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# A diffusive predator–prey model in heterogeneous environment

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**Abstract**

In this paper, we demonstrate some special behavior of steady-state solutions to a predator–prey model due to the introduction of spatial heterogeneity. We show that positive steady-state solutions with certain prescribed spatial patterns can be obtained when the spatial environment is designed suitably. Moreover, we observe some essential differences of the behavior of our model from that of the classical Lotka–Volterra model that seem to arise only in the heterogeneous case.
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## 1. Introduction

To capture the influence of heterogeneous spatial environment on population models is not easy in general. Traditionally population models were considered in homogeneous spatial environments, and hence the coefficients appearing in the models are usually chosen to be positive constants. To include spatial variations of the environment, naturally these coefficients should be replaced by positive

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functions of the space variable $x$. However, the mathematical techniques developed to study these models are typically either not sensitive to this change (e.g., the bifurcation approach, the topological degree approach, the upper and lower solution argument), in which case the effects of heterogeneous spatial environment are difficult to observe in the mathematical analysis, or the techniques are too sensitive to this change (e.g., the various Lyapunov function techniques) and become unapplicable when the constant coefficients are replaced by functions.

It has been observed recently that, in general, the behavior of these models are very sensitive to certain coefficient functions becoming small in part of the underlying spatial region. This observation was successfully used in [D1,D2,D3] for the Lotka–Volterra competition model and in [DD] for the Lotka–Volterra predator–prey model to reveal the effects of spatial heterogeneous environments on these models. It would be interesting to know whether this approach works for sufficiently different non-Lotka–Volterra models. In this paper we examine such a predator–prey model and demonstrate that the approach indeed yields interesting results that reveal certain effects of the heterogeneous spatial environment on the model. Moreover, as will become clear later, the effects turn out to be significantly different in a number of ways from those observed in the Lotka–Volterra predator–prey model in the study of [DD].

As in [DD,D1,D2,D3], to make the ideas more transparent, we have restricted our consideration to the simplest forms of the corresponding Holling–Tanner models in order to avoid excessive technicalities. We believe that our techniques are applicable to more general models.

Let us now be more precise. The model to be studied in this paper is the following diffusive predator–prey system:

$$\begin{cases} u_t - d_1\Delta u = \lambda u - \alpha u^2 - \beta uv, & x\in\Omega,\ \ t>0, \\ v_t - d_2\Delta v = \mu v\Big(1 - \delta\dfrac{v}{u}\Big), & x\in\Omega,\ \ t>0, \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ \ t>0, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $R^N$ with smooth boundary $\partial\Omega$, $d_1, d_2, \lambda, \mu, \alpha, \beta, \delta$ are continuous positive functions of $x\in\bar{\Omega}$. This system describes the interaction of a prey species $u$ and a predator species $v$ in a given spatial region $\Omega$; the Neumann boundary condition means that no species can pass across the boundary of $\Omega$.

The main part of this paper is concerned with the steady-state solutions of (1.1), though some special cases of the parabolic problem are also considered. We are interested in revealing new phenomena caused by the introduction of inhomogeneous spatial environment, in particular the existence of steady-state solutions with certain prescribed spatial patterns. We also want to reveal some essential differences between (1.1) and the following classical Lotka–Volterra predator–prey model:

$$
\begin{cases}
u_t - d_1\Delta u = \lambda u - \alpha u^2 - \beta uv, & x \in \Omega, \quad t > 0, \\
v_t - d_2\Delta v = \mu v - a v^2 + buv, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0,
\end{cases} \tag{1.2}
$$

where $\alpha, \beta, a$ and $b$ are positive continuous functions, $\lambda, \mu$ are continuous functions not necessarily positive.

To put our discussions into perspective, let us mention that (1.1) and (1.2) are special cases (i.e., $\gamma = 0$) of the following well-known Holling–Tanner-type predator–prey models (see [M,R]):

$$
\begin{cases}
u_t - d_1\Delta u = \lambda u - \alpha u^2 - \dfrac{\beta uv}{1 + \gamma u}, & x \in \Omega, \quad t > 0, \\
v_t - d_2\Delta v = \mu v\left(1 - \delta \dfrac{v}{u}\right), & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0,
\end{cases} \tag{1.3}
$$

$$
\begin{cases}
u_t - d_1\Delta u = \lambda u - \alpha u^2 - \dfrac{\beta uv}{1 + \gamma u}, & x \in \Omega, \quad t > 0, \\
v_t - d_2\Delta v = \mu v - a v^2 + \dfrac{buv}{1 + \gamma u}, & x \in \Omega, \quad t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0.
\end{cases} \tag{1.4}
$$

It is well known that predator–prey models are generally rather sensitive to the changes of their reaction terms, i.e., the dynamical behavior may change drastically under small changes of the right-hand sides of the equations. This already occurs in the ODE models. For example, for the corresponding ODE models of (1.3) and (1.4), it is known that their dynamics is simple in the case $\gamma = 0$ where the unique positive equilibrium $(u^*, v^*)$ (when exists) attracts all the positive solutions as $t \to \infty$, but when $\gamma > 0$, stable limiting cycles may exist (see [Hz,HH1,HH2]).

