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# CONCENTRATION PHENOMENA IN A NONLOCAL QUASI-LINEAR PROBLEM MODELLING PHYTOPLANKTON II: LIMITING PROFILE*

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**Abstract.** This is Part II of our study on the positive steady state of a quasi-linear reaction-diffusion system in one space dimension introduced by Klausmeier and Litchman for the modelling of the distributions of phytoplankton biomass and its nutrient. In Part I, we proved nearly optimal existence and nonexistence results. In Part II, we obtain complete descriptions of the profile of the solutions when the coefficient of the drifting term is large, rigorously proving the numerically observed phenomenon of concentration of biomass for this model. Moreover, we reveal four critical numbers for the model and provide further insights to the problem being modelled.

**Key words.** quasi-linear, nonlocal dependence, phytoplankton, concentration phenomenon, reaction-diffusion equation

**AMS subject classifications.** 35J55, 35J65, 92D25

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## 1. Introduction.

We continue our investigation in [DH] on the problem

$$
\begin{cases}
-[d_1 u_x + \sigma c(x) u]_x = [g(x) - m]u, & 0 < x < 1,\\
-d_2 v_{xx} = -g(x) u, & 0 < x < 1,\\
d_1 u_x + \sigma c(x) u = 0, & x = 0,\ 1,\\
v_x(0) = 0,\ v_x(1) = \beta[v_0 - v(1)], &
\end{cases} \tag{1.1}
$$

where $d_1$, $d_2$, $\sigma$, $m$, $v_0$, and $\beta$ are positive constants,

$$
g(x) = f(\min\{\alpha v(x), w(x)\}), \qquad f(s) = \frac{rs}{K_I + s},
$$

and

$$
w(x) = w_0 \exp\left[-A_0 x - A \int_0^x u(s)ds\right],
$$

with $\alpha$, $r$, $K_I$, $w_0$, $A$, and $A_0$ positive constants. We are interested in positive solutions of (1.1), namely, $u > 0$ and $v > 0$ in $[0, 1]$. From (1.1) it is easy to see that for any such solution $v$ is an increasing function. Clearly $w$ is a decreasing function. The function $c(x)$ is defined by

$$
c(x) = \frac{x - x_0}{\delta + |x - x_0|},
$$

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where $\delta > 0$ is a small constant and $x_0 \in [0,1]$ is uniquely determined by the following description:

$$\min\{\alpha v(x), w(x)\} = \alpha v(x) \quad \forall x \in [0, x_0); \quad \min\{\alpha v(x), w(x)\} = w(x) \quad \forall x \in (x_0, 1].$$

(Due to the monotonicity of $v(x)$ and $w(x)$, such $x_0$ always exists.)

Problem (1.1) is a rescaled version of a model proposed by Klausmeier and Litchman in [KL] for the study of phytoplankton in a one-dimensional water column, where $u(x)$ represents the distribution of phytoplankton biomass, $v(x)$ stands for the distribution of nutrient, and $x$ denotes the depth in the water column, with $x = 0$ at the surface and $x = 1$ at the bottom. The term $\sigma c(x)$ is used to describe the active movement of the biomass towards spatial location with optimal growth condition. Klausmeier and Litchman [KL] use this system to model the concentration phenomenon of phytoplankton in lakes and oceans, and the numerical analysis in [KL] demonstrates that, for large $\sigma$, the biomass function $u(x)$ concentrates at a certain level $x = x_*$, while the nutrient function $v(x)$ is close to a piecewise linear function. They then treat $u$ as a constant multiple of the $\delta$-function concentrating at $x_*$ and propose a game theoretical model to determine the location of $x_*$. We refer the reader to Part I [DH] for further details regarding the background of (1.1).

Here we rigorously prove the existence of such a concentration phenomenon and obtain exact formulas for the determination of $x_*$ and the total biomass. In doing so, we reveal the existence of four critical values $v_{**} < v_* < v^* < v^{**}$ for $v_0$ (the nutrient level at the sediment) such that

(i) $x_* = 0$ when $v_0 \geq v^*$, $x_* \in (0,1)$ when $v_0 \in (v_*, v^*)$, and $x_* = 1$ when $v_0 \leq v_*$;

(ii) the total biomass increases with $v_0$ in the range $v_{**} < v_0 < v^{**}$, but stays constant for $v_0 \geq v^{**}$ or $v_0 \leq v_{**}$ (but with $v_0$ above a certain level so that the biomass can survive).

In order to give a more detailed description of these results, we first recall the main results of Part I [DH], where we proved the following two theorems.

Theorem 1.1. *There exist* $0 < m_* \leq m^* < \infty$ *such that* (1.1) *has a positive solution for* $m \in (0, m_*)$ *and has no positive solution for* $m > m^*$.

The values of $m_*$ and $m^*$ depend on the parameters in (1.1). To stress their dependence on $\sigma$, we write $m_* = m_*(\sigma)$, $m^* = m^*(\sigma)$.

Theorem 1.2.

$$\lim_{\sigma\to\infty} m_*(\sigma) = \lim_{\sigma\to\infty} m^*(\sigma) = f(\min\{\alpha v_0, w_0\}).$$

To investigate the limiting profile of the positive solutions of (1.1) as $\sigma \to \infty$, we will fix $m$ such that $0 < m < f(\min\{\alpha v_0, w_0\})$ and let $\sigma_n$ be an increasing sequence of positive numbers converging to $\infty$. By Theorems 1.1 and 1.2, for all large $n$, (1.1) with $\sigma = \sigma_n$ has at least one positive solution. Suppose that $(u_n, v_n)$ is such a solution. We will analyze the behavior of $(u_n, v_n)$ as $n \to \infty$. This will be done in the following two sections.

In section 2, we find all the possible limiting profiles that a subsequence of $\{(u_n, v_n)\}$ can have; in particular, we find the limiting equations governing these possible limiting profiles. More precisely, let $x_n \in [0,1]$ be uniquely determined by

$$\begin{cases} \min\{\alpha v_n(x), w_n(x)\} = \alpha v_n(x) & \text{for } x \in [0, x_n), \\ \min\{\alpha v_n(x), w_n(x)\} = w_n(x) & \text{for } x \in (x_n, 1], \end{cases}$$

where

$$w_n(x) = w_0 \exp\left[-A_0 x - A\int_0^x u_n(s)ds\right].$$

We consider the following possibilities:

$$\text{(i) } x_n \to x_* \in (0,1), \qquad \text{(ii) } x_n \to 0, \qquad \text{(iii) } x_n \to 1.$$

The cases (ii) and (iii) are each further divided into two subcases, namely, for case (ii),

$$\text{(a1) } \sigma_n^{1/2} x_n \to \infty, \qquad \text{(a2) } \sigma_n^{1/2} x_n \to a_* \in [0,\infty);$$

for case (iii),

$$\text{(b1) } \sigma_n^{1/2}(1-x_n) \to \infty, \qquad \text{(b2) } \sigma_n^{1/2}(1-x_n) \to b_* \in [0,\infty).$$

One easily sees that, subject to a subsequence, the above are all the possible behaviors of the sequence $\{x_n\}$. Eventually we will show in section 3 that the limit of the entire sequence $\{x_n\}$ always exists and that this limit is completely determined by the value of $v_0$, which in turn allows us to completely determine the profiles of $u_n$ and $v_n$ for large $n$. But in order to prove these facts, we need to first find all the possible limiting profiles of $\{(u_n, v_n)\}$ and the limiting equations that govern these profiles for each of the above listed cases. The main results of section 2 are summarized below.

If case (i) occurs, we show (see Lemma 2.3) that as $n \to \infty$, subject to a subsequence, $u_n \to 0$ uniformly in $[0, x_* - \epsilon] \cup [x_* + \epsilon, 1]$ for any small $\epsilon > 0$ and

$$\int_0^1 u_n(x)dx \to \tau_* C_0, \qquad C_0 := \int_{-\infty}^{\infty} e^{-x^2/(2\delta d_1)}dx = \sqrt{2\delta d_1 \pi},$$

$$v_n \to v_0 - \frac{\tau_*}{d_2} m C_0 (1 + \beta^{-1} - \max\{x, x_*\})$$

uniformly in $[0,1]$, where $x_* \in (0,1)$ and $\tau_* > 0$ are determined by

$$\text{(1.2)} \qquad \begin{cases} w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} = \alpha\left[v_0 - \dfrac{\tau_*}{d_2} m C_0 (1 + \beta^{-1} - x_*)\right], \\ m = \displaystyle\int_0^1 f\big(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, C_0 y\}}\big)dy. \end{cases}$$

If case (ii)(a1) occurs, we show (see Lemma 2.4) that the above conclusions hold with $x_* = 0$; in particular,

$$\text{(1.3)} \qquad w_0 e^{-A\tau_*(C_0/2)} = \alpha\left[v_0 - \frac{\tau_*}{d_2} m C_0 (1+\beta^{-1})\right],$$

and

$$\text{(1.4)} \qquad m = \int_0^1 f\big(w_0 e^{-A\tau_* \max\{C_0/2, C_0 y\}}\big)dy.$$

Since (1.4) uniquely determines $\tau_* > 0$, we can substitute this $\tau_*$ into (1.3) to obtain a special value for $v_0$, say $v_0 = v_*$.

Similarly, if case (iii)(b1) occurs, we can show (Lemma 2.5) that the conclusions of case (i) hold except that $x_* = 1$; in particular,

$$w_0 e^{-A_0 - A\tau_*(C_0/2)} = \alpha\left[v_0 - \frac{\tau_*}{d_2} mC_0(1+\beta^{-1}-1)\right], \tag{1.5}$$

and

$$m = \int_0^1 f\big(w_0 e^{-A_0 - A\tau_* \max\{C_0/2, C_0 y\}}\big)dy. \tag{1.6}$$

Analogously, $\tau_* > 0$ is uniquely determined by (1.6), and one can then use (1.5) to obtain a special value for $v_0$, say $v_0 = v^*$.

If case (ii)(a2) occurs, we show (Lemma 2.4) that as $n \to \infty$, subject to a subsequence, $u_n \to 0$ uniformly in $[\epsilon, 1]$ for any small $\epsilon > 0$,

$$\int_0^1 u_n(x)dx \to \tau_* C(a_*), \qquad C(a_*) = \int_{-a_*}^{\infty} e^{-x^2/(2\delta d_1)}dx,$$

$$v_n \to v_0 - \frac{\tau_*}{d_2} mC(a_*)(1+\beta^{-1}-x)$$

uniformly in $[0,1]$, where $a_* \in [0,\infty)$ and $\tau_* > 0$ are determined by

$$m = \int_0^1 f\big(w_0 e^{-A\tau_* \max\{C(a_*)-C_0/2, C(a_*)y\}}\big)dy, \tag{1.7}$$

and

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2} mC(a_*)(1+\beta^{-1})\Big) = w_0 e^{-A\tau_*[C(a_*)-C_0/2]} \quad \text{if } a_* > 0, \tag{1.8}$$

$$\alpha\left(v_0 - \frac{\tau_*}{d_2} m\left(\frac{C_0}{2}\right)(1+\beta^{-1})\right) \geq w_0 \quad \text{if } a_* = 0. \tag{1.9}$$

If case (iii)(b2) occurs, we show (Lemma 2.5) that as $n \to \infty$, subject to a subsequence, $u_n \to 0$ uniformly in $[0, 1-\epsilon]$ for any small $\epsilon > 0$,

$$\int_0^1 u_n(x)dx \to \tau_* \tilde{C}(b_*), \qquad \tilde{C}(b_*) = \int_{-\infty}^{b_*} e^{-x^2/(2\delta d_1)}dx = C(-b_*),$$

and

$$v_n \to v_0 - \frac{\tau_*}{d_2}\beta^{-1}\tilde{C}(b_*),$$

where $b_* \in [0,\infty)$ and $\tau_* > 0$ are determined by

$$m = \int_0^1 f\big(w_0 e^{-A_0 - A\tau_* \max\{C_0/2, \tilde{C}(b_*)y\}}\big)dy \tag{1.10}$$

and

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2\beta} C(b_*)\Big) = w_0 e^{-A_0 - A\tau_* C_0/2} \quad \text{if } b_* > 0, \tag{1.11}$$

$$\alpha\left(v_0 - \frac{\tau_*}{d_2\beta}\left(\frac{C_0}{2}\right)\right) \leq w_0 e^{-A_0 - A\tau_* C_0/2} \quad \text{if } b_* = 0. \tag{1.12}$$

In section 3, through careful analysis of the limiting equations (1.2)–(1.12), we show that the entire sequence $\{x_n\}$ always converges to a point $x_* \in [0,1]$, that exactly one of the cases considered in section 2 occurs, and that in each case the limit of the entire sequence in the conclusion exists. More precisely, if $v_* < v_0 < v^*$ (recall that $v_*$ and $v^*$ are defined above in cases (ii)(a1) and (iii)(b1), respectively), then case (i) must occur, and (1.2) uniquely determines $x_*$ and $\tau_*$. If $v_0 = v_*$, then case (ii)(a1) occurs; if $v_0 = v^*$, then case (iii)(b1) occurs. If $v_0 > v^*$, then case (ii)(a2) must happen, and if $v_0 < v_*$, then case (iii)(b2) must happen. Moreover, our analysis on the limiting total biomass $\lim_{n\to\infty}\int_0^1 u_n(x)dx$ reveals two further critical values of $v_0$, $v_{**} < v_*$ and $v^{**} > v^*$ such that this limiting total biomass is strictly increasing with $v_0$ for $v_0$ in the range $v_{**} \le v_0 \le v^{**}$ but remains constant (i.e., no longer changes with $v_0$) when $v_0 \ge v^{**}$ or when $v_0 \le v_{**}$. See Theorems 3.1–3.3 for more accurate descriptions of these results.

