

### Brief solutions to selected problems in homework 03

1. Section 10.4: Solutions, common mistakes and corrections:

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$$
$$\ln(1+x) \sim x \text{ (when } x \rightarrow 0)$$
$$\ln\left(1 + \frac{1}{n^2}\right) \sim \frac{1}{n^2}$$

compare to  $\sum \frac{1}{n^2}$

$\Rightarrow$  converge  $\ast$  ✓

Figure 1: Solution to Section 10.4, problem 16

$$\sum_{n=1}^{\infty} \frac{1}{2n^{\frac{1}{2}} + n^{\frac{1}{3}}}$$
$$a_n = \frac{1}{2n^{\frac{1}{2}} + n^{\frac{1}{3}}}$$
$$b_n = \frac{1}{n^{\frac{1}{2}}} \text{ (} p = \frac{1}{2} \text{)} \Rightarrow \text{(div.)}$$
$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{2n^{\frac{1}{2}}}$$

$\therefore b_n \text{ div}$

$\therefore a_n \text{ (div.)}$

Figure 2: Solution to Section 10.4, problem 17

$$f = \frac{1}{\ln(\ln n)} \quad g = \frac{1}{\ln n}$$

$$\lim_{n \rightarrow \infty} \frac{g}{f} = \lim_{n \rightarrow \infty} \frac{\ln(\ln n)}{\ln n} = \lim_{n \rightarrow \infty} \frac{\ln(\ln n)}{\ln n} = 0$$

$\rightarrow f \gg g$

$$\therefore \sum_{n=3}^{\infty} \frac{1}{n} = \infty \quad \therefore \sum_{n=3}^{\infty} \frac{1}{\ln n} = \infty$$

10.4.15

$$\rightarrow \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} = \infty$$

Figure 3: Solution to Section 10.4, problem 27

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \ln n}, \quad \text{令 } a_n = \frac{1}{\sqrt{n} \ln n}, \quad b_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n} \ln n} = \infty$$

$$\sum_{n=2}^{\infty} b_n \text{ div} \Rightarrow \sum_{n=2}^{\infty} a_n \text{ div.}$$

Figure 4: Solution to Section 10.4, problem 29

$$1^\circ \frac{1}{n!} = \frac{1}{1 \times 2 \times \dots \times (n-1)n} < \frac{1}{n(n-1)}$$

$$2^\circ \text{ Let } \langle a_n \rangle = \frac{1}{n(n-1)}, \langle b_n \rangle = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n(n-1)} = 1$$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ (con)} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \text{ (con)}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n!} \text{ (con)}$$

Figure 5: Solution to Section 10.4, problem 34

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \text{ take } g(n) = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{g(n)} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$$

$$\Rightarrow \text{since } \sum_{n=1}^{\infty} \frac{1}{n} \text{ div. } \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \text{ div.}$$

Figure 6: Solution to Section 10.4, problem 45

$$\sum_{n=1}^{\infty} \frac{1}{n \sqrt[n]{n}} \quad \lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{n \rightarrow \infty} e^{-\frac{1}{n}} = e^{-0} = e^0 = 1$$

$$\text{Suppose } a_n = \frac{1}{n \sqrt[n]{n}}, b_n = \frac{1}{n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n \sqrt[n]{n}}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = 1$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n}$  is a p-series with  $n=1$ , it diverges, so  $\sum a_n$  diverges by the limit comparison test.

Figure 7: Solution to Section 10.4, problem 51

$$\sum A_n = \frac{(\ln n)^g}{n^p} \quad b_n = \frac{1}{n^r} \quad 1 < r < p$$

$$\lim_{n \rightarrow \infty} \frac{A_n}{b_n} = \frac{(\ln n)^g}{n^p} \cdot n^r = \frac{(\ln n)^g}{n^{p-r}}$$

$$p-r > 0 \Rightarrow \frac{(\ln n)^g}{n^{p-r}} \rightarrow 0 \text{ for all } g$$

$$\lim_{n \rightarrow \infty} \frac{A_n}{b_n} = 0 \quad \because \sum b_n = \sum \frac{1}{n^r} \text{ conv. } (r > 1)$$

$$\therefore \sum A_n \text{ conv.}$$

Figure 8: Solution to Section 10.4, problem 61

10.4.62

$$\sum_{n=2}^{\infty} \frac{\ln n^g}{n^p}, \text{ compare with } \sum_{n=2}^{\infty} \frac{1}{n^r}, \quad 0 < p < r < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\frac{1}{n^r}}{\frac{\ln n^g}{n^p}} = \lim_{n \rightarrow \infty} \frac{\ln n^{-g}}{n^{r-p}}$$

