L'Hôpital's Rule

How to evaluate "$\lim \frac{0}{0}$" and "$\lim \frac{\infty}{\infty}$"

Eg: $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

<u>Thm</u> If f and g are differentiable on $(a-\delta, a+\delta)$, $f(a)=g(a)=0$ and $g'(x) \neq 0$ for $x \neq a$

If $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$

If the limit does not exist (eg: oscillatory), then l'Hopital's rule does not apply.

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

$\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ does not exist $\not\Rightarrow$ $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ does not exist

Eg1 $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta}$

$$\lim_{\theta \to \theta} \frac{(\sin\theta)'}{\theta'} = \lim_{\theta \to 0} \frac{\cos\theta}{1} = 1$$

$$\therefore \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \lim_{\theta \to 0} \frac{\cos\theta}{1} = 1$$

Eg2 $\lim\limits_{\theta \to 0} \dfrac{1-\cos\theta}{\theta^2} = ?$

Sol $= \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{2\theta} = \lim\limits_{\theta \to 0} \dfrac{\cos\theta}{2} = \dfrac{1}{2}$

Eg 3: $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\ ?\ \left(\text{"}\dfrac{0}{0}\text{"}\right)$

Sol, $=\lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2}\quad\left(\text{"}\dfrac{0}{0}\text{"}\right)$

$=\lim\limits_{x\to 0}\dfrac{\sin x}{6x}\quad\left(\text{"}\dfrac{0}{0}\text{"}\right)$

$=\lim\limits_{x\to 0}\dfrac{\cos x}{6}\quad\left(\text{not "}\dfrac{0}{0}\text{"},\ \text{stop!}\right)$

$=\dfrac{1}{6}$

Eg 4: $\lim_{x \to 0} \frac{1-\cos x}{x+x^2} = ?$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

Sol: $= \lim_{x \to 0} \frac{\sin x}{1+2x}$ $\left(\begin{array}{l}\text{not "}\frac{0}{0}\text{"} \\ \therefore \text{ stop}\end{array}\right)$

$\neq \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$

incorrect

$= \frac{0}{1}$ #

correct

Eg5: $\lim_{x\to\infty} x\sin\frac{1}{x} = ?\ (\infty\cdot 0)$

$$= \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \quad \left(\text{"}\frac{0}{0}\text{"}\right)$$

$$= \lim_{x\to\infty} \frac{-\cancel{\frac{1}{x^2}}\cos\left(\frac{1}{x}\right)}{-\cancel{\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1$$

$\frac{\infty}{\infty}$ version of l'Hôpital's Rule.

If $\lim\limits_{x \to a, a^+, a^-, \pm\infty} f(x) = \pm\infty$

and $\lim\limits_{x \to a, a^+, a^-, \pm\infty} g(x) = \pm\infty$

and $\lim\limits_{x \to a, a^+, a^-, \pm\infty} \frac{f'(x)}{g'(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$

Then $\lim\limits_{x \to a, a^+, a^-, \pm\infty} \frac{f(x)}{g(x)} = \lim\limits_{x \to a, a^+, a^-, \pm\infty} \frac{f'(x)}{g'(x)}$

l'Hôpital's Rule also applies to $\lim\limits_{\substack{x\to a^{\pm} \\ x\to\pm\infty \\ x\to a}}$ "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

Eg6: $\lim\limits_{x\to\infty} \dfrac{x^{4.5}}{e^{\frac{x}{100}}}$ (Similarly for any $\lim\limits_{x\to\infty} \dfrac{x^{M}}{e^{\varepsilon x}}$ $M>0$, $\varepsilon>0$)

Sol: $= \lim\limits_{x\to\infty} \dfrac{4.5\, x^{3.5}}{\frac{1}{100} e^{\frac{x}{100}}}$ (still "$\frac{\infty}{\infty}$")

$= \lim\limits_{x\to\infty} \dfrac{4.5\cdot 3.5\, x^{2.5}}{\frac{1}{100}\frac{1}{100} e^{\frac{x}{100}}}$

$= \cdots\cdots = \lim\limits_{x\to\infty} \dfrac{\cdots\, x^{-0.5}}{\cdots\, e^{\frac{x}{100}}} = 0$

Eg 7: $\lim_{x\to\infty} \frac{\ln x}{x^{\frac{1}{2}}} = ?$ "$\frac{\infty}{\infty}$"

Sol: $\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{\frac{-1}{2}}}$ ("$\frac{0}{\infty} = 0$")

$= \lim_{x\to\infty} 2x^{\frac{-1}{2}} = 0$

Same result for $\lim_{x\to\infty} \frac{\ln x}{x^k}$ for any $k > 0$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{kx^{k-1}} = \lim_{x\to\infty} \frac{1}{kx^k} = 0$

Rm $a, b, c > 0$

$$\lim_{x\to\infty} \frac{e^{ax}}{x^b} = \left(\lim_{x\to\infty}\left(\frac{e^{\frac{a}{b}x}}{x}\right)\right)^b = \infty$$

$$\lim_{x\to\infty} \frac{(\ln x)^c}{x^b} = \left(\lim_{x\to\infty} \frac{\ln x}{x^{\frac{b}{c}}}\right)^c = 0$$

In this sense, we write

$$e^{ax} \gg x^b \gg (\ln x)^c$$

as $x \to \infty$

for any $a, b, c > 0$

Proposition: for any $a, b, c > 0$

$$(\ln x)^{a} \ll x^{b} \ll e^{cx}$$

as $x \to +\infty$ (背)

and (let y = 1/x)

$$(|\ln y|)^{a} \ll y^{-b} \ll e^{\frac{c}{y}}$$

as $y \to 0^{+}$ (背)

where $f(x) \ll g(x)$

means

$$\lim \frac{f(x)}{g(x)} = 0$$

"$\infty-\infty$",

Eg8: $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$ ("$\infty-\infty$")

$$= \lim_{x\to 0}\frac{x-\sin x}{x\sin x} \quad \left(\text{"}\frac{0}{0}\text{"}\right)$$

$$= \lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x} \quad \left(\text{"}\frac{0}{0}\text{"}\right)$$

$$= \lim_{x\to 0}\frac{\sin x}{\cos x+\cos x-x\sin x} = \frac{0}{2}$$

$$= 0$$

Eg 9: $\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = ?$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

$\left(\text{Note } \lim_{x\to 0^+} e^{-\frac{1}{x}} = \lim_{x\to 0^+} \frac{1}{e^{\frac{1}{x}}} = \frac{1}{e^{\infty}} = 0\right)$

<u>Sol:</u> $= \lim_{x\to 0^+} \frac{\frac{1}{x^2} e^{-\frac{1}{x}}}{1} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^2}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

$= \lim_{x\to 0^+} \frac{\frac{1}{x^2} e^{-\frac{1}{x}}}{2x} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{2x^3}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

$= \cdots\cdots$ does not work

Instead, we write

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}} \quad \left(\text{"}\frac{\infty}{\infty}\text{"}\right)$$

$$= \lim_{x\to 0^+} \frac{\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}\, e^{\frac{1}{x}}}$$

$$= \lim_{x\to 0^+} \frac{1}{e^{\frac{1}{x}}} = 0$$

$\underline{\underline{\text{Rm}}}$ $\displaystyle\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^k}$, $k>0$

$= 0$ (homework)