In section 4 we give biological interpretations of our main results and compare our rigorous limiting equations with the game theoretical model of [KL].

Though the proofs are rather involved, they consist mainly of elementary mathematical analysis; most of the proofs in section 2 and all of the arguments in section 3 can be understood with sound knowledge of calculus and real analysis.

**2. The limiting equations.** We will keep using the notation of Part I [DH]. It turns out that the techniques used in the proof of Theorem 3.1 in Part I are not quite suitable for our purpose here. We will introduce some different techniques.

Suppose that $0 < m < f(\min\{\alpha v_0, w_0\})$ and $\sigma_n$, $(u_n, v_n)$ are as given in the introduction above. Suppose $c_{v_n,w_n}(x) = C_{x_n}(x)$, $x_n \in [0,1]$. By passing to a subsequence we may assume that $x_n \to x_* \in [0,1]$. Then

$$C_{x_n} = \frac{x - x_n}{\delta + |x - x_n|} \to C_{x_*}$$

in $C^1([0,1])$.

In order to obtain useful equations to determine the profiles of $u_n$ and $v_n$, we need to stretch the variable $x$ appropriately. We define

$$\Phi_n(x) = \exp\left[-\frac{\sigma_n}{2d_1}\int_{x_n}^{x} C_{x_n}(s)ds\right]$$

and

$$\Psi_n(x) = u_n(x)/\Phi_n(x).$$

By a direct computation we obtain

$$\begin{cases} -d_1\Psi_n'' + \sigma_n\Gamma_n(x)\Psi_n = \big[f(\min\{\alpha v_n, w_n\}) - m\big]\Psi_n, & x \in (0,1), \\ d_1\Psi_n' + (\sigma_n/2)C_{x_n}\Psi_n = 0, & x = 0,1, \end{cases}$$

where

$$\Gamma_n(x) := \frac{\sigma_n(x-x_n)^2 - 2d_1\delta}{4d_1(\delta + |x - x_n|)^2}.$$

Let us introduce the stretched variable $y = \sigma^{1/2}(x - x_n)$ and define

$$V_n(y) := \Psi_n\big(\sigma_n^{-1/2}y + x_n\big), \qquad C_n(y) := \sigma_n^{1/2}C_{x_n}\big(\sigma_n^{-1/2}y + x_n\big) = \frac{y}{\delta + \sigma_n^{-1/2}|y|},$$

$$a_n := -\sigma_n^{1/2}x_n, \qquad b_n := \sigma_n^{1/2}(1 - x_n),$$

and

$$F_n(y) := f\big(\min\{\alpha v_n(\sigma_n^{-1/2}y + x_n), w_n(\sigma_n^{-1/2}y + x_n)\}\big).$$

Then

$$(2.1)\quad \begin{cases} -d_1 V_n'' + \dfrac{y^2 - 2d_1\delta}{4d_1(\delta + \sigma_n^{-1/2}|y|)^2} V_n = \sigma_n^{-1}\big[F_n(y) - m\big]V_n, & y \in (a_n, b_n), \\ d_1 V_n' + (1/2)C_n V_n = 0, & y = a_n, b_n. \end{cases}$$

In the discussions below, we will consider the cases $x_* \in (0,1)$, $x_* = 0$, and $x_* = 1$ separately.

LEMMA 2.1. *Suppose $x_n \to x_* \in (0,1)$, and set $\tilde{V}_n(y) = V_n(y)/\|V_n\|_{L^\infty([a_n,b_n])}$. Then*

$$\tilde{V}_n \to V_0 \quad \textit{in } C^1(J) \textit{ for any finite interval } J \subset (-\infty, \infty),$$

*where $V_0(y) = \exp[-\frac{y^2}{4d_1\delta}]$ is the unique solution of*

$$-d_1 V'' = \frac{2d_1\delta - y^2}{4d_1\delta^2} V, \quad 0 < V \le 1,\ V(0) = 1,\ V'(0) = 0.$$

*Proof.* Since $x_* \in (0,1)$, we have $a_n \to -\infty$ and $b_n \to \infty$ as $n \to \infty$. Let us note that, for $y \in [a_n, -(2d_1\delta)^{1/2} - \epsilon]$ with $\epsilon > 0$ sufficiently small and all large $n$, the first equation in (2.1) implies that $V_n''(y) > 0$. Since $d_1 V_n'(a_n) = -(1/2)C_n(a_n)V_n(a_n) \ge 0$, we deduce that $V_n'(y) > 0$ in $(a_n, -(2d_1\delta)^{1/2} - \epsilon]$ for all large $n$. Hence $V_n$ is increasing in this range. Similarly, we can see that $V_n(y)$ is decreasing in the range $y \in [(2d_1\delta)^{1/2} + \epsilon, b_n]$ for all large $n$. Therefore $\max V_n = V_n(y_n)$ for some $y_n \in [-(2d_1\delta)^{1/2} - \epsilon, (2d_1\delta)^{1/2} + \epsilon]$, and $\tilde{V}_n(y) = V_n(y)/V_n(y_n)$. We may assume that $y_n \to y^*$ as $n \to \infty$. We now define

$$\tilde{F}_n(y) := \frac{2d_1\delta - y^2}{4d_1(\delta + \sigma_n^{-1/2}|y|)^2} + \sigma_n^{-1}\big[F_n(y) - m\big].$$

Then $\tilde{V}_n(y_n) = 1$, and

$$(2.2)\quad \begin{cases} -d_1 \tilde{V}_n'' = \tilde{F}_n \tilde{V}_n, \quad 0 < \tilde{V}_n \le 1, & y \in (a_n, b_n), \\ d_1 \tilde{V}_n' + (1/2)C_n \tilde{V}_n = 0, & y = a_n,\ b_n. \end{cases}$$

Since $\{\tilde{F}_n\}$ is uniformly bounded over any bounded interval and $0 \le \tilde{V}_n \le 1$, we may apply the interior $L^p$ theory (see [GT]) to (2.2) and use the Sobolev imbedding theorem and a standard diagonal argument to conclude that, by passing to a subsequence, $\tilde{V}_n \to \tilde{V}$ in $C^1(J)$ for any bounded interval $J$, and $\tilde{V}$ satisfies

$$(2.3)\quad -d_1 \tilde{V}'' = \frac{2d_1\delta - y^2}{4d_1\delta^2}\tilde{V}, \quad 0 < \tilde{V} \le 1 \text{ in } (-\infty, \infty),\ \tilde{V}(y^*) = 1,\ \tilde{V}'(y^*) = 0.$$

By the monotonicity property of $V_n(y)$ observed earlier, we know that $\tilde{V}(y)$ is nondecreasing in $(-\infty, -(2d_1\delta)^{1/2})$ and is nonincreasing in $((2d_1\delta)^{1/2}, \infty)$. We can now use (2.3) to conclude that $\tilde{V}'(y)$ is positive and increasing in $(-\infty, -(2d_1\delta)^{1/2})$, reaching a positive maximum at $y = -(2d_1\delta)^{1/2}$; then is decreasing in $(-(2d_1\delta)^{1/2},$

$(2d_1\delta)^{1/2})$, reaching a negative minimum at $y = (2d_1\delta)^{1/2}$; and for $y > (2d_1\delta)^{1/2}$, is increasing and stays negative. Therefore $V'(y)$ has a unique zero at some $y_0 \in \big(-(2d_1\delta)^{1/2}, (2d_1\delta)^{1/2}\big)$, which is the unique maximum point of $\tilde{V}$. Thus $y_0 = y^*$. In other words, $\tilde{V}(y)$ is increasing in $(-\infty, y^*)$ and is decreasing in $(y^*, 0)$. It then follows from an elementary analysis that $\tilde{V}$ decays to 0 as $|y| \to \infty$, and there exists $C_1, C_2 > 0$ such that

$$\tilde{V}(y),\, |\tilde{V}'(y)| \le C_1 e^{-C_2|y|} \quad \forall y \in (-\infty, \infty).$$

We now multiply $\tilde{V}(-y)$ to (2.3), integrate over $[y^*, \infty)$, and then apply integration by parts. Since $\tilde{V}(-y)$ satisfies the differential equation in (2.3), we deduce

$$\tilde{V}'(-y^*)\tilde{V}(y^*) + \tilde{V}'(y^*)\tilde{V}(-y^*) = 0.$$

It follows that $\tilde{V}'(-y^*) = 0$. Since $y^*$ is the only zero of $\tilde{V}'$, we must have $y^* = -y^*$, that is, $y^* = 0$. By the uniqueness theorem of initial value problems of ordinary differential equations, we must have $\tilde{V} = V_0$, the unique solution of (2.3) with $y^* = 0$. A simple calculation confirms that the function $\exp[-\frac{y^2}{4d_1\delta}]$ solves the equation for $V_0$. Hence, by uniqueness,

$$V_0(y) = \exp\left[-\frac{y^2}{4d_1\delta}\right].$$

Since $V_0$ is uniquely determined, it follows that the entire original sequence $\{\tilde{V}_n\}$ converges to $V_0$. $\square$

Using the monotonicity of $\tilde{V}_n$ and the fact that $V_0(y) \to 0$ as $|y| \to \infty$, we see that Lemma 2.1 implies

$$\big\|\Psi_n(\cdot)/\|\Psi_n\|_\infty - V_0\big(\sigma_n^{1/2}(\cdot - x_n)\big)\big\|_{L^\infty([0,1])} \to 0 \quad \text{as } n \to \infty, \tag{2.4}$$

since for large $n$ the function

$$\Psi_n(x)/\|\Psi_n\|_\infty - V_0\big(\sigma_n^{1/2}(x - x_n)\big)$$

is uniformly small at those values of $x \in [0,1]$ such that $\sigma_n^{1/2}(x - x_n)$ stays bounded (by Lemma 2.1), and, by the properties of $\tilde{V}_n$ and $V_0$, the values of the function at the remaining $x \in [0,1]$ are also small.

We now denote $\tilde{\Psi}_n(x) = \Psi_n(x)/\|\Psi_n\|_\infty$ and consider the function

$$\tilde{u}_n(x) := \sigma_n^{1/2}\Phi_n(x)\tilde{\Psi}_n(x) = \left(\frac{\sigma_n^{1/2}}{\|\Psi_n\|_\infty}\right)u_n.$$

We will show that, for large $n$, $\tilde{u}_n$ behaves like the $\delta$-function concentrating at $x_*$. Indeed, we have the following result.