$r-p > 0 \Rightarrow n^{r-p} \rightarrow \infty$

If  $g < 0 \Rightarrow \ln n^{-g} \rightarrow \infty$   
but  $\ln n \ll n^{\epsilon} \Rightarrow \lim = 0$

If  $g > 0, \ln n^{-g} \rightarrow 0 \Rightarrow \lim = 0$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\frac{1}{n^r}}{\frac{\ln n^g}{n^p}} = 0 \quad \because \sum \frac{1}{n^r} \text{ div. } (0 < r < 1)$$

$$\Rightarrow \sum \frac{\ln n^g}{n^p} \text{ div.}$$

Figure 9: Solution to Section 10.4, problem 62

2. Section 10.5: Solutions, common mistakes and corrections:

$$\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$$

$$a_n = \frac{n^{\sqrt{2}}}{2^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{\sqrt{2}}}{2^{n+1}} \times \frac{2^n}{n^{\sqrt{2}}} = \left(1 + \frac{1}{n}\right)^{\sqrt{2}} \times \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n}\right)^{\sqrt{2}} \times \frac{1}{2} \right| = \frac{1}{2} < 1$$

By the ratio test,  $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$  conv. ✓

Figure 10: Solution to Section 10.5, problem 17

10.5.19

$$\sum_{n=1}^{\infty} n! (-e^{-n})$$

$$\rho = \lim_{n \rightarrow \infty} \frac{|(n+1)! e^{-n-1}|}{|n! e^{-n}|}$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{e} = \infty$$

By Ratio Test div. ✓

Figure 11: Solution to Section 10.5, problem 19

$$\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$$

$\lim_{n \rightarrow \infty} \left( \frac{n^{10}}{10^n} \right)^{\frac{1}{n}} = \frac{1}{10}$ 
  
 By root test

$$\sum_{n=1}^{\infty} \frac{n^{10}}{10^n} \text{ con}$$

Figure 12: Solution to Section 10.5, problem 21

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3} \quad (10.4.61)$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$  converge

$$L = \lim_{n \rightarrow \infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{\ln n}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0$$

converge

Figure 13: Solution to Section 10.5, problem 27

10.5.29

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n^2} \right) = \sum_{n=1}^{\infty} \frac{n-1}{n^2} = \sum_{n=1}^{\infty} \frac{1-\frac{1}{n}}{n}$$

when  $n > 1$   $\frac{1-\frac{1}{n}}{n} > 0$   $\frac{1}{n} > 0$

$$\lim_{n \rightarrow \infty} \frac{1-\frac{1}{n}}{n} = 0 \quad \text{We know } \sum_{n=1}^{\infty} \frac{1}{n} \text{ div}$$

so div by Limit Comparison

Figure 14: Solution to Section 10.5, problem 29

$$\left| \frac{-n}{(\ln n)^n} \right| = \frac{n}{(\ln n)^n} \quad (n \geq 2)$$

$$\lim_{h \rightarrow \infty} h^{\frac{1}{h}} = 1$$

$$\lim_{h \rightarrow \infty} \frac{h^{\frac{1}{h}}}{\ln h} = 0 \Rightarrow \sum_{n=2}^{\infty} \frac{-n}{(\ln n)^n} \text{ ~~div.~~ conv.}$$

Figure 15: Solution to Section 10.5, problem 29

$$a_n = \frac{n! \ln n}{n(n+2)!} = \frac{\ln n}{n(n+1)(n+2)}$$

$$\lim_{n \rightarrow \infty} \frac{\ln n / [n(n+1)(n+2)]}{\ln n / n^3} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3 + 3n^2 + 2n} = 1$$

By 10.f.61,  $\sum \frac{\ln n}{n^3}$  conv.

By limit Comparison Test,  $\sum a_n$  conv.

