Half-Angle substitution

$$\int \frac{P(\cos x, \sin x)}{Q(\cos x, \sin x)}\,dx \quad P, Q \text{ polynomials}$$

Let $t = \tan\frac{x}{2}$ $\quad dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$$dx = \frac{2\,dt}{1+t^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\tan\frac{x}{2}\cos^2\frac{x}{2} = \frac{2t}{1+t^2}$$

$$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\text{Ans} = \int \frac{P\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)}{Q\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)} \frac{2}{1+t^2}\,dt$$

$$= \int \frac{P_1(t)}{Q_1(t)}\,dt \text{ : partial fraction}$$

(Replace $t = \tan\frac{x}{2}$ in the end)

Eg 1 $\displaystyle\int \frac{1}{1-\cos x}\,dx$

Sol. $t = \tan\frac{x}{2}$, $dx = \dfrac{2dt}{1+t^2}$

$\cos x = \dfrac{1-t^2}{1+t^2}$

$$= \int \frac{1}{1-\frac{(1-t^2)}{1+t^2}} \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{t^2}\,dt = -\frac{1}{\tan(\frac{x}{2})} + C$$

Eg 2 $\displaystyle \int \frac{dx}{2+\sin x} = ?$

<u>Sol</u> $t = \tan\frac{x}{2}$, $dx = \frac{2\,dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$$= \int \frac{1}{2+\frac{2t}{1+t^2}} \, \frac{2\,dt}{1+t^2}$$

$$= \int \frac{1}{1+t+t^2} \, dt$$

$$= \int \frac{1}{\frac{3}{4}+\left(t+\frac{1}{2}\right)^2} \, dt$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(\tan\left(\frac{x}{2}\right)+\frac{1}{2}\right)\right) + C$$