Trigonometric Substitution

Designed for $\int f(x)\,dx$,
where $f(x)$ contains factors
of a^2+x^2, $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$.
$(a>0)$

a^2+x^2: $x = a\tan\theta$, $a^2+x^2 = a^2\sec^2\theta$
$dx = a\sec^2\theta\,d\theta$

$\sqrt{a^2-x^2}$: $x = a\sin\theta$, $a^2-x^2 = a^2\cos^2\theta$
$dx = a\cos\theta\,d\theta$

$\sqrt{x^2-a^2}$: $x = a\sec\theta$, $x^2-a^2 = a^2\tan^2\theta$
$dx = a\tan\theta\sec\theta\,d\theta$

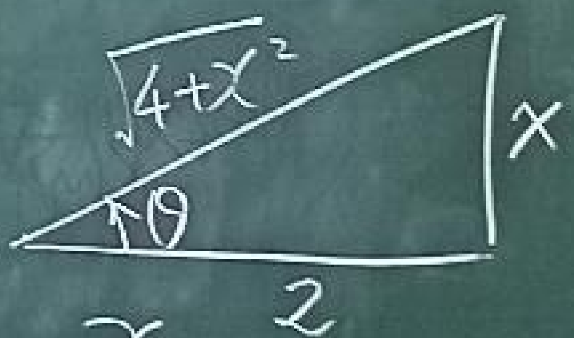
Eg 1 $\int \frac{dx}{\sqrt{4+x^2}}$ $\quad x = 2\tan\theta,\ dx = \sec^2\theta\, d\theta$

$\theta = \tan^{-1}\left(\frac{x}{2}\right) \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$= \int \frac{2\sec^2\theta}{2\sqrt{\sec^2\theta}}\, d\theta = \int \sec\theta\, d\theta$$

$\sqrt{\sec^2\theta} = \sec\theta$ if $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$= \frac{1}{2} \ln \frac{1+\sin\theta}{1-\sin\theta} + C$$



$\tan\theta = \frac{x}{2} \Longrightarrow \sin\theta = \frac{x}{\sqrt{4+x^2}}$

$$= \frac{1}{2} \ln\left(\frac{x+\sqrt{4+x^2}}{x-\sqrt{4+x^2}}\right) + C$$

$$= \frac{1}{2} \ln\left(\frac{(x+\sqrt{\ })(x+\sqrt{\ })}{(x-\sqrt{\ })(x+\sqrt{\ })}\right) + C = \ln\left(\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right) + C$$

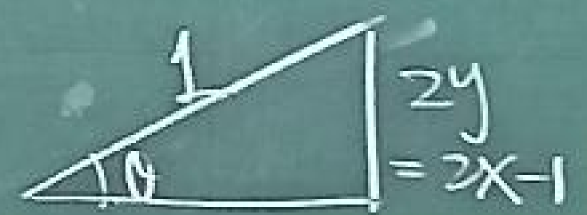
Eg2 $\int \sqrt{x-x^2}\,dx$

Sol $x-x^2$

$$= -(x^2-x) = -\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)$$

$$= \frac{1}{4}-\left(x-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 - y^2$$

$y = x-\frac{1}{2}$ $\quad dx = dy$



$y = \frac{1}{2}\sin\theta,\ dy = \frac{1}{2}\cos\theta\, d\theta$

$\theta = \sin^{-1}(2y) \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\Downarrow$$

$$= \int \sqrt{\left(\frac{\cos\theta}{2}\right)^2}\cos\theta\, d\theta \quad \left(\sqrt{\cos^2\theta} = \cos\theta\right)$$

$$= \int \frac{\cos^2\theta}{2}\, d\theta = \int \frac{1+\cos 2\theta}{4}\, d\theta = \frac{\theta}{4} + \frac{\sin 2\theta}{8} + C$$

$$= \frac{\sin^{-1}(2x-1)}{4} + \frac{1}{4}\underbrace{(2x-1)}_{\sin\theta}\left(\underbrace{\sqrt{1-(2x-1)^2}}_{\cos\theta}\right) + C$$

Eg3 $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}}$, $x < \frac{-2}{5}$

Sol $\sqrt{25x^2-4} = 5\sqrt{x^2-\left(\frac{2}{5}\right)^2}$

$x = \frac{2}{5}\sec\theta$, $x < \frac{-2}{5} \Leftrightarrow \frac{-\pi}{2} < \theta < 0$