Lemma 2.2. *For any given small $\epsilon > 0$, $|x - x_n| \ge \epsilon$ implies*

$$0 < \tilde{u}_n(x) \le \sigma_n^{1/2} \exp\left[-\frac{\sigma_n}{4(\delta+1)d_1}\epsilon^2\right] \to 0. \tag{2.5}$$

*Moreover, when $x_n \to x_* \in (0,1)$,*

$$\lim_{n\to\infty} \int_0^1 \tilde{u}_n(x)dx = C_0 := \int_{-\infty}^{\infty} e^{-x^2/(2\delta d_1)}dx = \sqrt{2\delta d_1 \pi}. \tag{2.6}$$

*Proof.* For any given small $\epsilon > 0$, there exists $\delta_0 = \delta_0(\epsilon) > 0$ small so that, when $|x - x_n| \le \delta_0$,

$$\exp\Big[-\frac{\sigma_n}{4\delta d_1}(x-x_n)^2\Big] \le \Phi_n(x) \le \exp\Big[-\frac{\sigma_n(1-\epsilon)}{4\delta d_1}(x-x_n)^2\Big].$$

For any $x \in [0,1]$, we have

$$\exp\Big[-\frac{\sigma_n}{4\delta d_1}(x-x_n)^2\Big] \le \Phi_n(x) \le \exp\Big[-\frac{\sigma_n}{4(\delta+1) d_1}(x-x_n)^2\Big].$$

Since $\tilde\Psi_n \le 1$, for $|x - x_n| \ge \epsilon$, we have

$$\tilde u_n(x) \le \sigma_n^{1/2} \exp\Big[-\frac{\sigma_n}{4(\delta+1)d_1}\epsilon^2\Big] \to 0.$$

This proves (2.5). Moreover, we have

$$\begin{aligned}\int_0^1 \tilde u_n(x)dx &= \int_{x_n-\epsilon}^{x_n+\epsilon} \sigma_n^{1/2}\Phi_n(x)\tilde\Psi_n(x)dx + o(1)\\ &= \int_{-\epsilon\sigma_n^{1/2}}^{\epsilon\sigma_n^{1/2}} \Phi_n(x_n + \sigma_n^{-1/2}y)\tilde V_n(y)dy + o(1)\\ &= \int_{-\infty}^{\infty} \exp\Big[-\frac{y^2}{4d_1\delta}\Big]V_0(y)dy + o(1)\\ &= \int_{-\infty}^{\infty} \exp\Big[-\frac{y^2}{2d_1\delta}\Big]dy + o(1).\end{aligned}$$

Hence (2.6) holds. For later application, let us also note from the above argument that

$$\lim_{n\to\infty}\int_{x_n}^1 \tilde u_n(x)dx = \lim_{n\to\infty}\int_0^{x_n}\tilde u_n(x)dx = C_0/2. \tag{2.7}$$ ∎

Denote $\tau_n := \|\Psi_n\|_\infty \sigma_n^{-1/2}$. We find that

$$u_n(x) = \tau_n \tilde u_n(x).$$

LEMMA 2.3. *Suppose that $x_n \to x_* \in (0,1)$. Then $\{\tau_n\}$ has a subsequence, still denoted by itself, such that $\tau_n \to \tau_* > 0$. Moreover, $\tau_*$ and $x_*$ must satisfy*

$$w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} = \alpha\Big[v_0 - \frac{\tau_*}{d_2} m C_0(1+\beta^{-1} - x_*)\Big] \tag{2.8}$$

*and*

$$m = \int_0^1 f\big(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, C_0 y\}}\big)dy. \tag{2.9}$$

*Furthermore, by possibly passing to a further subsequence, $u_n \to 0$ in $C([0,1]\setminus[x_* - \epsilon, x_* + \epsilon])$, for all $\epsilon > 0$, and*

$$v_n(x) \to v_0 - \frac{\tau_*}{d_2} m C_0(1+\beta^{-1} - \max\{x, x_*\}) \tag{2.10}$$

*uniformly in* $[0,1]$.

*Proof.* By passing to a subsequence, we have two possible cases:

$$\text{(i)}\quad \tau_n \to \infty, \qquad \text{(ii)}\quad \tau_n \to \tau_* \in [0,\infty).$$

*Step* 1. Case (i) cannot happen.

Suppose $\tau_n \to \infty$; we are going to derive a contradiction. Denote

$$f_n = f(\min\{\alpha v_n, w_n\}).$$

Since

$$w_n(x_n) \le w_0 e^{-A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds},$$

and by (2.7)

$$\int_0^{x_n} \tilde{u}_n(s)ds \to C_0/2 > 0,$$

we easily see that $w_n(x_n) \to 0$. It follows that

$$\|f_n\|_\infty = f_n(x_n) = f(w_n(x_n)) \to 0.$$

This implies that

$$\int_0^1 f_n \tilde{u}_n dx \to 0.$$

On the other hand, we may integrate the equation for $u_n$ to obtain

$$\int_0^1 [f_n(x) - m] u_n dx = 0,$$

which implies that

$$\int_0^1 [f_n(x) - m] \tilde{u}_n dx = 0.$$

Letting $n \to \infty$ and using (2.6), we obtain

$$mC_0 = \lim_{n\to\infty} \int_0^1 f_n \tilde{u}_n dx = 0,$$

which contradicts our assumption that $m > 0$. Therefore case (i) cannot happen.

*Step* 2. The limiting profile of $u_n$ and $v_n$.

We next consider case (ii), namely, $\tau_n \to \tau_* \in [0,\infty)$. In this case, due to (2.5), $u_n = \tau_n \tilde{u}_n \to 0$ in $C([0,1] \setminus [x_* - \epsilon, x_* + \epsilon])$, for all $\epsilon > 0$, and hence

$$\tau_n f_n \tilde{u}_n \to 0 \quad \text{uniformly in } [0, x_* - \epsilon] \cup [x_* + \epsilon, 1] \quad \forall \epsilon > 0. \tag{2.11}$$

Let $\zeta_n = v_0 - v_n$. Then

$$-d_2 \zeta_n'' = \tau_n f_n \tilde{u}_n \text{ in } (0,1), \quad \zeta_n'(0) = 0, \quad \zeta_n'(1) + \beta \zeta_n(1) = 0. \tag{2.12}$$

Since $v_n \geq 0$, we have $\zeta_n \leq v_0$. Since $\tau_n f_n \tilde{u}_n > 0$, from (2.12) and the maximum principle, we deduce that $\zeta_n > 0$. Hence we always have $0 < \zeta_n \leq v_0$. Therefore we can integrate (2.12) to obtain

$$\eta_n := \tau_n \int_0^1 f_n \tilde{u}_n dx = d_2[\zeta_n'(0) - \zeta_n'(1)] = d_2\beta\zeta_n(1) \in [0, d_2\beta v_0].$$

This implies that, by passing to a subsequence, we may assume that $\eta_n \to \eta_* \in [0, d_2\beta v_0]$.

Moreover, using (2.11), (2.12), and $\eta_n \to \eta_*$, we find that

$$\begin{aligned}
&\{\zeta_n'\} \text{ is a bounded sequence in } L^\infty([0,1]),\\
&\zeta_n'(x) \to 0 \text{ uniformly in } [0, x_* - \epsilon] \quad \forall \epsilon > 0,\\
&\zeta_n'(x) \to -\eta_*/d_2 \text{ uniformly in } [x_* + \epsilon, 1] \quad \forall \epsilon > 0.
\end{aligned}$$

Since, moreover, $0 \leq \zeta_n \leq v_0$, we conclude that $\{\zeta_n\}$ is precompact in $C([0,1])$. Hence, by passing to a subsequence, we may assume that $\zeta_n \to \zeta$ in $C([0,1])$.

On the other hand, we may apply the $L^p$ theory to (2.12) and the Sobolev imbedding theorem to find a further subsequence, still denoted by $\zeta_n$, such that $\zeta_n \to \tilde{\zeta}$ in $C^1(J)$ for any compact interval $J \subset [0, x_*) \cup (x_*, 1]$, and $\tilde{\zeta}$ satisfies (in the weak sense)

$$-d_2\tilde{\zeta}'' = 0 \text{ in } [0, x_*) \cup (x_*, 1], \qquad \tilde{\zeta}'(0) = 0, \qquad \tilde{\zeta}'(1) + \beta\tilde{\zeta}(1) = 0.$$

Clearly we must have $\tilde{\zeta} = \zeta$. Moreover, our earlier analysis on $\zeta_n$ implies that $\zeta'(x) = 0$ in $[0, x_*)$ and $\zeta'(x) = -\eta_*/d_2$ in $(x_*, 1]$. These properties uniquely determine $\zeta$:

$$\zeta(x) = (\eta_*/d_2)(1 + \beta^{-1} - \max\{x_*, x\}). \tag{2.13}$$

*Step* 3. $\tau_* > 0$.

Otherwise, $\tau_* = 0$ and hence $\eta_* = 0$. It follows that $\zeta = 0$ and $v_n \to v_0$ uniformly in $[0,1]$, and that

$$w_n(x) = w_0 e^{-A_0 x} e^{-A\tau_n \int_0^x \tilde{u}_n(s)ds} \to w_0 e^{-A_0 x} = w_*(x)$$

uniformly in $[0,1]$. This implies that

$$x_* = x_0^* \quad \text{ and } \quad f_n(x) \to f_0(x) := f(\min\{\alpha v_0, w_*\}) \text{ uniformly in } [0,1].$$

We may now integrate the equation for $u_n$ to obtain, as before,

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0.$$

Letting $n \to \infty$, we deduce

$$[f_0(x_0^*) - m]C_0 = 0,$$

which contradicts our assumption that $m < f(\min\{\alpha v_0, w_0\}) = f_0(x_0^*)$. Hence $\tau_* > 0$.

*Step* 4. The equations for $x_*$ and $\tau_*$.

We now set out to find the equations that determine $x_*$ and $\tau_*$. By (2.7),

$$w_n(x_n) = w_0 e^{-A_0 x_n} e^{-A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds} \to w_0 e^{-A_0 x_*} e^{-A\tau_*(C_0/2)}.$$

On the other hand,

$$w_n(x_n) = \alpha v_n(x_n) \to \alpha\big[v_0 - \zeta(x_*)\big].$$

Thus we necessarily have

$$w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} = \alpha\big[v_0 - \zeta(x_*)\big] = \alpha\big[v_0 - (\eta_*/d_2)(1+\beta^{-1} - x_*)\big]. \tag{2.14}$$

Moreover, using (2.5), (2.7), and the fact that $\alpha v_n \to \alpha(v_0 - \zeta)$ uniformly in $[0,1]$, we deduce

$$\int_0^{x_n} f(\alpha v_n)\tilde{u}_n dx \to (C_0/2) f\big(\alpha v_0 - \alpha\zeta(x_*)\big). \tag{2.15}$$

Using

$$w_n(x) = w_0 e^{-A_0 x} e^{-A\tau_n \int_0^x \tilde{u}_n(s)ds}$$

and the property of $\tilde{u}_n$, we obtain, for any small $\epsilon > 0$,

$$\begin{aligned}
&\int_{x_n}^1 f(w_n)\tilde{u}_n dx \\
&\quad = \int_{x_n}^1 f\big(w_0 e^{-A_0 x - A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds - A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\big)\tilde{u}_n dx \\
&\quad = \int_{x_n}^{x_*+\epsilon} f\big(w_0 e^{-A_0 x - A\tau_n (C_0/2)} e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\big)\tilde{u}_n dx + o(1) \\
&\quad = [1 + o_\epsilon(1)] \int_{x_n}^{x_*+\epsilon} f\big(w_0 e^{-A_0 x_* - A\tau_* (C_0/2)} e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\big)\tilde{u}_n dx + o(1) \\
&\quad = [1 + o_\epsilon(1)] \int_0^{[\int_{x_n}^1 \tilde{u}_n(s)ds]} f\big(w_0 e^{-A_0 x_* - A\tau_* (C_0/2)} e^{-A\tau_n y}\big) dy + o(1) \\
&\quad = [1 + o_\epsilon(1)] \int_0^{C_0/2} f\big(w_0 e^{-A_0 x_* - A\tau_* (C_0/2)} e^{-A\tau_* y}\big) dy + o(1),
\end{aligned}$$

where $o_\epsilon(1)$ represents a quantity that converges to 0 as $\epsilon \to 0$.