For the PDE models (1.3) and (1.4) in a homogeneous environment (i.e., when all the coefficient functions take constant values), the case $\gamma = 0$ does not seem to give rise to interesting phenomenon either. Indeed, for (1.2) it is well known that the constant positive steady-state $(u^*, v^*)$ attracts all the positive solutions of (1.2) as $t \to \infty$ (see [B,DR,L]); for (1.1), we will show that a similar result holds, at least when $\alpha/\beta$ is not small (see Section 2 below).

The main purpose of this paper is to closely examine (1.1) in a heterogeneous environment and reveal that, in contrast to the homogeneous case mentioned above, certain interesting phenomena do arise. Firstly, we show that positive steady-state solutions of (1.1) with certain prescribed spatial patterns can be obtained if the coefficient functions are chosen suitably (see Remark 3.19). To achieve this, we use various elliptic estimates to show that if a degeneracy appears in the model, i.e., if $\alpha(x)$ vanishes in a subdomain of $\Omega$, then the model undergoes an essential change of its behavior; this enables us to perturb the degenerate $\alpha(x)$ by $\alpha(x)+\varepsilon$ and obtain the desired patterned solutions. This strategy is adapted from [D1,D2,D3] where it was used for the competition model. The technical difficulties of this paper, however, are considerably different from that in [D1,D2,D3].

Secondly, by comparing our results here with those obtained in [DD], we reveal that in a heterogeneous environment, the behavior of (1.1) and (1.2) exhibits some essential differences which do not seem to appear in the homogeneous case (see Remarks 3.14 and 3.20 for details).

The rest of this paper is organized as follows. In the short Section 2, we apply a Lyapunov function argument to show that, in the homogeneous case, the dynamics of (1.1) is simple, at least when $\alpha/\beta$ is not small. Section 3 constitutes the main body of the paper, where the heterogeneous case of (1.1) is carefully examined, through the use of various elliptic estimates, topological degree theory, and boundary blow-up solutions.

We would like to remark that our results for the heterogeneous case of (1.1) are mainly on its steady-state solutions. A deep understanding of the global dynamical behavior of (1.1) in this case seems a very difficult and interesting problem, awaiting for further studies.

Finally, we note that the effects of heterogeneous environment on *competition* models have been considered in several recent papers, see, for example [AC,CCH,D1,D2,D3,HLM,HMP], the survey paper [D4], and the references therein.

## 2. The homogeneous case

We assume throughout this section that all the coefficient functions in (1.1) are positive constants. By replacing $u$ by $u/\delta$, and $\alpha$ by $\alpha\delta$, we readily see that (1.1) is reduced to

$$\begin{cases} u_t - d_1\Delta u = u(\lambda - \alpha u - \beta v), & x\in\Omega, \quad t>0,\\ v_t - d_2\Delta v = \mu v\left(1-\dfrac{v}{u}\right), & x\in\Omega, \quad t>0,\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, \quad t>0. \end{cases} \tag{2.1}$$

Clearly,

$$(u^*, v^*) = \left(\frac{\lambda}{\alpha+\beta}, \frac{\lambda}{\alpha+\beta}\right)$$

is the only constant positive equilibrium of (2.1).

Let $(u(x,t), v(x,t))$ be a positive solution of (2.1). A simple comparison argument yields $0<u(x,t)<U(x,t)$ for all $t>0$ and $x\in\Omega$, where $U$ is the unique solution of

$$U_t - d_1 \Delta U = \lambda U - \alpha U^2 \quad \text{in } \Omega \times (0, \infty), \quad U_\nu|_{\partial\Omega \times (0,\infty)} = 0, \quad U(x,0) = u(x,0).$$

It is well known that $U(x,t) \to \lambda/\alpha$ as $t \to \infty$ uniformly in $x$. From these facts, it follows by standard comparison arguments that $u(x,t)$ and $v(x,t)$ exist and remain positive for all $t > 0$, and

$$\overline{\lim_{t\to\infty}}\ u(x,t) \leqslant \lambda/\alpha, \quad \overline{\lim_{t\to\infty}}\ v(x,t) \leqslant \lambda/\alpha.$$

Adapting the Lyapunov function in [HH1], we define

$$V(u,v) = \int \frac{u-u^*}{u^2}\, du + c \int \frac{v - v^*}{v}\, dv,$$
$$W(t) = \int_\Omega V(u(x,t), v(x,t))\, dx,$$

where $c > 0$ is a constant to be determined later, and $(u(x,t), v(x,t))$ is an arbitrary positive solution of (2.1).

