Trigonometric integrals (8.3)

$$\int \cos^m x \sin^n x \, dx \quad \left(\int c^m s^n dx\right)$$

If $m = 2k+1$ $\int c^{2k} s^n c \, dx = \int (1-s^2)^k s^n ds$

if $n = 2l+1$ $\int c^m s^{2l} s \, dx = -\int c^m (1-c^2)^l dc$

If $m = 2k$, $n = 2l$

Method 1 (k, l small), $C_2 \equiv \cos 2x$

$$\int c^{2k} s^{2l} dx = \int \left(\frac{1+C_2}{2}\right)^k \left(\frac{1-C_2}{2}\right)^l dx$$

$$= \int \left(a_0 + a_1 C_2 + a_2 C_2^2 + \cdots + a_{k+l} C_2^{k+l}\right) dx$$

(redue from $c^{2k} s^{2l}$ to C_2^{k+l})

Method 2 ($m=2k$, $n=2\ell$)

Case 1: $k \le 1$, $\ell \le 1 \to$ Method 1

Assume $k>1$, we will derive formula $\int C^{2k} S^{2\ell} dx \to \int C^{2k-2} S^{2\ell} dx$

$$= \int C^{2k-1} S^{2\ell} C\,dx = \int C^{2k-1} S^{2\ell}\, dS$$

$$= \frac{1}{2\ell+1} \int C^{2k-1}\, dS^{2\ell+1}$$

$$= \frac{1}{2\ell+1} \Big(C^{2k-1} S^{2\ell+1} - \int \underbrace{S^{2\ell+1}\, dC^{2k-1}}_{(*)} \Big)$$

$$(*) = (2k-1) \int S^{2\ell+1} C^{2k-2} \underset{\substack{\shortparallel \\ -S\,dx}}{dC}$$

$$\therefore \int C^{2k} S^{2l}\, dx$$

$$= \frac{1}{2l+1}\left(C^{2k-1} S^{2l+1} + (2k+1)\left(\int C^{2k-2} S^{2l} dx - \int C^{2k} S^{2l} dx \right)\right)$$

$$(S^2 = 1 - C^2)$$

$$\Rightarrow \left(1 + \frac{2k-1}{2l+1}\right) \int C^{2k} S^{2l}\, dx$$

$$= \frac{1}{2l+1} C^{2k-1} S^{2l+1} + \frac{2k-1}{2l+1} \int C^{2k-2} S^{2l}\, dx$$

Reduce from $\int C^{2k} S^{2l} dx$ to $\int C^{2k-2} S^{2l} dx$

Recall $\begin{pmatrix} C = \cos x, \ S = \sin x \\ C_2 = \cos 2x, \text{ etc} \end{pmatrix}$

$m, n, k, \ell > 0$

$$\int C^m S^{2\ell+1} dx = -\int C^m (1-C^2)^\ell dC$$

$$\int \frac{S^{2\ell+1}}{C^m} dx = -\int \frac{(1-C^2)^\ell}{C^m} dC$$

$$\int C^{2k+1} S^n dx = \int (1-S^2)^k S^n dS$$

$$\int \frac{C^{2k+1}}{S^n} dx = \int \frac{(1-S^2)^k}{S^n} dS$$

<u>Rm</u> $\displaystyle\int \frac{C^m}{S^{2\ell+1}} dx = \int \frac{C^m S \, dx}{S^{2\ell+2}}$

$$= -\int \frac{C^m}{(1-C^2)^{\ell+1}} dC \quad \left(\begin{array}{c}\text{Section 8.5} \\ \text{partial fractions}\end{array}\right)$$

Similarly for $\displaystyle\int \frac{S^n}{C^{2k+1}} dx$

If both m, n are even

$$\int C^m S^n dx = \frac{C^{m+1} S^{n-1}}{m+n} + \frac{n-1}{m+n} \int C^m S^{n-2} dx$$

$$\left(\begin{array}{c}\text{Table} \\ \text{in textbook}\end{array}\right) = \frac{C^{m-1} S^{n+1}}{m+n} + \frac{m-1}{m+n} \int C^{m-2} S^n dx$$