$\theta = \sec^{-1}\left(\frac{5x}{2}\right)$

$dx = \frac{2}{5}\tan\theta\sec\theta\, d\theta$

$$\left.\begin{aligned}\sqrt{x^2-\left(\tfrac{2}{5}\right)^2} &= \tfrac{2}{5}\sqrt{\sec^2\theta-1} = \tfrac{2}{5}\sqrt{\left(\tfrac{5x}{2}\right)^2-1}\\ &= \tfrac{2}{5}|\tan\theta|\\ \tfrac{-\pi}{2}<\theta<0 \Leftrightarrow \tfrac{2}{5}|\tan\theta| &= -\tfrac{2}{5}\tan\theta\end{aligned}\right\} \Rightarrow \tan\theta = -\sqrt{\left(\tfrac{5x}{2}\right)^2-1}$$

$$\therefore \int\frac{dx}{\sqrt{25x^2-4}} = \int\frac{\frac{2}{5}\tan\theta\sec\theta\, d\theta}{5\cdot\left(\frac{-2}{5}\tan\theta\right)}$$

$$= \frac{-1}{5}\int\sec\theta\, d\theta = \frac{-1}{5}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{-1}{5}\ln\left|\frac{5x}{2}-\sqrt{\left(\frac{5x}{2}\right)^2-1}\right| + C = \frac{-1}{5}\ln\left(\frac{-5x}{2}+\sqrt{\left(\frac{5x}{2}\right)^2-1}\right) + C$$

8.5 Partial fraction for $\int \frac{f(x)}{g(x)}dx$

where $f(x)$ and $g(x)$ are polynomials

Step 1. Make sure $\deg f < \deg g$

If not, $\frac{f}{g} = P + \frac{\tilde{f}}{g}$, $\deg \tilde{f} < \deg g$

Step 2 Find Prime factors of g

$(x-r_i)^{m_i}$, $\left[(x-a_j)^2+b_j^2\right]^{n_j}$, $i, j = 1, 2, \dots$

Eg: $g(x) = (x-1)\cdot(x-2)^3\cdot(x-5)^2\cdot(x^2+1)(x^2+3)^5$

Step 3. $(x-r)^m$ in $g \Rightarrow \frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots \frac{A_m}{(x-r)^m}$ in $\frac{f}{g}$

$((x-a)^2+b^2)^n$ in $g \Rightarrow \frac{B_1x+C_1}{(x-a)^2+b^2} + \cdots + \frac{B_nx+C_n}{((x-a)^2+b^2)^n}$ in $\frac{f}{g}$

Eg 1 $\displaystyle\int \frac{x^4}{(x-1)^3}\,dx$

Sol $\displaystyle \frac{x^4}{(x-1)^3} = (x+3) + \frac{6x^2-8x+3}{(x-1)^3}$

$$\frac{6x^2-8x+3}{(x-1)^3} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3}$$

Find A_1, A_2, A_3

$\Rightarrow 6x^2-8x+3 = A_1(x-1)^2 + A_2(x-1) + A_3$

$x \leftarrow 1 \Rightarrow A_3 = 1$; $\left.\frac{d}{dx}\right|_{x\leftarrow 1} \Rightarrow A_2 = 4$;

$\left.\frac{d^2}{dx^2}\right|_{x\leftarrow 1} \Rightarrow A_1 = 6$

$\therefore$ Ans $= \frac{x^2}{2} + 3x + \ln|x-1| - \frac{4}{(x-1)} - \frac{1}{2(x-1)^2} + C$

Another method

$$
\begin{array}{r|rrr}
 & 6 & -2 & \\
\hline
1 \quad -1 & 6 & -8 & 3 \\
 & 6 & -6 & \\
\hline
 & & -2 & 3 \\
 & & -2 & 2 \\
\hline
 & & & 1
\end{array}
$$

$$\Longleftrightarrow (6x^2-8x+3) = (6x-2)(x-1)+1$$

$$= (6(x-1)+4)(x-1)+1$$

$$= 6(x-1)^2+4(x-1)+1$$

Eg2 $\displaystyle\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,dx$

Sol. $\deg(x^2+1) < \deg(x-1)(x-2)(x-3)$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-2)} + \frac{A_3}{(x-3)}$$