Thus

$$\int_{x_n}^1 f(w_n)\tilde{u}_n(x) dx \to \int_0^{C_0/2} f\big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_* y}\big) dy \tag{2.16}$$

as $n \to \infty$.

Combining (2.15) and (2.16), we obtain

$$\begin{aligned}
\eta_* &= \lim_{n\to\infty} \tau_n \int_0^1 f_n \tilde{u}_n dx \\
&= \tau_* \Big[(C_0/2) f\big(\alpha v_0 - \alpha\zeta(x_*)\big) + \int_0^{C_0/2} f\big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_* y}\big) dy\Big].
\end{aligned} \tag{2.17}$$

Moreover, we may integrate the equation for $u_n$ to obtain

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0.$$

Letting $n \to \infty$ and using (2.15), (2.16), we obtain

$$mC_0 = (C_0/2) f(\alpha v_0 - \alpha\zeta(x_*)) + \int_0^{C_0/2} f\big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_* y}\big) dy.$$

This combined with (2.17) yields

$$\eta_* = \tau_* m C_0 \tag{2.18}$$

and combined with (2.14) gives

$$\begin{aligned} m &= (1/2) f(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)}) + C_0^{-1} \int_0^{C_0/2} f\big(w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} e^{-A\tau_* y}\big) dy \\ &= C_0^{-1} \int_0^{C_0} f\big(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, y\}}\big) dy \\ &= \int_0^1 f\big(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, C_0 y\}}\big) dy; \end{aligned}$$

thus (2.9) is proved. Equation (2.8) and (2.10) clearly follow from (2.13), (2.14), and (2.18). ∎

We now consider the case $x_* = 0$. By passing to a subsequence, we have two subcases:

$$\text{(a1) } a_n := \sigma_n^{1/2} x_n \to \infty, \qquad \text{(a2) } a_n \to a_* \in [0, \infty).$$

Lemma 2.4. *In subcase* (a1), *all of the conclusions in Lemmas* 2.2 *and* 2.3 *hold. In subcase* (a2), $\{\tau_n\}$ *has a subsequence, still denoted by itself, such that* $\tau_n \to \tau_* > 0$. *Moreover,* $\tau_*$ *and* $a_*$ *must satisfy*

$$m = \int_0^1 f\big(w_0 e^{-A\tau_* \max\{C(a_*) - C_0/2, C(a_*) y\}}\big) dy \tag{2.19}$$

*and*

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2} m C(a_*)(1 + \beta^{-1})\Big) = w_0 e^{-A\tau_*[C(a_*) - C_0/2]} \qquad \text{if } a_* > 0, \tag{2.20}$$

$$\alpha\left(v_0 - \frac{\tau_*}{d_2} m \left(\frac{C_0}{2}\right)(1 + \beta^{-1})\right) \geq w_0 \qquad \text{if } a_* = 0, \tag{2.21}$$

*where*

$$C(a_*) := \int_{-a_*}^{\infty} \exp\Big[-\frac{y^2}{2 d_1 \delta}\Big] dy.$$

*Furthermore, by possibly passing to a further subsequence,* $u_n \to 0$ *in* $C([\epsilon, 1])$, *for all* $\epsilon \in (0, 1)$,

$$\lim_{n\to\infty} \int_0^{x_n} \tilde{u}_n(x) dx = C(a_*) - C_0/2, \qquad \lim_{n\to\infty} \int_0^1 \tilde{u}_n(x) dx = C(a_*), \tag{2.22}$$

*and*

$$v_n(x) \to v_0 - \frac{\tau_*}{d_2} m C(a_*)(1 + \beta^{-1} - x) \tag{2.23}$$

*uniformly in* $[0,1]$.

*Proof.* In subcase (a1), we may repeat the arguments used for the case $x_* \in (0,1)$ above to see that all the conclusions there (with $x_*$ replaced by 0) remain valid; the proofs carry over with minor modifications.

Consider now subcase (a2). In this case, we may use interior and boundary $L^p$ estimates and the Sobolev imbedding theorem to conclude that, by passing to a subsequence, $\|\tilde{V}_n - \tilde{V}\|_{C^1([a_n,M])} \to 0$ for all $M > 0$, where $\tilde{V}$ satisfies, instead of (2.3),

$$\begin{cases} -d_1\tilde{V}'' = \dfrac{2d_1\delta - y^2}{4d_1\delta^2}\tilde{V}, \quad 0 < \tilde{V} \le 1 \text{ in } (-a_*, \infty), \\ d_1\tilde{V}'(-a_*) - \dfrac{a_*}{2\delta}\tilde{V}(-a_*) = 0, \quad \tilde{V}(y^*) = 1,\ \tilde{V}'(y^*) = 0. \end{cases} \tag{2.24}$$

Note that as before $\tilde{V}$ is decreasing in $[(2d_1\delta)^{1/2}, \infty)$. This and (2.24) imply that $\tilde{V}$ converges to 0 as $y \to \infty$. Moreover, an elementary consideration shows that

$$|\tilde{V}'(y)|, \tilde{V}(y) \le C_1 e^{-C_2 y}$$

for some $C_1$, $C_2 > 0$, and all $y > 0$.

We will show that $y^* = 0$ and $\tilde{V}$ is again the unique solution of (2.3) with $y^* = 0$, namely $V_0$. Since $V_0$ and $|V_0'|$ are bounded from above by a function of the form $C_1 e^{-C_2|y|}$, we can multiply the first equation in (2.24) by $V_0$, integrate over $[y^*, \infty)$, and use integration by parts to deduce

$$d_1[\tilde{V}V_0' - \tilde{V}'V_0]\big|_{y^*}^{\infty} = 0.$$

It follows that $V_0'(y^*) = 0$, which implies that $y^* = 0$. Therefore, by the uniqueness of initial value problems of the ordinary differential equations, we deduce $\tilde{V} \equiv V_0$. Let us note that a direct calculation shows

$$d_1 V_0'(y) + \frac{y}{2\delta} V_0(y) = 0 \quad \text{for every } y \in (-\infty, \infty).$$

Therefore (2.24) does not introduce any restriction for $a_*$.

Since now $\sigma_n^{1/2} x_n \to a_*$, instead of (2.6), we have

$$\lim_{n\to\infty} \int_0^{x_n} \tilde{u}_n(x)dx = C(a_*) - C_0/2, \qquad \lim_{n\to\infty} \int_0^1 \tilde{u}_n(x)dx = C(a_*), \tag{2.25}$$

where

$$C(a_*) := \int_{-a_*}^{\infty} \exp\left[-\frac{y^2}{4d_1\delta}\right] V_0(y)dy = \int_{-a_*}^{\infty} \exp\left[-\frac{y^2}{2d_1\delta}\right] dy.$$

We proceed as in the case $x_* \in (0,1)$ and have two possibilities for $\tau_n$ as before. We show that, in the current case, we still cannot have $\tau_n \to \infty$. Arguing indirectly, we assume that $\tau_n \to \infty$.

Then in the case $a_* > 0$, we have $C(a_*) - C_0/2 > 0$, and hence

$$w_n(x_n) \le w_0 e^{-A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds} \to 0.$$

It follows that

$$\|f_n\|_\infty = f_n(x_n) = f(w_n(x_n)) \to 0$$

and

$$\int_0^1 f_n \tilde{u}_n dx \to 0.$$

If $a_* = 0$, then $C(a_*) - C_0/2 = 0$ and

$$\begin{aligned}\int_0^1 f_n(x)\tilde{u}_n(x)dx &= \int_{x_n}^1 f(w_n(x))\tilde{u}_n(x)dx + o(1)\\ &\le \int_{x_n}^1 f\big(w_0 e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\big)\tilde{u}_n dx + o(1)\\ &= \int_0^{[\int_{x_n}^1 \tilde{u}_n(s)ds]} f\big(w_0 e^{-A\tau_n y}\big)dy + o(1)\\ &\le \epsilon f(w_0) + \int_\epsilon^{C_0/2} f\big(w_0 e^{-A\tau_n y}\big)dy + o(1)\\ &= \epsilon f(w_0) + o(1) \quad \forall \epsilon \in (0, C_0/2).\end{aligned}$$

Therefore we always have

$$\int_0^1 f_n \tilde{u}_n dx \to 0 \text{ as } n \to \infty.$$

As before, we may integrate the equation for $u_n$ to obtain

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0.$$

Letting $n \to \infty$ and using the above estimate, we deduce

$$-mC(a_*) = 0,$$

a contradiction to our assumption that $m > 0$. Therefore we cannot have $\tau_n \to \infty$.

Thus we can only have the case $\tau_n \to \tau_*$. Then much as before we deduce $u_n \to 0$ in $C([\epsilon, 1])$ for all $\epsilon \in (0, 1)$, and

$$\zeta_n \to \zeta := (\eta_*/d_2)(1 + \beta^{-1} - x)$$

in $C([0, 1]) \cap C^1([\epsilon, 1])$ for all $\epsilon \in (0, 1)$. If $\tau_* = 0$, we can deduce as before that $m = f_0(x_0^*)$, a contradiction to our initial assumption on $m$. Therefore $\tau_* > 0$.

If $a_* = 0$, we first choose $y_n \in (x_n, 1)$ such that $y_n \to 0$ and $\int_{y_n}^1 \tilde{u}_n(x)dx \to 0$, and then we have

$$\begin{aligned}\int_0^1 f_n(x)\tilde{u}_n(x)dx &= \int_{x_n}^{y_n} f(w_n(x))\tilde{u}_n(x)dx + o(1)\\ &= \int_{x_n}^{y_n} f\big(w_0 e^{-A_0 x - A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds - A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\big)\tilde{u}_n dx + o(1)\\ &= \int_{x_n}^{y_n} f\big(w_0 e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\big)\tilde{u}_n dx + o(1)\\ &= \int_{x_n}^1 f\big(w_0 e^{-A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\big)\tilde{u}_n dx + o(1)\\ &= \int_0^{C_0/2} f\big(w_0 e^{-A\tau_* y}\big)dy + o(1).\end{aligned}$$

If $a_* > 0$, then $x_n > 0$ and $w_n(x_n) = \alpha v_n(x_n)$. From

$$v_n(x_n) \to v_0 - \zeta(0)$$

and

$$w_n(x_n) = w_0 e^{-A_0 x_n - A\tau_n \int_0^{x_n} \tilde{u}_n dx} \to w_0 e^{-A\tau_*[C(a_*)-C_0/2]}$$

we obtain

$$\alpha[v_0 - \zeta(0)] = w_0 e^{-A\tau_*[C(a_*)-C_0/2]}.$$

Moreover, similar to the above,

$$\begin{aligned}\int_{x_n}^1 f_n(x)\tilde{u}_n(x) &= \int_{x_n}^{y_n} f(w_n(x))\tilde{u}_n(x)dx + o(1)\\ &= \int_{x_n}^{y_n} f\big(w_0 e^{-A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds - A\tau_n \int_{x_n}^{x} \tilde{u}_n(s)ds}\big)\tilde{u}_n dx + o(1)\\ &= \int_0^{C_0/2} f\big(w_0 e^{-A\tau_*[C(a_*)-C_0/2]-A\tau_* y}\big)dy + o(1),\end{aligned}$$

and

$$\begin{aligned}\int_0^{x_n} f_n(x)\tilde{u}_n(x)dx &= \int_0^{x_n} f(\alpha v_n(x))\tilde{u}_n(x)dx\\ &= f\big(\alpha[v_0 - \zeta(0)]\big)[C(a_*) - C_0/2] + o(1)\\ &= [C(a_*) - C_0/2] f\big(w_0 e^{-A\tau_*[C(a_*)-C_0/2]}\big) + o(1).\end{aligned}$$

Therefore we always have

$$\int_0^1 f_n \tilde{u}_n dx \to \int_0^{C(a_*)} f\big(w_0 e^{-A\tau_* \max\{[C(a_*)-C_0/2],y\}}\big)dy. \tag{2.26}$$

We may now use

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0$$

to obtain

$$mC(a_*) = \int_0^{C(a_*)} f\big(w_0 e^{-A\tau_* \max\{C(a_*)-C_0/2,y\}}\big)dy.$$

Therefore

$$m = \int_0^1 f\big(w_0 e^{-A\tau_* \max\{C(a_*)-C_0/2,C(a_*)y\}}\big)dy,$$

and (2.19) is proved.