Important identities

(i) $C_2 = 2C^2 - 1 = 1 - 2S^2$

$\Rightarrow 1 + C_2 = 2C^2, \ 1 - C_2 = 2S^2$

(ii) $\int S_m S_n dx, \ \int S_m C_n dx, \ \int C_m C_n dx$

Use $2 S_m S_n = C_{m-n} - C_{m+n}$

$2 S_m C_n = S_{m-n} + S_{m+n}$

$2 C_m C_n = C_{m-n} + C_{m+n}$

Remark

(i) $S_2 = 2SC$

(ii) $\sqrt{1+\sin x} = ?$

$$\begin{cases} \sin\left(x \pm \frac{\pi}{2}\right) = \pm\cos x \\ \cos\left(x \pm \frac{\pi}{2}\right) = \mp \sin x \end{cases}$$

$$= \sqrt{1-\cos\left(x-\frac{\pi}{2}\right)}$$

$$= \sqrt{2\sin^2\left(\frac{x}{2}-\frac{\pi}{4}\right)}$$

$$= \pm\sqrt{2}\sin\left(\frac{x}{2}-\frac{\pi}{4}\right)$$

Similarly

$$\sqrt{1-\sin x} = \sqrt{1+\cos\left(x-\frac{\pi}{2}\right)}$$

$$= \pm\sqrt{2}\cos\left(\frac{x}{2}-\frac{\pi}{4}\right)$$

Eg 1 $\int S^2 C^4 \, dx = ?$

(Method I) $m=4,\ n=2$

$$= \frac{C^3 S^3}{4+2} + \frac{4-1}{4+2}\int C^2 S^2 \, dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{2}\,\frac{1}{4}\int (2CS)^2 \, dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{8}\int S_2^2 \, dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{8}\int \frac{1-C_4}{2} \, dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{16}\left(x - \frac{S_4}{4}\right) + \tilde{C}$$

$$= \frac{\cos^3 x \sin^3 x}{6} + \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) + \tilde{C}$$

Method II

$$C^4 = \left(\frac{1+C_2}{2}\right)^2 = \frac{1}{4}\left(1+2C_2+\frac{1+C_4}{2}\right)$$

$$S^2 = \frac{1-C_2}{2}$$

$$C^4S^2 = \frac{1}{16}(1-C_2)\left(3+4C_2+C_4\right)$$

$$= \frac{1}{16}\left(\begin{array}{l}3+4C_2+C_4\\ \quad -3C_2-4C_2^2-C_2C_4\end{array}\right)$$

$$= \frac{1}{16}\left(3+C_2+C_4-2(1+C_4)-\frac{1}{2}(C_2+C_6)\right)$$

$$= \frac{1}{16}\left(1+\frac{C_2}{2}-C_4-\frac{C_6}{2}\right)$$

$$\text{Ans} = \frac{1}{16}\left(x+\frac{\sin 2x}{4}-\frac{\sin 4x}{4}-\frac{\sin 6x}{12}\right)+\tilde{C}$$

Rm: After reduction

(m,n)

$= (2,0)$ $\int c^2\,dx = \int \frac{1+c_2}{2}\,dx$

$(0,2)$ $\int s^2\,dx = \int \frac{1-c_2}{2}\,dx$

$(2,2)$ $\int c^2 s^2\,dx = \frac{1}{4}\int s_2^2\,dx$

$= \frac{1}{4}\int \frac{1-c_4}{2}\,dx$

$$\int t^m e^n dx \ : \ t = \tan x, \ e = \sec x$$

$$dt = e^2 dx \quad de = te\,dx$$

(i) $$\int t^{2k+1} e^n dx = \int t^{2k} e^{n-1} (te\,dx)$$

$$= \int (e^2 - 1)^k e^{n-1} de$$

Rm: $$\int t^{2k+1} e^n dx = \int \frac{S^{2k+1}}{C^{2k+1+n}} dx = -\int \frac{(1-C^2)^k}{C^{2k+1+n}} dC$$

