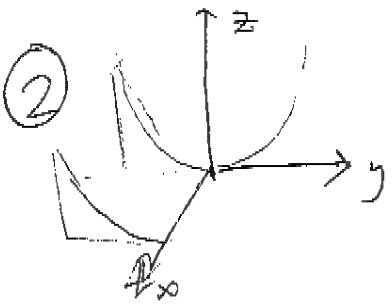
① $\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$

$\Rightarrow \begin{cases} y = -2\lambda_1 + 2x\lambda_2 \\ x = \lambda_1 + 2y\lambda_2 \\ 2z = 2z\lambda_2 \\ y - 2x = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases} \Rightarrow$ or $\lambda_2 = 1 \Rightarrow \begin{cases} (x, y, z, \lambda_1, \lambda_2) = (0, 0, 0, 0, \pm 1) \\ \text{or} \quad z = 0 \Rightarrow \left(\pm\frac{1}{\sqrt{5}}, \pm\frac{2}{\sqrt{5}}, 0, \frac{\mp 3}{5\sqrt{5}}, \frac{2}{5}\right) \end{cases}$

Extreme values $= f(0, 0, \pm 1) = 1$

and $f\left(\pm\frac{1}{\sqrt{5}}, \pm\frac{2}{\sqrt{5}}, 0\right) = \frac{2}{5}$

② 

$$\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dx\, dz$$

$$\int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\, dy\, dz$$

③ $u = xy \qquad v = y/x \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}$

$x = \left(\frac{u}{v}\right)^{\frac{1}{2}} \qquad y = (uv)^{\frac{1}{2}}$

$$\iint_R \sqrt{\frac{y}{x}} + \sqrt{xy}\, dx\, dy = \int_{\frac{1}{4}}^{1} \int_1^4 (\sqrt{v} + \sqrt{u}) \cdot \frac{1}{2v}\, du\, dv$$

$$= \int_{\frac{1}{4}}^{1} \int_1^4 \frac{1}{2} v^{-\frac{1}{2}}\, du\, dv + \left(\int_1^4 \sqrt{u}\, du\right)\left(\int_{\frac{1}{4}}^{1} \frac{1}{2v}\, dv\right) = \frac{3}{2} + \frac{7\ln 4}{3}$$

④ $\iint_{\substack{y=x^2 \\ 0\le x\le 2 \\ 0\le z\le 3}} x\,d\sigma = \int_{z=0}^{3}\int_{x=0}^{2} \frac{x}{\cos\theta}\,dx\,dz = \int_0^3\int_0^2 x\sqrt{1+4x^2}\,dx\,dz = \frac{1}{4}\left(17^{\frac{3}{2}}-1\right)$

$\cos\theta = \vec{n}\cdot\hat{j} = \frac{1}{\sqrt{1+4x^2}}$

⑤ $x = 2\cos t$, $y = 5\sin t$, $0 \le t \le \pi$

$$Ans = \int_C -y\,dx + 3x\,dy = \int_0^{\pi} (y\cdot\dot{x} + 3x\cdot\dot{y})\,dt = 4\pi$$

⑥ Stokes Thm, $C = \{x^2+y^2=4,\ z=0\}$ $\quad \nabla\times F = (0, 1-y, z)$, $\nabla\cdot F = 1$

$$\oint_C F\cdot T\,ds = \iint_S \nabla\times F\cdot\vec{n}\,d\sigma$$

Divergence Thm: $S_1 = \{x^2+y^2+z^2=1,\ z\ge 0\}$

$S_2 = \{x^2+y^2\le 1,\ z=0\}$

$$\iiint_D \nabla\cdot F\,dv = \iint_{\partial D} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_1} F\cdot\vec{n}\,d\sigma + \iint_{S_2} F\cdot n\,d\sigma$$

Integrals evaluation:

For Stoke's Thm:

$x = 2\cos t$, $y = 2\sin t$, $z = 0$ $\quad 0\le t\le 2\pi$

$$\oint_C F\cdot T\,ds = \int_0^{2\pi} x\cdot\frac{dx}{dt}\,dt = \frac{1}{2}\int_0^{2\pi}\frac{d(x^2)}{dt}\,dt = 0$$

$$\iint_S \nabla \times F \cdot \vec{n}\, d\sigma = \iint \frac{(y(1-y)+z^2)}{z}\, dA$$

$\nabla \times F = (0, 1-y, z)$

$\vec{n} = \frac{1}{2}(x, y, z)$

$\cos\theta = \frac{z}{2}$ on S

$$= \int_0^{2\pi}\int_0^2 \left(\frac{r\sin\theta}{\sqrt{4-r^2}} + - \frac{r^2\sin^2\theta}{\sqrt{4-r^2}} + \frac{4-r^2}{\sqrt{4-r^2}} \right) r\, dr\, d\theta$$

$$= \text{I} + \text{II} + \text{III}$$

$$\text{I} = \left(\int_0^2 \frac{r}{\sqrt{4-r^2}} r\, dr \right) \cdot \int_0^{2\pi} \sin\theta\, d\theta = 0$$

$$\text{II} = \left(\int_0^2 \frac{-r^2}{\sqrt{4-r^2}} r\, dr \right) \cdot \left(\int_0^{2\pi} \sin^2\theta\, d\theta \right) = -\pi \int_0^2 \frac{r^2}{\sqrt{4-r^2}} r\, dr$$

$$\text{III} = 2\pi \int_0^2 \frac{4-r^2}{\sqrt{4-r^2}} r\, dr$$

$$\text{II} + \text{III} = \pi \int_0^2 \frac{8-3r^2}{\sqrt{4-r^2}} r\, dr$$

Let $4-r^2 = s^2$

$-2r\, dr = 2s\, ds$

$$= \pi \int_{s=2}^{s=0} \frac{-4+3s^2}{s} (-s)\, ds$$

$$= \pi \int_0^2 (4-3s^2)\, ds = \pi(8-8) = 0$$

$\therefore$ Stokes Theorem verified

For divergence Theorem, $\nabla \cdot F = 1$

$$\iiint_D \nabla \cdot F \, dV = \frac{1}{2}(\text{Volume of unit ball}) = \frac{2\pi}{3}$$

$$\iint_{S_2} F \cdot n \, d\sigma = \iint_{x^2+y^2 \le 1} F \cdot (0,0,-1) \, dA = -\iint_{x^2+y^2 \le 1} xy \, dx dy = 0$$

by symmetry

$$\iint_{S_1} F \cdot \vec{n} \, d\sigma = \iint_{x^2+y^2 \le 1} \frac{(x^2 + xz + 2xyz)}{z} \, dx dy$$

$\vec{n} = (x, y, z)$

$\cos\theta = z$

$$= \int_0^{2\pi} \int_0^1 \frac{r^2 \cos^2\theta}{\sqrt{1-r^2}} \, r \, dr \, d\theta + \iint_{x^2+y^2 \le 1} (x + 2xy) \, dx dy$$

$$= I + II.$$

$II = 0$ by symmetry

$$I = \int_0^{2\pi} \cos^2\theta \, d\theta \cdot \int_0^1 \frac{r^2}{\sqrt{1-r^2}} \, r \, dr \qquad (1-r^2 = s^2)$$

$$= \pi \cdot \int_{s=1}^{s=0} \frac{1-s^2}{s} (-s) \, ds$$

$$= \pi \int_0^1 (1-s^2) \, ds = \frac{2\pi}{3}$$

$\therefore$ divergence Thm verified