We thus obtain

$$\eta_* = \tau_* \lim_{n\to\infty} \int_0^1 f_n \tilde{u}_n dx = \tau_* m C(a_*).$$

Therefore,

$$v_n(x) \to v_0 - \zeta = v_0 - \frac{\tau_*}{d_2} mC(a_*)(1 + \beta^{-1} - x)$$

uniformly in $[0, 1]$; that is, (2.23) holds.

Let us note that (2.22) was already proved in (2.25). So it remains to prove (2.20) and (2.21). If $a_* > 0$, then $x_n > 0$, and we necessarily have $\alpha v_n(x_n) = w_n(x_n)$. Recall that

$$w_n(x_n) \to w_0 e^{-A\tau_*[C(a_*)-C_0/2]}, \qquad v_n(x_n) \to v_0 - \zeta(0).$$

Hence

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2} mC(a_*)(1 + \beta^{-1})\Big) = w_0 e^{-A\tau_*[C(a_*)-C_0/2]}.$$

If $a_* = 0$, then $x_n = 0$ is possible, and so we have $\alpha v_n(x_n) \geq w_n(x_n)$ in general, and instead of the above identity we should have

$$\alpha \left( v_0 - \frac{\tau_*}{d_2} m \left( \frac{C_0}{2} \right) (1 + \beta^{-1}) \right) \geq w_0.$$

Thus (2.20) and (2.21) are established. The proof is now complete. ∎

Finally we consider the case $x_* = 1$. By passing to a subsequence, we have two subcases:

$$\text{(b1) } b_n := \sigma_n^{1/2}(1 - x_n) \to \infty, \qquad \text{(b2) } b_n \to b_* \in [0, \infty).$$

Lemma 2.5. *In subcase* (b1), *all of the conclusions in Lemmas* 2.2 *and* 2.3 *hold. In subcase* (b2), $\{\tau_n\}$ *has a subsequence, still denoted by itself, such that* $\tau_n \to \tau_* > 0$. *Moreover,* $\tau_*$ *and* $b_*$ *must satisfy*

$$m = \int_0^1 f\big(w_0 e^{-A_0 - A\tau_* \max\{C_0/2, \tilde{C}(b_*)y\}}\big) dy \tag{2.27}$$

*and*

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2\beta}\tilde{C}(b_*)\Big) = w_0 e^{-A_0 - A\tau_* C_0/2} \quad \textit{if } \ b_* > 0, \tag{2.28}$$

$$\alpha \left( v_0 - \frac{\tau_*}{d_2\beta} \left( \frac{C_0}{2} \right) \right) \leq w_0 e^{-A_0 - A\tau_* C_0/2} \quad \textit{if } \ b_* = 0, \tag{2.29}$$

*where*

$$\tilde{C}(b_*) := \int_{-\infty}^{b_*} \exp\Big[-\frac{y^2}{2d_1\delta}\Big] dy = C(-b_*).$$

*Furthermore, by possibly passing to a further subsequence,* $u_n \to 0$ *in* $C([0, 1 - \epsilon])$ *for every* $\epsilon \in (0, 1)$,

$$\lim_{n\to\infty} \int_{x_n}^1 \tilde{u}_n(x)dx = \tilde{C}(b_*) - C_0/2, \qquad \lim_{n\to\infty} \int_0^1 \tilde{u}_n(x)dx = \tilde{C}(b_*), \tag{2.30}$$

$$v_n(x) \to v_0 - \zeta = v_0 - \frac{\tau_*}{d_2\beta}\tilde{C}(b_*) \tag{2.31}$$

*uniformly in* $[0,1]$.

*Proof.* In subcase (b1), we may repeat the arguments used in Lemmas 2.2 and 2.3 for the case $x_* \in (0,1)$ to see that all the conclusions there (with $x_*$ replaced by 1) remain valid; the proofs need only minor modifications.

We now consider subcase (b2). Then instead of (2.3) we have

$$\begin{cases} -d_1\tilde{V}'' = \dfrac{2d_1\delta - y^2}{4d_1\delta}\tilde{V}, \quad 0 < \tilde{V} \le 1 \text{ in } (-\infty, b_*), \\ \tilde{V}'(b_*) + \dfrac{b_*}{2\delta}\tilde{V}(b_*) = 0, \quad \tilde{V}(y^*) = 1,\ \tilde{V}'(y^*) = 0. \end{cases} \tag{2.32}$$

Note that as before $\tilde{V}$ is increasing in $(-\infty, -(2d_1\delta)^{1/2}]$. This and (2.32) imply that $\tilde{V}$ converges to 0 as $y \to -\infty$. Moreover, an elementary consideration shows that

$$|\tilde{V}'(y)|, \tilde{V}(y) \le C_1 e^{-C_2|y|}$$

for some $C_1$, $C_2 > 0$, and all $y < 0$.

As in the case for (2.24), we can similarly show that $y^* = 0$ and $\tilde{V} \equiv V_0$, the unique solution of (2.3) with $y^* = 0$. Moreover, (2.32) introduces no restriction for $b_*$.

Since $\sigma_n^{1/2}(1 - x_n) \to b_*$, instead of (2.6), we have

$$\lim_{n\to\infty} \int_{x_n}^1 \tilde{u}_n(x)dx = \tilde{C}(b_*) - C_0/2, \qquad \lim_{n\to\infty} \int_0^1 \tilde{u}_n(x)dx = \tilde{C}(b_*),$$

where

$$\tilde{C}(b_*) := \int_{-\infty}^{b_*} \exp\left[-\frac{y^2}{4d_1\delta}\right] V_0(y)dy = C(-b_*).$$

This establishes (2.30).

We proceed as in the case $x_* \in (0,1)$ and have two possibilities for $\tau_n$ as before. We show that in the current case, we still cannot have $\tau_n \to \infty$. Arguing indirectly, we assume that $\tau_n \to \infty$.

Since $\int_0^{x_n} \tilde{u}_n dx \to C_0/2$, we have

$$w_n(x_n) \le w_0 e^{-A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds} \to 0.$$

It follows that

$$\|f_n\|_\infty = f_n(x_n) \le f(w_n(x_n)) \to 0,$$

and

$$\int_0^1 f_n \tilde{u}_n dx \to 0.$$

As before, we may integrate the equation for $u_n$ to obtain

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0.$$

Letting $n \to \infty$ and using the above estimate, we deduce

$$-m\tilde{C}(b_*) = 0,$$

a contradiction to our assumption that $m > 0$. Therefore we cannot have $\tau_n \to \infty$.

Thus we can have only the case $\tau_n \to \tau_*$. Then much as before we deduce $u_n \to 0$ in $C([0, 1-\epsilon])$ for each $\epsilon \in (0,1)$ and $\zeta_n \to \zeta$ in $C([0,1]) \cap C^1([0,1-\epsilon])$, for all $\epsilon \in (0,1)$, with $\zeta$ satisfying

$$\zeta'' = 0 \text{ in } [0,1), \qquad \zeta' = 0 \text{ in } [0,1).$$

Hence $\zeta$ is a constant. To determine its value, we use

$$-d_2\zeta_n'(1) = \int_0^1 \tau_n f_n \tilde{u}_n dx \to \tau_* \tilde{C}(b_*)$$

and

$$\zeta_n'(1) + \beta\zeta_n(1) = 0$$

to deduce

$$-\frac{\tau_*}{d_2}\tilde{C}(b_*) + \beta\zeta = 0,$$

and hence

$$\zeta = \frac{\tau_*}{d_2\beta}\tilde{C}(b_*). \tag{2.33}$$

If $\tau_* = 0$, then $\zeta \equiv 0$, and hence $v_n \to v_0$ uniformly in $[0,1]$ and

$$w_n(x) = w_0 e^{-A_0 x - A\tau_n \int_0^x \tilde{u}_n dx} \to w_0 e^{-A_0 x}$$

uniformly in $[0,1]$. Then we can deduce as before that $m = f_0(x_0^*)$, a contradiction to our initial assumption on $m$. Therefore $\tau_* > 0$.

We have

$$\begin{aligned}\int_0^{x_n} f_n(x)\tilde{u}_n(x)dx &= \int_0^{x_n} f(\alpha v_n(x))\tilde{u}_n(x)dx \\ &= (C_0/2) f\big(\alpha(v_0 - \zeta)\big) + o(1).\end{aligned}$$

If $b_* = 0$, then

$$\int_{x_n}^1 f_n(x)\tilde{u}_n(x)dx = o(1).$$

If $b_* > 0$, then $x_n > 0$ and $w_n(x_n) = \alpha v_n(x_n)$. From

$$v_n(x_n) \to v_0 - \zeta = v_0 - \frac{\tau_*}{d_2\beta}\tilde{C}(b_*)$$

and

$$w_n(x_n) = w_0 e^{-A_0 x_n - A\tau_n \int_0^{x_n} \tilde{u}_n dx} \to w_0 e^{-A_0 - A\tau_* C_0/2},$$

we obtain

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2\beta}\tilde{C}(b_*)\Big) = w_0 e^{-A_0 - A\tau_* C_0/2}. \tag{2.34}$$

Moreover,

$$\int_{x_n}^1 f_n(x)\tilde{u}_n(x)dx = \int_{x_n}^1 f(w_n(x))\tilde{u}_n(x)dx$$
$$= \int_{x_n}^1 f\Big(w_0 e^{-A_0 x - A\tau_n \int_0^{x_n} \tilde{u}_n(s)ds - A\tau_n \int_{x_n}^x \tilde{u}_n(s)ds}\Big)\tilde{u}_n dx$$
$$= \int_0^{\tilde{C}(b_*)-C_0/2} f\Big(w_0 e^{-A_0 - A\tau_* C_0/2 - A\tau_* y}\Big)dy + o(1).$$

Therefore, whether $b_* = 0$ or $b_* > 0$, we always have

$$\int_0^1 f_n(x)\tilde{u}_n(x)dx \to \int_0^{\tilde{C}(b_*)} f\Big(w_0 e^{-A_0 - A\tau_* \max\{C_0/2, y\}}\Big)dy. \tag{2.35}$$

We may now use

$$\int_0^1 [f_n(x) - m]\tilde{u}_n dx = 0$$

to obtain

$$m\tilde{C}(b_*) = \int_0^{\tilde{C}(b_*)} f\Big(w_0 e^{-A_0 - A\tau_* \max\{C_0/2, y\}}\Big)dy,$$

which gives (2.27).

Note that if $b_* = 0$, then $x_n = 1$ is possible, and we have only $w_n(x_n) \geq \alpha v(x_n)$, so instead of (2.27), we should have

$$\alpha\Big(v_0 - \frac{\tau_*}{d_2\beta}\tilde{C}(b_*)\Big) \leq w_0 e^{-A_0 - A\tau_* C_0/2}.$$

Thus we have established (2.28) and (2.29). Clearly (2.31) follows from (2.33) and the fact that $v_n \to v_0 - \zeta$ uniformly in $[0, 1]$. The proof is complete. ∎

**3. Limiting profile of the positive solutions.** We are now ready to state and prove our main results. We will show that the limiting equations obtained in the previous section uniquely determine $x_*$ and $\tau_*$, and the value of $v_0$ determines which set of limiting equations should be used for calculating $x_*$ and $\tau_*$. In this way, the asymptotic behavior of the positive solutions is completely determined.

Let us recall that $m$ is fixed such that

$$0 < m < f(\min\{\alpha v_0, w_0\}), \tag{3.1}$$

and $\sigma_n \to \infty$ is a sequence of positive numbers. Therefore by Theorems 1.1 and 1.2, problem (1.1) with $\sigma = \sigma_n$ has a positive solution $(u_n, v_n)$ for all large $n$. Recall that $C_0 > 0$ is given in (2.6), which is completely determined by $\delta$ and $d_1$. Due to (3.1) there exists a unique $\tau_0^* > 0$ such that

$$m = \int_0^1 f\Big(w_0 e^{-A\tau_0^* \max\{C_0/2, C_0 y\}}\Big)dy. \tag{3.2}$$

Let us then define

$$v^* = v^*(m) := \frac{w_0}{\alpha}e^{-A\tau_0^* C_0/2} + \frac{\tau_0^*}{d_2}mC_0(1 + \beta^{-1}). \tag{3.3}$$

Let $\underline{v}(m) > 0$ be uniquely determined by

$$m = f(\alpha \underline{v}(m)).$$

By (3.1), we always have $v_0 > \underline{v}(m)$.