(ii) $$\int t^m e^{2l} dx = \int t^m e^{2l-2} (e^2 dx)$$

$$= \int t^m (t^2+1)^{l-1} dt \quad \left(\begin{matrix} m \text{ can be} \\ \text{even or odd} \end{matrix} \right)$$

However $$\int t^m e^{2l} dx = \int \frac{S^m}{C^{m+2l}} dx$$

not easy if $m = \text{even}$

(iii) $\int t^{2k+1}\,dx = \int \frac{t^{2k+1}}{e} e\,dx = \int \frac{(e^2-1)^k}{e}\,de$

$$\int \frac{s^{2k+1}}{c^{2k+1}}\,dx = -\int \frac{(1-c^2)^k}{c^{2k+1}}\,dc$$

(iv) $\int t^{2k}\,dx = \int \frac{t^{2k} e^2}{e^2}\,dx = \int \frac{t^{2k}\,dt}{1+t^2}$

$$= \int \left((\text{Polynomial of } t^2) + \frac{\hat{c}}{1+t^2} \right) dt$$

$P(t^2)$

$$= \mathbb{P}(t^2) + \hat{c}\tan^{-1}(\tan x) + \tilde{c}$$

$$\alpha = \int t^{2k-2}(e^2-1)\,dx = \int t^{2k-2}(e^2\,dx) - \int t^{2k-2}\,dx$$

$$= \int t^{2k-2}\,dt - \int t^{2k-2}\,dx \left(\text{reduce from } \int t^{2k}\,dx \text{ to } \int t^{2k-2}\,dx \right)$$

$$\int t^{2k} e^{2l+1} dx \to \text{integration by parts}$$

In general

$$\int t^m e^n dx \;<\; \begin{array}{ll} \int t^{m-2} e^n dx & (1) \\ \int t^m e^{n-2} dx & (2) \end{array}$$

For (1): $\quad (t e dx = de)$

$$= \int t^{m-1} e^{n-1} de = \frac{1}{n} \int t^{m-1} de^n$$

$$= \frac{1}{n} \left(t^{m-1} e^n - \int e^n \underbrace{(m-1) t^{m-2} e^2 dx}_{d t^{m-1}} \right.$$

$$e^2 = t^2 + 1$$

$$= \frac{1}{n} \left(t^{m-1} e^n - (m-1) \int t^m e^n dx - (m-1) \int t^{m-2} e^n dx \right)$$

$$\Rightarrow \int t^m e^n dx = \frac{1}{n+m-1} t^{m-1} e^n - \frac{m-1}{n+m-1} \int t^{m-2} e^n dx$$

Similarly for (2).

$(e^2 dx = dt)$ $\quad = \frac{1}{n+m-1} t^{m+1} e^{n-2} + \frac{(n-2)}{n+m-1} \int t^m e^{n-2} dx$

$$(2k, 2l+1) \begin{array}{l} \nearrow (2k, 2l-1) \\ \searrow (2k-2, 2l+1) \end{array} \to \cdots (2,1) \to (0,1)$$

$$\int t^2 e^1 dx = \int \frac{s^2}{c^3} dx = \int \frac{s^2 c\, dx}{c^4} = \int \frac{s^2\, ds}{(1-s^2)^2} \quad (8,5)$$

$$\downarrow$$

$$\int e\, dx = \int \frac{1}{c} dx = \int \frac{c\, dx}{c^2} = \int \frac{ds}{1-s^2}$$