Figure 16: Solution to Section 10.5, problem 41

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1 \Rightarrow \sum \text{conv}$$

Figure 17: Solution to Section 10.5, problem 43

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{4^n \cdot 2^n \cdot n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdots (2n+1)}{4^{n+1} \cdot 2^{n+1} \cdot (n+1)!} \cdot \frac{4^n \cdot 2^n \cdot n!}{1 \cdot 3 \cdots (2n-1)}$$

$$= \lim_{n \rightarrow \infty} \frac{2n+1}{8 \cdot (n+1)} = \frac{1}{4}$$

thus converges. ✓

Figure 18: Solution to Section 10.5, problem 61

10.5.65

$$a_n = \begin{cases} \frac{n}{2^n} & \text{if } n \text{ is prime number} \\ \frac{1}{2^n} & \text{otherwise} \end{cases}$$

root test works but ratio test doesn't

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{n} \text{ or } \frac{n+1}{2^n}$$

$\Rightarrow a_n \leq \frac{1}{2^n}$  for every  $n$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{2^n} \text{ (Ratio test)}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n \rightarrow \infty} \frac{(n+1)}{2n} = \frac{1}{2} < 1$$

$\Rightarrow$  Since  $\sum_{n=1}^{\infty} \frac{n}{2^n}$  conv.  $\sum_{n=1}^{\infty} a_n$  conv. ✓

Figure 19: Solution to Section 10.5, problem 65

3. Section 10.6: Solutions, common mistakes and corrections:

(28)

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n} \rightarrow U_n$$

①  $U_n > 0$

②  $U_{n+1} < U_n$

$$U_n = \frac{1}{n \ln n}$$

$$U_{n+1} = \frac{1}{(n+1) \ln(n+1)} < U_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$$

Alteating test  $\Rightarrow$  conv.

但加绝对值:  $\sum_{n=2}^{\infty} \left( \frac{1}{n \ln n} \right)$  (div)  $\rightarrow (p=1)$

$\Rightarrow$  conditional conv. #

$\int_2^{\infty} \frac{1}{x \ln x} dx$   
 $= \int_{\ln 2}^{\infty} \frac{du}{u}$   
 $\ln x = u$

Figure 20: Solution to Section 10.6, problem 28

(29)

$$\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1}(n)}{n^2+1} = a_n$$

let  $f(x) = \frac{\tan^{-1} x}{1+x^2}$  wae

$f(n) = a_n, \forall n \in \mathbb{N}$

$f'(x) = \frac{1 - (\tan^{-1} x)^2}{(1+x^2)^2} < 0$

$a_n$  is positive,  $a_{n+1} < a_n$

check:  $\lim_{n \rightarrow \infty} \frac{\tan^{-1} n}{n^2+1} = \lim_{n \rightarrow \infty} \frac{\frac{\pi}{2}}{n^2+1} = 0$

Integral Test

$$\Rightarrow \int_1^{\infty} \frac{\tan^{-1} x}{x^2+1} dx = \lim_{b \rightarrow \infty} \left[ \frac{1}{2} (\tan^{-1}(x))^2 \right]_1^b$$

$$= \lim_{b \rightarrow \infty} \frac{1}{2} (\tan^{-1}(b))^2 - \frac{1}{2} (\tan^{-1}(1))^2$$

$$= \frac{1}{2} \left( \frac{\pi}{2} \right)^2 - \frac{1}{2} \left( \frac{\pi}{4} \right)^2$$

$\Rightarrow$  Absolute convergence

Figure 21: Solution to Section 10.6, problem 29

10.6.49

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \quad (a_n > 0)$$

Thm. If  $\sum (-1)^{n+1} a_n$  satisfies the hypothesis of Alternating series test, then  $|s - s_n| \leq a_{n+1} = |s_{n+1} - s_n|$ .

$$|s - s_4| \leq a_5 = \frac{1}{5} = 0.2$$

Figure 22: Solution to Section 10.6, problem 49

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2 + 3}$$

$$|s - s_n| < 0.001$$

Goal: find  $n$  such that  $a_{n+1} < 0.001$

$$a_{n+1} = \frac{1}{(n+1)^2 + 3} < 0.001$$

$$\Leftrightarrow (n+1)^2 > 997 \Rightarrow n \geq 31$$

Figure 23: Solution to Section 10.6, problem 53