$$\Rightarrow (x^2+1) = A_1(x-2)(x-3) + A_2(x-1)(x-3) + A_3(x-1)(x-2).$$

$x \leftarrow 1 \Rightarrow 2 = 2A_1$

$x \leftarrow 2 \Rightarrow 5 = -A_2$

$x \leftarrow 3 \Rightarrow 10 = 2A_3$

$\therefore$ Ans $= \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C$

Eg3 $\displaystyle\int \frac{x^4}{(x^2+1)^2(x-1)}\,dx$

Sol: $$\frac{x^4}{(x^2+1)^2(x-1)} = \frac{A}{x-1} + \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2}$$

$$\Rightarrow x^4 = A(x^2+1)^2 + (B_1x+C_1)(x^2+1)(x-1) + (B_2x+C_2)(x-1)$$

$$x \leftarrow 1 \Rightarrow A = \frac{1}{4}$$

$$\Rightarrow \frac{1}{4}(3x^4-2x^2-1) = (x-1)\left(\begin{array}{c}(B_1x+C_1)(x^2+1)\\ + B_2x + C_2\end{array}\right)$$

$$\Rightarrow \cancel{\tfrac{1}{4}}(3x^3+3x^2+x+1) = 4\left(\underbrace{(B_1x+C_1)}_{\text{商}}(x^2+1) + \underbrace{B_2x+C_2}_{\text{餘}}\right)$$

$$\begin{array}{rrr|rrrr}
 & & & & & 3 & 3 \\ \hline
1 & 0 & 1 & 3 & 3 & 1 & 1 \\
 & & & 3 & 0 & 3 & \\ \hline
 & & & & 3 & -2 & 1 \\
 & & & & 3 & 0 & 3
\end{array}$$

$$= (3x+3)(x^2+1) + (-2x-2)$$

$$\because\ 3x^3+3x^2+x+1=4\left(\begin{matrix}(\frac{3}{4}x+\frac{3}{4})(x^2+1)\\ -\frac{1}{2}(x+1)\end{matrix}\right)$$

$$\Rightarrow B_1x+C_1=\frac{3}{4}x+\frac{3}{4}$$

$$B_2x+C_2=-\frac{1}{2}(x+1)$$

$$Ans=\frac{1}{4}\ln|x-1|+\frac{3}{4}\underbrace{\int\frac{x}{x^2+1}dx}_{①}+\frac{3}{4}\underbrace{\int\frac{1}{x^2+1}dx}_{②}$$

$$-\frac{1}{2}\underbrace{\int\frac{x}{(x^2+1)^2}dx}_{③}-\frac{1}{2}\underbrace{\int\frac{1}{(x^2+1)^2}dx}_{④}$$

$$① = \frac{3}{8}\ln(1+x^2)+C$$

$$② = \frac{3}{4}\tan^{-1}x+C$$

$$③ = -\frac{1}{4}\int\frac{d(x^2+1)}{(x^2+1)^2}=\frac{1}{4}(x^2+1)^{-1}+C$$

④ $x=\tan\theta$

$dx=\sec^2\theta\, d\theta$, $④=-\frac{1}{2}\int\frac{\sec^2\theta\, d\theta}{\sec^4\theta}=\frac{-1}{2}\int\cos^2\theta\, d\theta$

Eg4 $\displaystyle\int \frac{1}{(x^2+1)(x-1)^2}\,dx$

Sol: $\displaystyle\frac{1}{(x^2+1)(x-1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{Bx+C}{x^2+1}$

$\Rightarrow 1 = A_1(x-1)(x^2+1) + A_2(x^2+1) + (Bx+C)(x-1)^2$

$x \leftarrow 1 \Rightarrow A_2 = \frac{1}{2}$

$-\frac{1}{2}x^2 + \frac{1}{2} = A_1(x-1)(x^2+1) + (Bx+C)(x-1)^2$

$\Rightarrow -\frac{1}{2}(x+1) = A_1(x^2+1) + (Bx+C)(x-1)$

$x \leftarrow 1 \Rightarrow -1 = 2A_1,\ A_1 = \frac{-1}{2}$

$\Rightarrow \frac{1}{2}x^2 - \frac{1}{2}x = (Bx+C)(x-1) \Rightarrow B = \frac{1}{2},\ C = 0$

Ans $= \frac{-1}{2}\ln|x-1| - \frac{1}{2}(x-1)^{-1} + \frac{1}{4}\ln(1+x^2) + C$