Denote

$$f(u,v) = u(\lambda - \alpha u - \beta v), \quad g(u,v) = \mu v(1 - v/u).$$

We have

$$\begin{aligned}
&V_u(u,v)f(u,v) + V_v(u,v)g(u,v)\\
&\quad = \frac{u-u^*}{u}(\lambda - \alpha u - \beta v) + c\mu(v - v^*)(1 - v/u)\\
&\quad = \frac{u-u^*}{u}(\alpha u^* + \beta v^* - \alpha u - \beta v) + c\mu(v - v^*)\frac{u - u^* + v^* - v}{u}\\
&\quad = -\alpha \frac{(u-u^*)^2}{u} + (c\mu - \beta)\frac{(u-u^*)(v-v^*)}{u} - c\mu \frac{(v-v^*)^2}{u}.
\end{aligned}$$

We now choose $c = \beta/\mu$ and obtain

$$V_u(u,v)f(u,v) + V_v(u,v)g(u,v) = -\alpha \frac{(u-u^*)^2}{u} - \beta \frac{(v-v^*)^2}{u}.$$

It follows that

$$\begin{aligned} W'(t) &= \int_\Omega (V_u(u(x,t),v(x,t))u_t + V_v(u(x,t),v(x,t))v_t)\,dx \\ &= \int_\Omega \left(\frac{u-u^*}{u^2} d_1\Delta u + c\frac{v-v^*}{v} d_2 \Delta v\right) dx - \int_\Omega \left(\alpha\frac{(u-u^*)^2}{u} + \beta\frac{(v-v^*)^2}{u}\right) dx \\ &= -\int_\Omega \left(d_1 \frac{2u^*-u}{u^3}|\nabla u|^2 + d_2\frac{v^*}{v^2}|\nabla v|^2 + \alpha\frac{(u-u^*)^2}{u} + \beta\frac{(v-v^*)^2}{u}\right) dx. \end{aligned}$$

Suppose that $\alpha > \beta$. Then $2u^* = 2\lambda/(\alpha+\beta) > \lambda/\alpha$. Since $u(x,t) < U(x,t)$ and $U(x,t) \to \lambda/\alpha$ as $t \to \infty$, we can find $T > 0$ large such that $U(x,t) < 2\lambda/(\alpha+\beta)$ for all $t \geqslant T$ and all $x \in \Omega$. Therefore, $w'(t) \leqslant 0$ for all $t > T$ and equality holds only if $(u,v) \equiv (u^*, v^*)$. Together with some standard arguments based on the boundedness of $(u,v)$ and parabolic regularity, this proves the following result.

**Proposition 2.1.** *When $\alpha > \beta$, $(u^*, v^*)$ attracts every positive solution of* (2.1).

Next, we show how the restriction $\alpha > \beta$ can be relaxed by using a different Lyapunov function. Define

$$V^*(u,v) = \int \frac{u^2 - (u^*)^2}{u^2}\,du + c\int \frac{v-v^*}{v}\,dv,$$

with $c > 0$ to be chosen later. We have

$$\begin{aligned} &V_u^*(u,v)f(u,v) + V_v^*(u,v)g(u,v) \\ &\quad = \frac{u^2-(u^*)^2}{u}(\lambda - \alpha u - \beta v) + c\mu(v-v^*)(1-v/u) \\ &\quad = \frac{1}{u}(-\alpha(u+u^*)\xi^2 + [c\mu - \beta(u+u^*)]\xi\eta - c\mu\eta^2), \\ &\xi = u - u^*, \quad \eta = v - v^*. \end{aligned}$$

If

$$[c\mu - \beta(u+u^*)]^2 - 4\alpha(u+u^*)c\mu < 0, \tag{2.2}$$

then the quadratic form

$$-\alpha(u+u^*)\xi^2 + [c\mu - \beta(u+u^*)]\xi\eta - c\mu\eta^2$$

is negative unless $\xi = \eta = 0$.

We now show that it is possible to choose $c>0$ so that (2.2) holds under less restrictive conditions than $\alpha>\beta$. To this end, we rewrite (2.2) as

$$(\mu c)^2 - 2(u+u^*)(\beta+2\alpha)(\mu c) + \beta^2(u+u^*)^2 < 0. \tag{2.3}$$

We find that (2.3) holds if and only if $\mu c \in (c_1, c_2)$ where

$$c_1 = c_1(u) = (u+u^*)\left(\beta+2\alpha - \sqrt{(\beta+2\alpha)^2-\beta^2}\right),$$
$$c_2 = c_2(u) = (u+u^*)\left(\beta+2\alpha + \sqrt{(\beta+2\alpha)^2-\beta^2}\right).$$

