$

$P_1(x) = \frac{1}{2} - \frac{1}{4}(x-2)$

$P_2(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2$

$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$

②

$f(x) = \frac{1}{x} = \frac{1}{2+(x-2)}$

$= \frac{1}{2} \frac{1}{1 + \frac{(x-2)}{2}}$

$= \frac{1}{2} \frac{1}{1 - \frac{(2-x)}{2}}$

let $|\frac{2-x}{2}| < 1$

$\Rightarrow f(x) = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{2-x}{2}\right)^k$

Figure 14: Solution to Section 10.8, problem 05

$f(x) = \sin 3x$ $f(0) = 0$

$f'(x) = 3 \cos 3x$ $f'(0) = 3$

$f''(x) = -9 \sin 3x$ $f''(0) = 0$

$f'''(x) = -27 \cos 3x$ $f'''(0) = -27$

$0 + 3x + 0 + \frac{-27}{3!} x^3 + \dots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1} x^{2n+1}}{(2n+1)!}$ ✓

Figure 15: Solution to Section 10.8, problem 15