When $m < f(w_0 e^{-A_0})$, we can find a unique $\tau_1^* > 0$ such that

$$m = \int_0^1 f\big(w_0 e^{-A_0 - A\tau_1^* \max\{C_0/2, C_0 y\}}\big) dy. \tag{3.4}$$

We now define

$$\begin{aligned} v_* &= v_*(m) \\ &:= \begin{cases} \dfrac{w_0}{\alpha} e^{-A_0 - A\tau_1^* C_0/2} + \dfrac{\tau_1^*}{d_2} m C_0 \beta^{-1} & \text{if } m < f(w_0 e^{-A_0}), \\ \underline{v}(m) & \text{if } f(w_0 e^{-A_0}) \le m < f(w_0). \end{cases} \end{aligned} \tag{3.5}$$

It is easily seen that $v_*(m)$ is continuous in $m$.

As we will see below, to completely determine the asymptotic profile of $(u_n, v_n)$, it is necessary to distinguish the cases $v_0 \in [v_*(m), v^*(m)]$, $v_0 > v^*(m)$, and $v_0 < v_*(m)$.

THEOREM 3.1. *Suppose that* $v_0 > \underline{v}(m)$ *and*

$$v_*(m) \le v_0 \le v^*(m). \tag{3.6}$$

*Then the system* (2.8) *and* (2.9), *namely,*

$$\begin{cases} w_0 e^{-A_0 x_* - A\tau_*(C_0/2)} = \alpha \left[ v_0 - \dfrac{\tau_*}{d_2} m C_0 (1 + \beta^{-1} - x_*) \right], \\ m = \displaystyle\int_0^1 f\big(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, C_0 y\}}\big) dy, \end{cases}$$

*has a unique solution pair* $(x_*, \tau_*)$ *satisfying* $x_* \in [0,1]$ *and* $\tau_* > 0$. *Moreover,*

$$u_n \to 0 \text{ in } C([0,1] \setminus [x_* - \epsilon, x_* + \epsilon]) \quad \forall\, \epsilon > 0, \qquad \int_0^1 u_n dx \to \tau_* C_0,$$

$$v_n(x) \to v_0 - \frac{\tau_*}{d_2} m C_0 (1 + \beta^{-1} - \max\{x, x_*\}) \quad \text{uniformly in } [0,1].$$

*Furthermore,* $x_* = 0$ *if* $v_0 = v^*(m)$, $x_* \in (0,1)$ *if* $v_*(m) < v_0 < v^*(m)$, *and* $x_* = 1$ *if* $v_0 = v_*(m)$.

*Proof.* Using the notation of the previous section, by passing to a subsequence, $x_n \to x_* \in [0,1]$. By possibly passing to a further subsequence, the behavior of $(u_n, v_n)$ as $n \to \infty$ is then determined by Lemmas 2.2, 2.3 (if $x_* \in (0,1)$), Lemma 2.4 (if $x_* = 0$ and subcases (a1) and (a2) occur), and Lemma 2.5 (if $x_* = 1$ and subcases (b1) and (b2) happen).

If we can show that $x_*$ and $\tau_*$ are uniquely determined by the value of $v_0$, then the corresponding results in the previous section would hold not only for a subsequence, but for the entire original sequence, and hence the behavior of $(u_n, v_n)$ as $n \to \infty$ would be completely determined.

The rather long proof below is broken into several steps.

*Step* 1. Subcases (a2) and (b2) do not happen

First we observe that subcase (a2) does not happen. Indeed, if this case occurs, then since $C(a_*) < C_0/2$, we see (as explained below) from a careful comparison of (2.19) and (3.2) that

$$\tau_* > \tau_0^*, \quad \tau_* C(a_*) < \tau_0^* C_0/2, \quad \tau_*[C_0/2 + C(a_*)] > \tau_0^* C_0.$$

In the comparison, we can deduce these inequalities one at a time, in the above order, and the previous inequalities are used for obtaining the next inequality. For example, to deduce $\tau_* C(a_*) < \tau_0^* C_0/2$ from $\tau_* > \tau_0^*$, we observe that $\tau_* C(a_*) \geq \tau_0^* C_0/2$ and $\tau_* > \tau_0^*$ would imply

$$\begin{aligned}\tau_* \max\{C(a_*), [C_0/2 + C(a_*)]y\} &\geq \max\{\tau_0^* C_0/2, [\tau_* C_0/2 + \tau_0^* C_0/2]y\} \\ &\geq \tau_0^* \max\{C_0/2, C_0 y\}\end{aligned}$$

with strict inequality holding in the last step for $y \in [1/2, 1]$, which is impossible when one compares (2.19) with (3.2).

It then follows from (2.20) and (2.21) that $v_0 > v^*(m)$, contradicting (3.6).

Similarly, if subcase (b2) happens, then from (2.27) we deduce

$$\tau_* > \tau_1^* \quad \text{and} \quad \tau_*[C_0/2 + \tilde{C}(b_*)] < \tau_1^* C_0,$$

which imply, by (2.28) and (2.29), that $v_0 < v_*(m)$, again contradicting (3.6). Therefore subcase (b2) cannot happen.

Thus, by our discussion in the previous section, we have the cases where (2.8) and (2.9) hold. To show that (2.8) and (2.9) have a unique solution $(x_*, \tau_*)$ satisfying $x_* \in [0, 1]$ and $\tau_* > 0$, we establish a procedure to uniquely find $x_*$ and $\tau_*$. In the discussion below, we will treat $v_0 > 0$ as a varying parameter.

*Step* 2. A procedure to solve (2.8) and (2.9).

It is useful to use the new variable

$$\lambda = A_0 x_* + A\tau_* C_0/2.$$

Then

$$x_* = (\lambda - A\tau_* C_0/2)/A_0,$$

and (2.8) can be rewritten as

$$\frac{w_0}{\alpha} e^{-\lambda} = v_0 - \frac{\tau_*}{d_2} m C_0 \left(1 + \beta^{-1} - \frac{\lambda - A\tau_* C_0/2}{A_0}\right)$$

or

$$\frac{mC_0}{d_2 A_0} \tau_* \left[(1 + \beta^{-1})A_0 - \lambda + A(C_0/2)\tau_*\right] = v_0 - \frac{w_0}{\alpha} e^{-\lambda}.$$

We now consider the quadratic equation of $\tau$:

$$\frac{mC_0}{d_2 A_0} \tau \left[(1 + \beta^{-1})A_0 - \lambda + A(C_0/2)\tau\right] = v_0 - \frac{w_0}{\alpha} e^{-\lambda}. \tag{3.7}$$

For each $v_0 > 0$, let $\lambda_0(v_0)$ denote the minimal nonnegative $\lambda$ such that $v_0 - \frac{w_0}{\alpha} e^{-\lambda} \geq 0$. Clearly

$$\lambda_0(v_0) = 0 \text{ if } v_0 \geq w_0/\alpha, \ \lambda_0(v_0) \text{ is decreasing in } (0, w_0/\alpha], \ \lim_{v_0 \to 0} \lambda_0(v_0) = \infty. \tag{3.8}$$

For each $v_0 > 0$ and $\lambda \geq \lambda_0(v_0)$, the quadratic equation (3.7) has a maximal zero, which we denote by $\tau(\lambda, v_0)$. It is easily seen that $\tau(\lambda, v_0) \geq 0$ and

$$\text{when } v_0 \leq w_0/\alpha,\ \tau(\lambda_0(v_0), v_0) = \max\left\{0, \frac{\lambda_0(v_0) - A_0(1+\beta^{-1})}{AC_0/2}\right\}, \tag{3.9}$$

$$\tau(\lambda, v_0) \text{ is increasing in } \lambda \text{ and in } v_0, \quad \lim_{v_0 \to \infty} \tau(\lambda, v_0) = \infty \text{ for fixed } \lambda \geq 0. \tag{3.10}$$

Since $\lambda_0(w_0/\alpha) = 0$, by (3.9), $\tau(\lambda_0(w_0/\alpha), w_0/\alpha) = 0$. Let us consider the continuous function

$$M(v_0) = \int_0^1 f\big(w_0 e^{-\lambda_0(v_0) - A\tau(\lambda_0(v_0), v_0)\max\{0, C_0 y - C_0/2\}}\big) dy.$$

The above observation shows that $M(w_0/\alpha) = f(w_0) > m$. By (3.8), we have $M(v_0) \to 0$ as $v_0 \to 0$. By (3.10), we deduce $M(v_0) \to 0$ as $v_0 \to \infty$. Hence from the continuity of $M(v_0)$ we can find $v_{min}$ and $v_{max}$ such that

$$0 < v_{min} < w_0/\alpha < v_{max} < \infty,$$

$$M(v_0) > m \quad \forall v_0 \in (v_{\min}, v_{max}), \qquad M(v_{min}) = M(v_{max}) = m.$$

Now for each $v_0 \in (v_{min}, v_{max})$,

$$m < \int_0^1 f\big(w_0 e^{-\lambda_0(v_0) - A\tau(\lambda_0(v_0), v_0)\max\{0, C_0 y - C_0/2\}}\big) dy.$$

This and the monotonicity of $\tau(\lambda, v_0)$ in $\lambda$ imply that for such $v_0$ we can find a unique $\lambda_* = \lambda_*(v_0) > \lambda_0(v_0)$ such that

$$m = \int_0^1 f\big(w_0 e^{-\lambda_* - A\tau(\lambda_*, v_0)\max\{0, C_0 y - C_0/2\}}\big) dy.$$

Clearly $v_0 \to \lambda_*(v_0)$ is continuous and

$$\lambda_*(v_{min} + 0) = \lambda_0(v_{min}), \qquad \lambda_*(v_{max} - 0) = \lambda_0(v_{max}).$$

So we may define

$$\lambda_*(v_{min}) = \lambda_0(v_{min}), \qquad \lambda_*(v_{max}) = \lambda_0(v_{max}).$$

We claim that the function $T(v_0) := \tau(\lambda_*(v_0), v_0)$ is increasing in $[v_{min}, v_{max}]$. Otherwise, we can find $v_{min} \leq s_1 < s_2 \leq v_{max}$ such that $T(s_1) \geq T(s_2)$. Since

$$\int_0^1 f\big(w_0 e^{-\lambda_*(s_1) - AT(s_1)\max\{0, C_0 y - C_0/2\}}\big) = \int_0^1 f\big(w_0 e^{-\lambda_*(s_2) - AT(s_2)\max\{0, C_0 y - C_0/2\}}\big),$$

$T(s_1) \geq T(s_2)$ implies that $\lambda_*(s_1) \leq \lambda_*(s_2)$, which implies, by the monotonicity of $\tau(\lambda, v_0)$,

$$T(s_1) = \tau(\lambda_*(s_1), s_1) < \tau(\lambda_*(s_2), s_2) = T(s_2).$$

This contradiction proves the claimed monotonicity of $T(v_0)$.

We show next that $T(v_{max}) > \tau_0^*$. Since $v_{max} > w_0/\alpha$, we have $\lambda_0(v_{max}) = 0$ and hence

$$m = M(v_{max}) = \int_0^1 f\big(w_0 e^{-A\tau(0,v_{max})\max\{0,C_0y-C_0/2\}}\big)dy.$$

By (3.2),

$$m = \int_0^1 f\big(w_0 e^{-A\tau_0^* C_0/2 - A\tau_0^*\max\{0,C_0y-C_0/2\}}\big)dy.$$

Comparing the above two expressions, we obtain $\tau(0, v_{max}) > \tau_0^*$. Hence

$$T(v_{max}) = \tau(\lambda_*(v_{max}), v_{max}) = \tau(\lambda_0(v_{max}), v_{max}) = \tau(0, v_{max}) > \tau_0^*,$$

as we wanted.