To choose a proper $c$, we need to find out when the inequality $c_1(\lambda/\alpha) < c_2(0)$ holds, i.e.,

$$\left(\frac{\lambda}{\alpha}+\frac{\lambda}{\alpha+\beta}\right)\left(\beta+2\alpha-\sqrt{(\beta+2\alpha)^2-\beta^2}\right) < \frac{\lambda}{\alpha+\beta}\left(\beta+2\alpha+\sqrt{(\beta+2\alpha)^2-\beta^2}\right).$$

This is equivalent to, after some simple calculations,

$$(\beta+\alpha)(\beta+2\alpha)^2 < 4\alpha(\beta+3\alpha)^2,$$

or, writing $s = \alpha/\beta$,

$$h(s) = 32s^3 + 16s^2 - s - 1 > 0.$$

Since $h(0) = h'(0) = -1$, the cubic $h(s)$ has a unique positive zero $s_0$ and $h(s)>0$ when $s>s_0$. Since $h\left(\frac{1}{4}\right) > 0 > h\left(\frac{1}{5}\right)$, we conclude that $s_0 \in \left(\frac{1}{4}, \frac{1}{5}\right)$.

Now suppose $\alpha/\beta > s_0$. Then $c_1(\lambda/\alpha) < c_2(0)$ holds and hence we can choose $\varepsilon>0$ small so that $c_1\left(\frac{\lambda}{\alpha}+\varepsilon\right) < c_2(0)$. We now choose $c>0$ such that

$$c_1\left(\frac{\lambda}{\alpha}+\varepsilon\right) < \mu c < c_2(0).$$

Then

$$c_1(u) \leqslant c_1\left(\frac{\lambda}{\alpha}+\varepsilon\right) < \mu c < c_2(0) \leqslant c_2(u) \quad \forall u \in \left[0, \frac{\lambda}{\alpha}+\varepsilon\right].$$

Therefore, for this choice of $c$, (2.3) holds for $u \in [0, \frac{\lambda}{\alpha}+\varepsilon]$. It follows that

$$Z(u,v) := V_u^* f + V_v^* g \leqslant 0 \quad \forall u \in \left[0, \frac{\lambda}{\alpha}+\varepsilon,\right]$$

and equality holds only if $(u,v) = (u^*, v^*)$.

Define

$$W^*(t) = \int_\Omega V^*(u(x,t), v(x,t))\,dx.$$

Then

$$\frac{d}{dt}W^*(t) = \int_\Omega \left( -d_1 \frac{2u^*}{u^3}|\nabla u|^2 - cd_2 \frac{v^*}{v^2}|\nabla v|^2 + Z(u,v) \right) dx.$$

Since $u(x,t) < U(x,t) \to \lambda/\alpha$, we can find $T > 0$ such that $u(x,t) \leqslant \frac{\lambda}{\alpha} + \varepsilon$ for $t > T$. Thus,

$$\frac{d}{dt}W^*(t) \leqslant 0 \quad \text{for } t > T \text{ and equality holds only if } (u,v) \equiv (u^*, v^*).$$

This proves the following result.

**Theorem 2.2.** *Suppose* $\alpha/\beta > s_0 \in \left(\frac{1}{4}, \frac{1}{5}\right)$, *then* $(u^*, v^*)$ *attracts every positive solution of* (2.1).

**Remark 2.3.** We conjecture that the conclusion of Theorem 2.2 is valid for all positive constants $\alpha$ and $\beta$.

## 3. The heterogeneous case

As this section is rather long, we divide it into three subsections. Section 3.1 gives some general results which are obtained from rather standard methods. Section 3.2 considers a degenerate case and reveals that the degeneracy can cause the system to undergo an essential change of behavior. This is used in Section 3.3 to construct solutions with prescribed patterns.

### *3.1. Some general results*

We now consider the case that all the coefficients in (1.1) are continuous positive functions on $\bar{\Omega}$. Given any continuous positive function pair $(u_0(x), v_0(x))$ over $\bar{\Omega}$, let $(u(x,t), v(x,t))$ be the unique solution of (1.1) satisfying

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)).$$

Standard theory of parabolic equations implies that the solution exists as long as it is bounded (in the $L^\infty$-norm, for example). A simple comparison argument shows that the solution remains positive and $0 < u(x,t) < U(x,t)$ for $t > 0$, $x \in \Omega$, where $U$ is the unique solution to

$$\begin{cases} U_t - d_1 \Delta U = \lambda U - \alpha U^2 & \text{in } \Omega \times (0, \infty), \\ U_\nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ U(x,0) = u_0(x). \end{cases} \tag{3.1}$$

From well-known results on the logistic model, we know that

$$U(x,t) \to U^*(x) \quad \text{as } t \to \infty \text{ uniformly in } x, \tag{3.2}$$

where $U^*$ is the unique positive steady-state of (3.1). By the maximum principle we have $U^*(x) > 0$ on $\bar{\Omega}$.

