Rate of Growth

<u>Def</u>

$f(x)$ grows $\begin{cases} \text{faster than} \\ \text{at the same rate as} \\ \text{slower than} \end{cases}$ $g(x)$

as $x \to \infty$

if $$\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \begin{cases} \infty \\ L, \; 0 < L < \infty \\ 0 \end{cases}$$

Eg 1 $-5x^5+7x^4-2x^2-1$

grows at the same rate

as x^5, as $x \to \infty$

Eg 2. $f(x)=e^{0.01x}$

$g(x)=x^7$, $h(x)=(\ln x)^{1000}$

$f(x)$ grows faster than $g(x)$

$g(x)$ grows faster than $h(x)$

as $x \to \infty$

Since $\lim_{x\to\infty}\frac{g(x)}{f(x)}=0$, $\lim_{x\to\infty}\left(\frac{h(x)}{g(x)}\right)^{\frac{1}{1000}}=0$, Def. $f(x) = o(g(x))$ as $x \to \infty$, 0, 0^+, $\vdots$

if $\lim\limits_{\substack{x \to \infty \\ 0, 0^+}} \frac{f(x)}{g(x)} = 0$

(i.e. f "$<$" g)

Def $f(x) = O(g(x))$ as $x \to \infty$, 0, 0^+ ..

(if $\left|\frac{f(x)}{g(x)}\right| \leq M$ as $x \to \infty$, 0, 0^+)

if there exists $M > 0$, $N > 0$, $(\delta > 0)$

such that $\left|\frac{f(x)}{g(x)}\right| \leq M$ for all $x > N$, $x \in (-\delta, \delta)$, $x \in (0, \delta)$

(i.e. f "$\leq$" g)

Remark

(i) $\lim \frac{f}{g} = 0 \iff f = o(g)$

(ii) $\lim \left|\frac{f}{g}\right| = \underset{\neq 0}{L} \;\; \begin{matrix} \Longrightarrow \\ \not\Longleftarrow \end{matrix} \;\; \begin{cases} f = O(g) \\ g = O(f) \end{cases}$

(iii) $\lim \left|\frac{f}{g}\right| = \begin{cases} 0 \\ L \end{cases} \Longrightarrow f = O(g)$

$\left(f = o(g) \Longrightarrow f = O(g) \right)$

Eg. $f(x) = x, \quad g(x) = x/(2+\sin x)$

$\Longrightarrow \left|\frac{f(x)}{g(x)}\right| = 2 + \sin x \; \begin{matrix} \leq 3 \\ \geq 1 \end{matrix} \Longrightarrow \begin{cases} f = O(g) \\ g = O(f) \end{cases}$

But $\lim\limits_{x\to\infty} \left|\frac{f(x)}{g(x)}\right|$ does not exist as $x \to \infty$

Rm $f = o(1) \iff \lim f = 0$

$f = O(1) \iff |f(x)| \le M \left(\begin{array}{l} \forall x \ge N \\ \text{or } \forall\, 0 < |x-c| < \delta \end{array} \right)$

$$\begin{array}{l} f = o(F) \\ g = o(G) \end{array} \Rightarrow \begin{array}{l} fg = o(FG) \\ f \pm g = o(|F| + |G|) \end{array}$$

$$\begin{array}{l} f = O(F) \\ g = O(G) \end{array} \Rightarrow \begin{array}{l} fg = O(FG) \\ f \pm g = O(|F| + |G|) \end{array}$$

Application:

(1) $f'(a) = m$

$$\iff f(x) = \underbrace{f(a) + m(x-a)}_{L(x)} + \underbrace{\varepsilon(x-a)}_{= o(1)(x-a) = o(x-a)}, \quad \lim \varepsilon = 0$$

(2) If $f''(x)$ exists on $a - \delta < x < a + \delta$

$$\Rightarrow |f(x) - L(x)| = \left| \tfrac{1}{2} f''(c)(x-a)^2 \right| \le \frac{M}{2}(x-a)^2$$

one can also write $f(x) - L(x) = O((x-a)^2)$

Eg 3

$$-5x^5+7x^4-2x-1=O(x^5)$$

as $x\to\infty$

Eg 4

$$(\ln x)^{1000}=o(x^7)$$

$$x^7=o(e^{0.001x}) \quad \text{as } x\to\infty$$

Eg 5

$$(\ln x)^{1000}=o(x^{-2})$$

$$x^{-2}=o\left(e^{\frac{1}{x^2}}\right) \quad \text{as } x\to 0^+$$

Eg6 $x^p = o(x^q)$

(i) as $x \to \infty$ if $p < q$

(ii) as $x \to 0$ if $p > q$

Eg7 $x + \sin x = O(x)$ as $x \to \infty$

Eg8 $e^x + x^3 = O(e^x)$

as $x \to \infty$

8. Techniques of integration

8.2 Integration by parts.

$$\int f(x)\, g'(x)\, dx = \int f(x)\, dg(x) = ?$$

Ans: $\frac{d}{dx}\int f g' \, dx = f g' = (fg)' - f' g$

$$= \frac{d}{dx}\left(fg - \int f' g \, dx \right)$$

$$\Rightarrow \int f g' \, dx = fg - \int f' g \, dx$$

or $\int f \, dg = fg - \int g \, df$

Useful if $\int g \, df = \int g f' dx$ is easy to compute.