$$= \frac{1}{2} \int \left(\frac{1}{s+1} - \frac{1}{s-1} \right) dx$$

$$= \frac{1}{2} \ln \left| \frac{1+\sin x}{\sin x - 1} \right| + \tilde{C}$$

$$= \ln \sqrt{\frac{1+\sin x}{1-\sin x}} + \tilde{C}$$

or $\int e\, dx = \int \frac{e(t+e)}{t+e} = \int \frac{d(t+e)}{t+e}$

$$= \ln |\tan x + \sec x| + \tilde{C}$$

$t = \tan x,\ e = \sec x$

Eg1 $\int t^3 e^2 dx = \int t^3 dt$

or $= \int t^2 e\, de = \int (e^2 - 1) e\, de$

or $= \int \frac{S^3}{C^5} dx = -\int \frac{(1-C^2)\, dC}{C^5}$

Eg2 $\int t^2 e\, dx$

$= \int t\, de = e \cdot t - \int e\, dt$

$= et - \int e^3 dx = et - \int (t^2+1) e\, dx$

$= et - \int t^2 e\, dx - \int e\, dx$

$\Rightarrow \int t^2 e = \frac{et}{2} - \frac{1}{2} \int \frac{1}{\cos x} dx = \cdots$

Eg3 $\int e^6 \, dx$

$= \int e^4 \, dt = \int (t^2+1)^2 \, dt$

$= \int (t^4 + 2t^2 + 1) \, dt$

Eg4 $\int t^4 \, dx$

$= \int t^2 (e^2 - 1) \, dx = \int t^2 \, dt - \int t^2 \, dx$

$= \int t^2 \, dt - \int (e^2 - 1) \, dx$

$= \int t^2 \, dt - \int dt - \int dx = \cdots$

Eg5 $\int t^5 \, dt \to \int t^3 \, dt \to \int t \, dt$

or $= \int \frac{S^5}{C^5} \, dx = -\int \frac{S^4 \, dC}{C^5} = -\int \frac{(1-C^2)^2}{C^5} \, dC$

$\int s^m c^n dx \longleftrightarrow \int t^p e^q dx$

$m, n \in \mathbb{Z}$ $\quad p = m,\ q = -m-n$

$m = p,\ n = -p-q$

① $\int \frac{s^4}{c^3} dx = \int \frac{s^4 c\, dx}{c^4} = \int \frac{s^4 ds}{(1-s^2)^2} = \int \frac{s^4}{(1-s)^2(1+s)^2} ds$

② $\int s^4 c^{-3} dx = \int s^3 c^{-3} (s\, dx) = -\int s^3 c^{-3} dc = \frac{1}{2}\int s^3 dc^{-2}$

$= \frac{1}{2}\left(s^3 c^{-2} - \int ds^3 c^{-2}\right) = \frac{1}{2}\left(s^3 c^{-2} - 3\underbrace{\int s^2 c^{-2} c\, dx}_{(*)}\right),\quad (*) = \int \frac{s^2}{c} dx = \int \frac{s^2 ds}{1-s^2}$

Another method

$\int \frac{s^4}{c^3} dx = \int \frac{t^4}{e} dx = \begin{cases} \int \frac{t^3}{e^2}(te\, dx) = \int \frac{t^3}{e^2} de = \int t^3 e^{-2} de & ① \\ \int \frac{t^4 (e^2 dx)}{e^3} = \int \frac{t^4}{e^3} dt = \int t^4 e^{-3} de & ② \end{cases}$

$\int \frac{t^2(e^2-1)}{e} dx = \underbrace{\int \frac{t^2 dt}{e}}_{?} - \int \frac{t^2}{e} dx$ ③

$\frac{t^2}{e} dx = \frac{s^2}{c^3}$

① $= -\int t^3 de^{-1}$

$= -\left(t^3 e^{-1} - \int 3t^2 e\, dx\right)$, $\quad 3t^2 e = 3\frac{s^2}{c^3}$

= need another integration by parts.

② $= \frac{-1}{2}\int t^4 de^{-2}$

$= \frac{-1}{2}\left(t^4 e^{-2} - \int 4t^3 dx\right)$, $\quad 4t^3 dx = 4\frac{s^3}{c^3} dx \to$ easier

tan ①