If we denote by $V(x,t)$ the unique solution of

$$\begin{cases} V_t - d_2 \Delta V = \mu V \left(1 - \delta \dfrac{V}{U}\right) & \text{in } \Omega \times (0, \infty), \\ V_\nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ V(x,0) = v_0(x), \end{cases}$$

we find from the comparison principle that $0 < v(x,t) < V(x,t)$ for $t > 0$ and $x \in \Omega$. Moreover, using (3.2), one easily shows that $V(x,t) \to V^*(x)$ as $t \to \infty$ uniformly in $x$, where $V^*$ is the unique positive solution of

$$-d_2 \Delta V = \mu V \left(1 - \delta \frac{V}{U^*}\right) \quad \text{in} \quad \Omega, \;\; V_\nu|_{\partial\Omega} = 0.$$

Therefore, we have

$$\overline{\lim_{t \to \infty}}\; u(x,t) \leqslant U^*(x), \quad \overline{\lim_{t \to \infty}}\; v(x,t) \leqslant V^*(x). \tag{3.3}$$

Unfortunately, we are not able to go much further from (3.3) about the long-time behavior of (1.1). From now on, we will mainly consider the positive steady-state of (1.1). We will obtain existence and some interesting spatial properties for the positive steady-states under suitable assumptions of the coefficient functions. This is based on various elliptic estimates, topological degree theory and the use of boundary blow-up solutions. We suspect that (1.1) has a unique positive steady-state which attracts every positive solution as $t \to \infty$.

It turns out that the spatial behavior of the steady-states is very sensitive to $\alpha(x)$ being small. To simplify the mathematical presentation, we will from now on assume that all the coefficient functions are positive constants, except $\alpha$, which is a nonconstant function of $x$. As we are concerned with steady-states only, we need only study the positive solutions of the elliptic system

$$\begin{cases} -d_1\Delta u = \lambda u - \alpha(x)u^2 - \beta uv, & x\in\Omega, \\ -d_2\Delta v = \mu v\left(1 - \delta\dfrac{v}{u}\right), & x\in\Omega, \\ u_\nu = v_\nu = 0, & x\in\partial\Omega. \end{cases} \tag{3.4}$$

By some simple change of scales, (3.4) can be reduced to the following simpler form:

$$\begin{cases} -\Delta u = \lambda u - \alpha(x)u^2 - \beta uv, & x\in\Omega, \\ -\Delta v = \mu v\left(1 - \dfrac{v}{u}\right), & x\in\Omega, \\ u_\nu = v_\nu = 0, & x\in\partial\Omega. \end{cases} \tag{3.5}$$

We would like to remark that our techniques in the rest of this paper work as well without these simplifications, but using form (3.5) greatly simplifies the notations in our later discussions.

Let us recall that in (3.5), $\lambda, \mu, \beta$ are positive constants, and $\alpha(x)$ is a continuous positive function over $\bar{\Omega}$.

**Theorem 3.1.** *Problem* (3.5) *always has a positive solution.*

**Proof.** We will use a continuation and topological degree argument. Let $(u, v)$ be an arbitrary positive solution of the following problem with parameter $t\in[0,1]$:

$$\begin{cases} -\Delta u = \lambda u - \alpha(x)u^2 - t\beta uv, & x\in\Omega, \\ -\Delta v = \mu v\left(1 - \dfrac{v}{u}\right), & x\in\Omega, \\ u_\nu = v_\nu = 0, & x\in\partial\Omega. \end{cases} \tag{3.6}$$

A simple comparison argument shows

$$0 < u \leqslant u_\lambda^*, \quad 0 < v \leqslant ||u||_\infty \leqslant ||u_\lambda^*||_\infty,$$

where $u_\lambda^*$ denotes the unique positive solution of

$$-\Delta u = \lambda u - \alpha(x)u^2 \quad \text{in } \Omega, \ \ u_\nu|_{\partial\Omega} = 0.$$

By standard elliptic regularity, $u, v\in W^{2,p}(\Omega)\ \forall p > 1$. Hence, $u, v\in C^1(\bar{\Omega})$.