We now consider $T(v_{min})$. We have two different cases: $m < f(w_0 e^{-A_0})$ and $m \geq f(w_0 e^{-A_0})$. First consider the case $m < f(w_0 e^{-A_0})$. We show that $T(v_{min}) < \tau_1^*$ in this case. Since $\lambda_*(v_{min}) = \lambda_0(v_{min})$ we have

$$T(v_{min}) = \tau(\lambda_0(v_{min}), v_{min}).$$

Hence, by (3.9),

$$T(v_{min}) = \max\left\{0, \frac{\lambda_0(v_{min}) - A_0(1+\beta^{-1})}{AC_0/2}\right\}.$$

If $\frac{\lambda_0(v_{min})-A_0(1+\beta^{-1})}{AC_0/2} \leq 0$, then $T(v_{min}) = 0 < \tau_1^*$. If $\frac{\lambda_0(v_{min})-A_0(1+\beta^{-1})}{AC_0/2} > 0$, then

$$T(v_{min}) = \frac{\lambda_0(v_{min}) - A_0(1+\beta^{-1})}{AC_0/2},$$

and hence

$$\begin{aligned} m &= \int_0^1 f\big(w_0 e^{-\lambda_0(v_{min}) - AT(v_{min})\max\{0,C_0y-C_0/2\}}\big)dy \\ &= \int_0^1 f\big(w_0 e^{-A_0(1+\beta^{-1}) - AT(v_{min})C_0/2 - AT(v_{min})\max\{0,C_0y-C_0/2\}}\big)dy \\ &= \int_0^1 f\big(w_0 e^{-A_0(1+\beta^{-1}) - AT(v_{min})\max\{C_0/2,C_0y\}}\big)dy. \end{aligned}$$

Comparing this with (3.4), we find that $T(v_{min}) < \tau_1^*$.

With the above properties of $T(v_0)$, we can uniquely determine $v_*$ and $v^*$ with $v_{min} < v_* < v^* < v_{max}$ such that

$$T(v^*) = \tau_0^*, \qquad T(v_*) = \tau_1^*.$$

We claim that $v^* = v^*(m)$ and $v_* = v_*(m)$. Indeed, from

$$m = \int_0^1 f\big(w_0 e^{-\lambda_*(v^*) - AT(v^*)\max\{0,C_0y-C_0/2\}}\big)dy$$

and $T(v^*) = \tau_0^*$, we easily see by comparing with (3.2) that $\lambda_*(v^*) = \tau_0^* AC_0/2$. Hence

$$\tau_0^* = T(v^*) = \tau(\lambda_*(v^*), v^*) = \tau(\tau_0^* AC_0/2, v^*).$$

By the definition of $\tau(\lambda, v_0)$, the above identity means that $\tau = \tau_0^*$ solves (3.7) with $\lambda = \tau_0^* AC_0/2$ and $v_0 = v^*$. Therefore we may compare with (3.3) to deduce $v^* = v^*(m)$. Similarly, we can show that $v_* = v_*(m)$.

Since $T$ is monotone, for each $v_0 \in [v_*(m), v^*(m)]$, $T(v_0) \in [\tau_1^*, \tau_0^*]$. Hence we can compare (3.2) and (3.4) with

$$m = \int_0^1 f\big(w_0 e^{-\lambda_*(v_0) - AT(v_0)\max\{0, C_0 y - C_0/2\}}\big) dy$$

to find that, for such $v_0$, we necessarily have

$$AT(v_0)C_0/2 \le \lambda_*(v_0) \le A_0 + AT(v_0)C_0/2;$$

otherwise we would arrive at contradictions to $T(v_0) \in [\tau_1^*, \tau_0^*]$. This implies that there exists a unique $x_* \in [0, 1]$ such that

$$\lambda_*(v_0) = A_0 x_* + AT(v_0)C_0/2.$$

Let $\tau_* = T(v_0)$; we find that $(x_*, \tau_*)$ solves (2.8) and (2.9).

We next consider the case $m \ge f(w_0 e^{-A_0})$. In this case, $v_*(m) = \underline{v}(m)$; moreover, we show that

$$T(v_{min}) = 0, \quad v_{min} = \underline{v}(m).$$

Indeed, from

$$T(v_{min}) = \tau(\lambda_*(v_{min}), v_{min}) = \tau(\lambda_0(v_{min}), v_{min}),$$

we obtain

$$\begin{aligned} m &= \int_0^1 f\big(w_0 e^{-\lambda_0(v_{min}) - A\tau(\lambda_0(v_{min}), v_{min})\max\{0, C_0 y - C_0/2\}}\big) dy \\ &\le f(w_0 e^{-\lambda_0(v_{min})}). \end{aligned}$$

It follows that $\lambda_0(v_{min}) \le A_0 < A_0(1+\beta^{-1})$. By (3.9), we deduce $\tau(\lambda_0(v_{min}), v_{min}) = 0$, that is, $T(v_{min}) = 0$. This gives

$$m = \int_0^1 f(w_0 e^{-\lambda_0(v_{min})}) dy = f(w_0 e^{\lambda_0(v_{min})}).$$

Hence

$$\alpha \underline{v}(m) = w_0 e^{-\lambda_0(v_{min})}.$$

On the other hand, since $v_{min} < w_0/\alpha$, by the definition of the function $\lambda_0$,

$$v_{min} - \frac{w_0}{\alpha} e^{-\lambda_0(v_{min})} = 0.$$

Therefore we have $v_{min} = \underline{v}(m)$.

We can now conclude that there exists a unique $v^* \in (v_{min}, v_{max})$ such that $T(v^*) = \tau_0^*$. We may then prove $v^* = v^*(m)$ as before. Since $T$ is monotone, for each $v_0 \in (v_*(m), v^*(m)] = (\underline{v}(m), v^*(m)]$, $T(v_0) \in (0, \tau_0^*]$. Hence we can compare (3.2) and $m \geq f(w_0 e^{-A_0})$ with (3.1) to deduce

$$AT(v_0)C_0/2 \leq \lambda_*(v_0) < A_0 + AT(v_0)C_0/2,$$

and there exists a unique $x_* \in [0, 1)$ such that

$$\lambda_*(v_0) = A_0 x_* + AT(v_0)C_0/2.$$

Let $\tau_* = T(v_0)$; we find that $(x_*, \tau_*)$ solves (2.8) and (2.9).

The above discussion shows that when (3.6) holds, (2.8) and (2.9) have at least one solution $(x_*, \tau_*)$ satisfying $x_* \in [0, 1]$ and $\tau_* > 0$, and such a solution can be found by following the above procedure.

*Step* 3. Uniqueness of $(x_*, \tau_*)$ and completion of the proof.

We next show that when (3.6) holds, (2.8) and (2.9) have a unique solution $(x_*, \tau_*)$ satisfying $x_* \in [0, 1]$ and $\tau_* > 0$. So let $(x_*, \tau_*)$ be an arbitrary solution of (2.8) and (2.9) with $v_0 \in [v_*(m), v^*(m)] \cap (\underline{v}(m), v^*(m)]$ and $x_* \in [0, 1]$, $\tau_* > 0$. Then $\tau_*$ must be the maximal zero of (3.7) with $\lambda = A_0 x_* + A\tau_* C_0/2 > 0$; this is the case because $v_0 - \frac{w_0}{\alpha} e^{-\lambda} > 0$, and thus the two zeros of (3.7) are of opposite sign. Therefore, using our earlier notation,

$$\tau_* = \tau(\lambda, v_0), \quad \lambda > \lambda_0(v_0).$$

Then (2.9) yields

$$\begin{aligned} m &= \int_0^1 f(w_0 e^{-A_0 x_* - A\tau_* \max\{C_0/2, C_0 y\}}) dy \\ &= \int_0^1 f(w_0 e^{-\lambda - A\tau(\lambda, v_0) \max\{0, C_0 y - C_0/2\}}) dy. \end{aligned}$$

Since $v_0 \in [v_*(m), v^*(m)] \cap (\underline{v}(m), v^*(m)] \subset (v_{min}, v_{max})$, in view of the above identity, our definition of $\lambda_*(v_0)$ implies that $\lambda = \lambda_*(v_0)$ and hence $\tau(\lambda, v_0) = T(v_0)$; that is, $\tau_* = \tau(\lambda, v_0) = T(v_0)$. This implies that the solution pair $(x_*, \tau_*)$ is the same as the one obtained through our procedure introduced above for solving (2.8) and (2.9). Hence there is a unique solution.

With $\tau_*$ and $x_*$ uniquely determined now, it is easily seen that our conclusions for $u_n$ and $v_n$ follow from Lemmas 2.2, 2.3, 2.4, and 2.5.

Moreover, from the above procedure for finding $(x_*, \tau_*)$, we easily see that $x_* = 0$ if $v_0 = v^*(m)$, $x_* \in (0, 1)$ if $v_*(m) < v_0 < v^*(m)$, and $x_* = 1$ if $v_0 = v_*(m)$.

The proof of the theorem is now complete. ∎

Next we consider the case that $v_0 > v^*(m)$. Let $0 < \lambda_0 < \lambda^0$ be uniquely determined by

$$m = f(w_0 e^{-A\lambda_0}) = \int_0^1 f\big(w_0 e^{-A\lambda^0 y}\big) dy. \tag{3.11}$$

For each $\lambda \in [0, \lambda_0]$, we can find a unique $\Gamma = \Gamma(\lambda)$ such that

$$m = \int_0^1 f\big(w_0 e^{-A \max\{\lambda, \Gamma y\}}\big) dy. \tag{3.12}$$

Moreover, it is easily seen that $\lambda \mapsto \Gamma(\lambda)$ is a continuous decreasing function with

$$\Gamma(\lambda_0) = \lambda_0, \qquad \Gamma(0) = \lambda^0.$$

Therefore we can find a unique $\lambda_*^0 \in (0, \lambda_0)$ such that

$$\Gamma(\lambda_*^0) = 2\lambda_*^0.$$

Comparing with (3.2), we find that actually

$$\lambda_*^0 = \tau_0^* C_0/2. \tag{3.13}$$

We define

$$\Lambda(\lambda) := \frac{w_0}{\alpha} e^{-A\lambda} + \frac{\Gamma(\lambda)}{d_2} m(1 + \beta^{-1}).$$

Clearly $\Lambda(\lambda)$ is a decreasing function on $[0, \lambda_0]$ with

$$\Lambda(0) = \frac{w_0}{\alpha} + \frac{\lambda^0}{d_2} m(1 + \beta^{-1}), \qquad \Lambda(\lambda_*^0) = \frac{w_0}{\alpha} e^{-A\lambda_*^0} + \frac{2\lambda_*^0}{d_2} m(1 + \beta^{-1}).$$

Due to (3.13), we find that

$$\Lambda(\lambda_*^0) = v^*(m).$$

THEOREM 3.2. *Suppose that*

$$v_0 > v^*(m) = \Lambda(\lambda_*^0). \tag{3.14}$$

*If* $v_0 < \Lambda(0)$ *and* $\lambda^* \in (0, \lambda_*^0)$ *is uniquely determined by* $v_0 = \Lambda(\lambda^*)$*, then*

$$u_n \to 0 \text{ in } C([\epsilon, 1]) \quad \forall \epsilon \in (0, 1), \qquad \int_0^1 u_n dx \to \Gamma(\lambda^*),$$

$$v_n(x) \to v_0 - \frac{\Gamma(\lambda^*)}{d_2} m(1 + \beta^{-1} - x) \text{ uniformly in } [0, 1].$$

*If* $v_0 \geq \Lambda(0)$*, then the above conclusions hold with* $\lambda^* = 0$.