Eg1 $\int \overset{\overset{g'}{f}}{x}\,\overset{\overset{f}{g'}}{\cos x}\,dx$

$$= \begin{cases} \int x \, d\sin x = x\sin x - \int \sin x \, dx \\ \int \cos x \, d\dfrac{x^2}{2} = \underbrace{\dfrac{x^2}{2}\cos x - \underbrace{\int \dfrac{x^2}{2} \, d\cos x}_{\text{worse}}}_{\text{NG}} \end{cases}$$

$$= x\sin x + \cos x + C$$

Check: $\dfrac{d}{dx}\left(x\sin x + \cos x\right)$

$$= \sin x + x\cos x - \sin x = \cos x$$

(correct)

Eg2 $\int \underbrace{\ln x}_{f} \underbrace{dx}_{dg}$, $(x>0)$

$$= x\ln x - \int x \, d\ln x$$

$$= x\ln x - \int x \frac{1}{x} dx$$

$$= x\ln x - x + C$$

check: $\frac{d}{dx}(x\ln x - x)$

$$= x \cdot \frac{1}{x} + 1 \cdot \ln x - 1$$

$= \ln x$ (correct)

Eg3. $\int x^2 e^x dx$

$= \int x^2 de^x$

$= x^2 e^x - \int e^x dx^2$

$= x^2 e^x - 2\int x e^x dx$

$= x^2 e^x - 2\int x de^x$

$= x^2 e^x - 2\left(x e^x - \int e^x dx\right)$

$= x^2 e - 2x e^x + 2e^x + C$

Eg4 $\int x^2 \ln x \, dx$, $x>0$

$$= \int \ln x \, d\frac{x^3}{3}$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \, d\ln x$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x} \, dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

<u>Check</u>:

$$\frac{d}{dx} = x^2 \ln x + \frac{x^3}{3}\cdot\frac{1}{x} - \frac{x^2}{3}$$

$$= x^2 \ln x \text{ (correct)}$$

Eg5: $\int e^x \cos x\,dx$

<u>Sol</u> $= \int e^x \, d\sin x$

$= e^x \sin x - \int \sin x \underbrace{e^x dx}_{de^x}$

$= e^x \sin x + \int e^x \, d(\cos x)$

$= e^x \sin x + e^x \cos x - \int \cos x \underbrace{e^x dx}_{de^x}$

$\Rightarrow 2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$

$\Rightarrow \int e^x \cos x\,dx = \frac{1}{2} e^x (\cos x + \sin x) + C$

Eg 6: $\int \cos^n x \, dx$, $n \geq 1$

Case 1. $n = 2k+1$.

$= \int \cos^{2k} x \cos x \, dx$

$= \int (1-\sin^2 x)^k \, d\sin x$

$= \int (1-s^2)^k \, ds$

If n = even $\quad c = \cos x$, $s = \sin x$

$= \int c^{n-1} c \, dx = \int c^{n-1} ds$

$= c^{n-1} s - \int s \, d(c^{n-1})$

$= c^{n-1} s - (n-1) \int s \, c^{n-2} \overbrace{(-s) dx}^{dc}$

$$= C^{n-1}S + (n-1)\int C^{n-2}(1-C^2)\,dx$$

$$\Rightarrow \int C^n dx = C^{n-1}S + (n-1)\int C^{n-2} dx - (n-1)\int C^n dx$$

$$\Rightarrow n\int C^n dx = C^{n-1}S + (n-1)\int C^{n-2} dx$$

$$\Rightarrow \int \cos^n x\,dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\int C^n dx \rightarrow \int C^{n-2} dx \xrightarrow{\cdots} \int C^2 dx \rightarrow \int 1\,dx$$

Similarly. (n = even)

$$\int S^n dx \rightarrow \int S^{n-2} dx \xrightarrow{\cdots} \int S^2 dx \rightarrow \int 1\,dx$$

Definite integrals

$$fg' = (fg)' - gf'$$

$$\int_a^b dx$$

$$\Rightarrow \int_a^b fg'\,dx = fg\Big|_a^b - \int_a^b gf'\,dx$$

or
$$\int_{x=a}^b f\,dg = fg\Big|_a^b - \int_{x=a}^b g\,df$$

Eg6: $\displaystyle\int_0^4 xe^x\,dx = \int_{x=0}^4 x\,de^x$

$$= xe^x\Big|_0^4 - \int_{x=0}^4 e^x\,dx$$

$$= xe^x\Big|_0^4 - e^x\Big|_0^4 = 3e^4 + 1$$