We now want to apply Lemma 2.1 of [LN] to obtain a positive lower bound for $u$ and $v$. But this requires $u, v\in C^2(\Omega)\cap C^1(\bar{\Omega})$. Since $\alpha(x)$ is only assumed to be continuous, we do not have enough regularity for $u$ in general (see [GT]). To overcome this difficulty, we first prove the result of Theorem 3.1 for $\alpha$ smooth, say in $C^1(\bar{\Omega})$. Under this extra assumption, by standard elliptic regularity, we find that $u, v\in C^2(\bar{\Omega})$. If $u(x_0) = \min_{\bar{\Omega}} u, v(y_0) = \min_{\bar{\Omega}} v$, then we can apply Lemma 2.1 of [LN] to conclude that

$$\lambda - \alpha(x_0)u(x_0) - t\beta v(x_0) \leqslant 0, \qquad 1 - \frac{v(y_0)}{u(y_0)} \leqslant 0.$$

It follows that

$$||\alpha||_\infty \min_{\bar{\Omega}} u + \beta||v||_\infty \geqslant \lambda, \quad \min_{\bar{\Omega}} v \geqslant \min_{\bar{\Omega}} u.$$

From the equation for $u$ we obtain,

$$-\Delta u = b(x)u \quad \text{in } \Omega, \quad u_\nu|_{\partial\Omega} = 0,$$

with $b(x) = \lambda - \alpha(x)u(x) - t\beta v(x)$ satisfying

$$||b||_\infty \leqslant \lambda + ||\alpha||_\infty ||u||_\infty + \beta||v||_\infty \leqslant \lambda + (||\alpha||_\infty + \beta)||u_\lambda^*||_\infty.$$

By the Harnack inequality in [LNT], we can find a constant $c_\lambda > 0$ such that,

$$\min_{\bar{\Omega}} u \geqslant c_\lambda \max_{\bar{\Omega}} u.$$

Therefore,

$$\begin{aligned}\lambda &\leqslant ||\alpha||_\infty \min_{\bar{\Omega}} u + \beta||v||_\infty \leqslant ||\alpha||_\infty \min_{\bar{\Omega}} u + \beta||u||_\infty \\ &\leqslant (||\alpha||_\infty + \beta c_\lambda^{-1}) \min_{\bar{\Omega}} u.\end{aligned}$$

It follows that

$$\min_{\bar{\Omega}} u \geqslant \lambda(||\alpha||_\infty + \beta c_\lambda^{-1})^{-1}.$$

Let us now define

$$O = O_\lambda = \{(u,v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : m_\lambda < u < M_\lambda, \ m_\lambda < v < M_\lambda\},$$

where

$$m_\lambda = (\lambda/2)(||\alpha||_\infty + \beta c_\lambda^{-1})^{-1}, \qquad M_\lambda = 2||u_\lambda^*||_\infty.$$

We find from the above discussion that for any $t \in [0,1]$, (3.6) has no solution $(u,v)$ in $\partial O$. When $t = 0$, (3.6) reduces to

$$\begin{cases} -\Delta u = \lambda u - \alpha(x)u^2, & x \in \Omega, \\ -\Delta v = \mu v\left(1 - \dfrac{v}{u}\right), & x \in \Omega, \\ u_\nu = v_\nu = 0, & x \in \partial\Omega, \end{cases} \tag{3.7}$$

which has a unique positive solution in $O$, namely, $(u, v) = (u^*, v^*)$, where $u^* = u^*_\lambda$ and $v^*$ is the unique positive solution of

$$-\Delta v = \mu v(1 - v/u^*) \quad \text{in } \Omega, \;\; v_\nu|_{\partial\Omega} = 0.$$

We now apply a degree argument to show that (3.5) has a positive solution. Denote

$$f(t, u, v) = u + u(\lambda - \alpha u - t\beta v), \quad g(u, v) = v + \mu v(1 - v/u)$$

and let $L = (-\Delta + I)^{-1}$ under Neumann boundary conditions. Define

$$A(t, u, v) = (Lf(t, u, v), Lg(u, v)).$$

It is easily checked by standard method that $A$ is completely continuous from $[0, 1] \times O$ to $C(\bar{\Omega}) \times C(\bar{\Omega})$, and $(u, v) \in O$ solves (3.6) if and only if it satisfies $(u, v) = A(t, u, v)$. By the choice of $O$, we have

$$(u, v) \neq A(t, u, v) \quad \forall t \in [0, 1] \quad \forall (u, v) \in \partial O.$$

Therefore, the topological degree $\deg(I - A(t, \cdot), O, 0)$ is well-defined and is independent of $t \in [0, 1]$. Since $(u^*, v^*)$ is the only fixed point of $A(0, \cdot)$ in $O$, we deduce

$$\deg(I - A(0, \cdot), O, 0) = \text{index } (I - A(0, \cdot), (u^*, v^*)).$$

A simple linearization analysis shows that $(u^*, v^*)$ is nondegenerate and linearly stable as a solution of (3.7). By the well-known Leray–Schauder formula, this yields

$$\text{index}(I - A(0, \cdot), (u^*, v^*)) = 1$$

Therefore, $\deg(I - A(1, \cdot), O, 0) = 1$. By the properties of the degree, $A(1, \cdot)$ has a fixed point in $O$, i.e., (3.5) has a positive solution in $O$. This proves our theorem under the extra assumption that $\alpha$ is $C^1$.