*Proof.* We first show that case (a2) happens. Let us start by observing that the cases leading to (2.8) and (2.9) (namely, cases (i), (ii)(a1), and (iii)(b1)) cannot happen. Indeed, in these cases, $(x_*, \tau_*)$ solves (2.8) and (2.9) with $x_* \in [0, 1]$ and $\tau_* > 0$. As in Step 3 of the proof of Theorem 3.1, denoting $\lambda = A_0 x_* + A\tau_* C_0/2$, we must have $\tau_* = \tau(\lambda, v_0)$ and $\lambda > \lambda_0(v_0)$. Then (2.9) gives

$$m = \int_0^1 f(w_0 e^{-\lambda - A\tau(\lambda, v_0) \max\{0, C_0 y - C_0/2\}}) dy.$$

Since $v_0 > v^*(m)$, we have either

$$v_0 > v_{max} \quad \text{or} \quad v_0 \in (v^*(m), v_{max}].$$

If $v_0 \in (v^*(m), v_{max}] \subset (v_{min}, v_{max})$, then the above identity implies that $\lambda = \lambda_*(v_0)$ and hence $\tau(\lambda, v_0) = T(v_0)$. From $v_0 > v^*(m)$ we now deduce $\tau_* = T(v_0) > \tau_0^*$, and hence we can compare (2.9) with (3.2) to deduce $x_* < 0$, a contradiction.

If $v_0 > v_{max}$, then by the monotonicity of $\tau(\cdot,\cdot)$, we deduce

$$\tau(\lambda, v_0) > \tau(\lambda, v_{max}) > \tau(0, v_{max}).$$

Therefore, recalling $\lambda_*(v_{max}) = \lambda_0(v_{\max}) = 0$, we obtain

$$\begin{aligned} m &= \int_0^1 f(w_0 e^{-\lambda - A\tau(\lambda,v_0)\max\{0,C_0y-C_0/2\}})dy \\ &< \int_0^1 f(w_0 e^{-A\tau(0,v_{max})\max\{0,C_0y-C_0/2\}})dy \\ &= m, \end{aligned}$$

again a contradiction. Therefore none of the cases that lead to (2.8) and (2.9) can happen. This implies that either (a2) or (b2) happens.

Next we show that case (b2) cannot happen. Otherwise, by (2.27) we obtain

$$m < f(w_0 e^{-A_0}).$$

Hence $\tau_1^* > 0$ is defined. Moreover, comparing (2.27) with (3.4), we obtain

$$\tau_* > \tau_1^*, \qquad \tau_* \tilde{C}(b_*) < \tau_1^* C_0,$$

which imply, by (2.28) and (2.29), that $v_0 < v_*(m) < v^*(m)$, contradicting (3.14).

Therefore we necessarily have case (a2). We now introduce the notation

$$\lambda = \tau_*[C(a_*) - C_0/2], \quad \Gamma = \tau_* C(a_*).$$

From (2.19), (2.20), and (2.21) we find that

$$m = \int_0^1 f\big(w_0 e^{-A\max\{\lambda,\Gamma y\}}\big)dy, \tag{3.15}$$

$$v_0 \geq \frac{w_0}{\alpha} e^{-A\lambda} + \frac{\Gamma}{d_2} m(1 + \beta^{-1}), \tag{3.16}$$

with equality holding if $a_* > 0$.

Suppose now that $v_0 \geq \Lambda(0)$. We claim that in this case we have $\lambda = 0$ and hence, by (3.15), $\Gamma = \Gamma(0) = \lambda^0$. Suppose for the sake of contradiction that $\lambda > 0$. From (3.15) and (3.11) we easily see that $\lambda \leq \lambda_0$. Now $C(a_*) - C_0/2 > 0$, and hence $a_* > 0$. Thus equality in (3.16) holds. By (3.15) we deduce $\Gamma = \Gamma(\lambda)$, and hence it follows from (3.16) that $v_0 = \Lambda(\lambda) < \Lambda(0)$, contradicting our assumption on $v_0$ above. Hence in this case, we have $\lambda = 0$ and thus

$$C(a_*) - C_0/2 = 0, \qquad \tau_* = \Gamma(0)/(C_0/2).$$

Next we suppose that $v^*(m) < v_0 < \Lambda(0)$. From (3.15) we deduce $\Gamma = \Gamma(\lambda)$ for some $\lambda \in [0, \lambda_0]$. We must have $\lambda > 0$ for otherwise, from (3.15) and (3.16), we deduce $\Gamma = \Gamma(0)$ and $v_0 \geq \Lambda(0)$, contradicting our current assumption on $v_0$. Therefore $\lambda > 0$ and hence $a_* > 0$, implying that equality in (3.16) holds. Recalling $\Gamma = \Gamma(\lambda)$, we thus obtain $v_0 = \Lambda(\lambda)$ and $\lambda = \lambda^*$. It follows that $\tau_*$ and $a_*$ in Lemma 2.4 are uniquely determined by

$$\tau_*[C(a_*) - C_0/2] = \lambda^*, \qquad \tau_* C(a_*) = \Gamma(\lambda^*),$$

namely,

$$\tau_* = \frac{\Gamma(\lambda^*) - \lambda^*}{C_0/2}, \qquad a_* = C^{-1}(\lambda^*/\tau_* + C_0/2).$$

The rest of the proof now follows from Lemma 2.4. □

We now consider the remaining case that $\underline{v}(m) < v_0 < v_*(m)$, which can happen only if $m < f(w_0 e^{-A_0})$. Suppose that $\lambda_0$, $\lambda^0$, $\lambda^0_*$, and $\Gamma(\lambda)$ are as in Theorem 3.2 but with $w_0$ there replaced by $w_0 e^{-A_0}$, and we denote them by $\tilde\lambda_0$, $\tilde\lambda^0$, $\tilde\lambda^0_*$, and $\tilde\Gamma(\lambda)$, respectively. Define

$$\Delta(\lambda) := \frac{w_0}{\alpha} e^{-A_0 - A\lambda} + \frac{\tilde\Gamma(\lambda)}{d_2} m\beta^{-1}.$$

Then $\Delta(\lambda)$ is a decreasing function over $[0, \tilde\lambda_0]$ with

$$\Delta(0) = \frac{w_0}{\alpha} e^{-A_0} + \frac{\tilde\lambda^0}{d_2} m\beta^{-1}, \qquad \Delta(\tilde\lambda^0_*) = \frac{w_0}{\alpha} e^{-A_0 - A\tilde\lambda^0_*} + \frac{\tilde\lambda^0_*}{d_2} m\beta^{-1} = v_*(m).$$

THEOREM 3.3. *Suppose that* $m < f(w_0 e^{-A_0})$ *and*

$$\underline{v}(m) < v_0 < v_*(m) = \Delta(\tilde\lambda^0_*). \tag{3.17}$$

*If* $v_0 > \Delta(0)$ *and* $\lambda_* \in (0, \tilde\lambda^0_*)$ *is uniquely determined by* $v_0 = \Delta(\lambda_*)$*, then*

$$u_n \to 0 \text{ in } C([0, 1-\epsilon])\ \forall \epsilon \in (0,1), \qquad \int_0^1 u_n dx \to \tilde\Gamma(\lambda_*),$$

$$v_n(x) \to v_0 - \frac{\tilde\Gamma(\lambda_*)}{d_2} m\beta^{-1} \text{ uniformly in } [0,1].$$

*If* $\underline{v}(m) < v_0 \le \Delta(0)$*, then the above conclusions hold with* $\lambda_* = 0$.

*Proof.* This is similar to the proof of Theorem 3.2. Here we can show that case (b2) must happen, and then we use Lemma 2.5. We omit the details. □

*Remark* 3.4. Our results in this section reveal an interesting property for the limiting total biomass $\lim_{n\to\infty} \int_0^1 u_n(x)dx$. First consider the case $v_*(m) \le v_0 \le v^*(m)$. By Theorem 3.1, in this case the above limit has value $\tau_* C_0$. By the proof of Theorem 3.1, we know that $\tau_* = T(v_0)$ with $T(v_0)$ a strictly increasing function of $v_0$. Therefore the limit is strictly increasing with $v_0$ for $v_0 \in [v_*(m), v^*(m)]$.

If $v_0 \in (v^*(m), \Lambda(0))$, then by Theorem 3.2, $\lim_{n\to\infty} \int_0^1 u_n(x)dx = \Gamma(\lambda^*) = \Gamma \circ \Lambda^{-1}(v_0)$. Since $\Gamma(\cdot)$ and $\Lambda(\cdot)$ are both strictly decreasing functions, $\Gamma\circ\Lambda^{-1}(\cdot)$ is strictly increasing, and hence the limit is strictly increasing with $v_0$ for $v_0 \in (v^*(m), \Lambda(0))$. However, by Theorem 3.2, this limit takes the same value $\Gamma(0)$ for all $v_0 \ge \Lambda(0)$.

If $v_0 \in (\Delta(0), v_*(m))$, then by Theorem 3.3, the limit takes value $\tilde\Gamma(\lambda_*) = \tilde\Gamma \circ \Delta^{-1}(v_0)$, which is strictly increasing in $v_0$, but it takes the same value $\tilde\Gamma(0)$ for all $v_0 \in (\underline{v}(m), \Delta(0)]$.

Therefore if we denote

$$v_{**} = v_{**}(m) := \Delta(0), \qquad v^{**} = v^{**}(m) := \Lambda(0),$$

then $\lim_{n\to\infty} \int_0^1 u_n(x)dx$ is strictly increasing with $v_0$ as $v_0$ varies in the range $v_{**} \le v_0 \le v^{**}$, but is constant for $v_0 \ge v^{**}$ or for $v_0 \in (\underline{v}(m), v_{**}]$.

**4. Biological interpretations.** In this section we compare our results with the game theoretical model in [KL] and explain the predictions that our theoretical results offer for the phytoplankton problem being modelled.

Since $u(x)$ and $v(x)$ are, respectively, rescaled versions of the biomass function $b(x)$ and the nutrient function $R(x)$ used in the original model of [KL] (see Part I for details), we will interpret $u(x)$ and $v(x)$ as representing the (steady) distributions of the biomass and nutrient at depth $x$ of a water column with surface at $x = 0$ and bottom at $x = 1$.

(i) First we note that, if we replace $\max\{C_0/2, C_0 y\}$ in (2.9) by $C_0$, then the system of equations for $(x_*, \tau_*)$ in Theorem 3.1 reduces to the game theoretical model of [KL], namely, equations (4) and (5) in [KL] with $\hat{B} = \tau_* C_0$. Thus the game theoretical model of [KL] is a simplified version of the rigorous limiting equations here. It captures the main properties of the limiting equations but only in the case that $v_* \leq v_0 \leq v^*$.

(ii) When $v_0 > v^* = v^*(m)$, from Theorem 3.2 and Step 1 of the proof of Theorem 3.1, we see that as $\sigma \to \infty$ the total biomass has limit

$$\Gamma(\lambda^*) = \tau_* C(a_*) > \tau_0^* C_0.$$

If we have simply used (2.8) and (2.9) with $x_* = 0$ to calculate the total biomass, we would have obtained the incorrect limit $\tau_0^* C_0$. Similarly, the limit of the total biomass in the case of Theorem 3.3 is less than the value one would have obtained by simply using (2.8) and (2.9) with $x_* = 1$.

(iii) Remark 3.4 shows that as $\sigma \to \infty$, the limit of the total biomass is strictly increasing with $v_0$ in the range $v_{**} \leq v_0 \leq v^{**}$. It is interesting to notice that the layer position of the biomass (in the limit) already reaches the surface at $v_0 = v^*$ (i.e., $x_* = 0$ when $v_0 = v_*$), but as the nutrient level at the sediment $v_0$ increases past the critical value $v^*$, though the layer remains at the surface, the total biomass keeps increasing until $v_0$ reaches a second critical value $v^{**}$, where the total biomass reaches a maximum, and then remains at this value even if $v_0$ is further increased. On the other hand, if $v_0$ is decreased to $v_*$, the layer of the biomass lowers to the bottom ($x_* = 1$), but as $v_0$ decreases past $v_*$, though the layer remains at the bottom, the total biomass keeps decreasing until $v_0$ reaches the critical value $v_{**}$, where the total biomass reaches its minimal value, and then remains at this minimal value until $v_0$ is so low ($v_0 \leq \underline{v}(m)$) that the phytoplankton can no longer survive.

(iv) Our results support the important predictions in [KL] that depth-regulating phytoplankton can form a thin layer in a poorly mixed water column and that the concentration of the limiting nutrient should be low and constant above the phytoplankton layer and linearly increasing with depth below the layer. The predictions in item (iii) above seem to provide new insights to this problem.

(v) We could fix $v_0$ and use a different parameter in the model, say the surface light level $w_0$, as a varying parameter to interpret the phenomena represented in items (i)–(iii) above.

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