If $\alpha$ is only continuous, we can find a sequence of $C^1$ functions $\alpha_n$ converging to $\alpha$ in the $L^\infty$-norm. By the above discussion, for each $n$, we can find a positive solution $(u_n, v_n)$ of (3.5) with $\alpha$ replaced by $\alpha_n$. Moreover, an inspection of our above proof leading to the a priori estimates for $(u, v)$ shows that we can find $0 < m < M < \infty$ independent of $n$ such that

$$m < u_n < M, \quad m < v_n < M \quad \forall n.$$

It follows that $u_n$ and $v_n$ are bounded in $W^{2,p}(\Omega)$ $\forall p > 1$. Hence we can find a subsequence of $(u_n, v_n)$ that converges in $C^1(\bar{\Omega})$ to some $(u, v)$ which is a positive solution of (3.5). This finishes the proof. □

In the case that the space dimension is one, i.e., $\Omega$ is a finite interval, we can use existing results to show that (3.5) has at most one positive solution.

**Theorem 3.2.** *Suppose that $\Omega$ is a finite interval, and that $\alpha(x)$ is a nonnegative continuous function that is not identically zero on $\Omega$. Then* (3.5) *has at most one positive solution. Moreover, any positive solution $(u_0, v_0)$ of* (3.5), *if exists, is nondegenerate, i.e., zero is not an eigenvalue of the linearized eigenvalue problem of* (3.5) *at $(u_0, v_0)$.*

**Proof.** Suppose that, apart from $(u_0, v_0)$, (3.5) has another positive solution $(u_1, v_1)$. Then it is easily checked that $(U, V) = (u_1 - u_0, v_1 - v_0)$ satisfies an equation of the form (1.1) in [LP] with $\tau = 0$. By Theorem 3.1 in [LP], we deduce $(U, V) = (0, 0)$. This proves the uniqueness part of the theorem. If zero is an eigenvalue of the linearized eigenvalue problem of (3.5) at some positive solution $(u_0, v_0)$, and $(\phi, \psi)$ the corresponding eigenfunction, then $(\phi, \psi)$ also satisfies an equation of the form (1.1) in [LP] with $\tau = 0$. By the same result of [LP], we deduce $(\phi, \psi) = (0, 0)$. This contradiction proves the nondegeneracy of $(u_0, v_0)$. $\square$

**Remark 3.3.** (i) Theorem 3.2 can also be proved by using the method of [H].

(ii) By the implicit function theorem, the nondegeneracy of the unique positive solution of (3.5) implies that it depends continuously on all the parameters in the problem.

### 3.2. *Effects of a degeneracy*

In this subsection, we consider the effect on (3.5) when $\alpha(x)$ is allowed to vanish on some parts of $\Omega$. More precisely, we assume throughout this subsection that $\alpha(x) = 0$ on $\bar{D} \subset \Omega$ and $\alpha(x) > 0$ on $\bar{\Omega} \backslash \bar{D}$, where $D = \bigcup_{j=1}^{m} D_j$, $D_1, \ldots, D_m$ are connected open sets with smooth boundaries and $\bar{D}_i \cap \bar{D}_j = \emptyset$ when $i \neq j$. We assume that the subscripts in $D_j$ are chosen so that

$$\lambda_1^{D_1} \leqslant \lambda_1^{D_2} \leqslant \cdots \leqslant \lambda_1^{D_m},$$

where $\lambda_1^{D_j}$ denotes the first eigenvalue of $-\Delta$ over $D_j$ under Dirichlet boundary conditions. We will reveal a crucial difference of the behavior of (3.5) between the case $\lambda < \lambda_1^{D_1}$ and the case $\lambda > \lambda_1^{D_1}$.

Let us first observe that if $\lambda \in (0, \lambda_1^{D_1})$, then by the main result in [O] (see also [FKLM]), the problem

$$-\Delta u = \lambda u - \alpha(x)u^2 \quad \text{in } \Omega, \; u_\nu|_{\partial\Omega} = 0,$$

still has a unique positive solution $u_\lambda^*$. It follows that the proof of Theorem 3.1 carries over to the present degenerate case. Therefore we have the following result.

**Theorem 3.4.** *If $\lambda \in (0, \lambda_1^{D_1})$, then* (3.5) *has a positive solution for every $\mu > 0$ and $\beta > 0$.*

In terms of the existence of positive solutions of (3.5), in view of Theorem 3.1 where $\alpha(x)$ is positive on $\Omega$, Theorem 3.4 suggests that the vanishing of $\alpha(x)$ on $D$ does not cause essential changes to the behavior of (3.5) in the case $\lambda < \lambda_1^{D_1}$; indeed, in either theorems, the existence of a positive solution is guaranteed for every $\mu > 0$ and $\beta > 0$.

In sharp contrast, we will show in the following that this is no longer the case once $\lambda > \lambda_1^{D_1}$. In fact, for any fixed $\lambda$ in this range, we will prove that for each $\mu \in (0, \lambda_1^{D_1})$, there exists a $\beta_{\lambda,\mu} > 0$ so that (3.5) has no positive solution when $0 < \beta < \beta_{\lambda,\mu}$. This implies that the dynamics of the model undergoes some deep changes when the value of $\lambda$ crosses $\lambda_1^{D_1}$.

For convenience of notation, we make the convention that $\lambda_1^{D_{m+1}} = \infty$. Let us fix $\mu \in (0, \lambda_1^{D_1})$ and suppose $\lambda \in (\lambda_1^{D_k}, \lambda_1^{D_{k+1}})$ for some $1 \leqslant k \leqslant m$. By Lemma 2.6 in [DL], for $\lambda$ in this range, the boundary blow-up problem

$$-\Delta u = \lambda u - \alpha(x)u^2 \quad \text{in } \Omega \backslash \left( \bigcup_{j=1}^{k} \bar{D}_j \right), \quad u_\nu|_{\partial\Omega} = 0, \; u|_{\bigcup_{j=1}^{k} (\partial D_j)} = \infty \tag{3.8}$$

has a minimal positive solution $U_\lambda$. Applying Lemma 2.3 in [DL], we find that if $(u, v)$ is a positive solution of (3.5), then

$$u(x) \leqslant U_\lambda(x) \quad \forall x \in \Omega \backslash \left( \bigcup_{j=1}^{k} \bar{D}_j \right).$$

Define

$$\alpha_\lambda(x) = \begin{cases} 0, & x \in \bigcup_{j=1}^{k} \bar{D}_j, \\ 1/U_\lambda(x), & x \in \Omega \backslash \left( \bigcup_{j=1}^{k} \bar{D}_j \right). \end{cases}$$

Clearly $\alpha_\lambda$ is continuous on $\bar{\Omega}$ and $\alpha_\lambda > 0$ on $\Omega \backslash \left( \bigcup_{j=1}^{k} \bar{D}_j \right)$. By our choice of $\mu$ and the main result of [O], the problem

$$-\Delta V = \mu V(1 - \alpha_\lambda(x)V) \quad \text{in } \Omega, \quad V_\nu|_{\partial\Omega} = 0 \tag{3.9}$$

has a unique positive solution $V_\lambda$.

We want to show that $v \leqslant V_\lambda$ if $(u, v)$ is a positive solution of (3.5). Indeed, we already know that $u(x) \leqslant U_\lambda(x)$ on $\Omega \backslash \left( \bigcup_{j=1}^{k} \bar{D}_j \right)$. Hence

$$1/u(x) \geqslant \alpha_\lambda(x) \quad \forall x \in \Omega.$$

It follows that

$$-\Delta v = \mu v(1 - v/u) \leqslant \mu v(1 - \alpha_\lambda(x)v) \quad \text{in } \Omega.$$

Thus, $v$ is a lower solution of (3.9). It is easily checked that for any constant $M>1$, $MV_\lambda$ is an upper solution of (3.9), and $MV_\lambda > v$ if $M$ is large enough. Therefore, $v \leqslant V_\lambda \leqslant MV_\lambda$ in $\Omega$.

Let us introduce some notations for our discussions to follow. We will use $\lambda_1^{\omega,N}(\phi)$ and $\lambda_1^{\omega}(\phi)$ to denote the first eigenvalues of the operator $-\Delta+\phi$ over $\omega$ under Neumann and Dirichlet boundary conditions, respectively. It is well known that

$$\lambda_1^{\omega,N}(\phi) < \lambda_1^{\omega}(\phi)$$

and both $\lambda_1^{\omega,N}(\phi)$ and $\lambda_1^{\omega}(\phi)$ are increasing with $\phi$ in the following sense:

$$\phi_1 \leqslant \phi_2 \text{ and } \phi_1 \not\equiv \phi_2 \text{ imply } \lambda_1^{\omega,N}(\phi_1) < \lambda_1^{\omega,N}(\phi_2),\ \lambda_1^{\omega}(\phi_1) < \lambda_1^{\omega}(\phi_2).$$

If $(u,v)$ is a positive solution of (3.5), then from the equation for $u$ we obtain

$$\lambda = \lambda_1^{\Omega,N}(\alpha u + \beta v) < \lambda_1^{\Omega}(\alpha u + \beta v) < \lambda_1^{D_i}(\alpha u + \beta v) = \lambda_1^{D_i}(\beta v).$$

Since $v \leqslant V_\lambda$, we obtain

$$\lambda < \lambda_1^{D_i}(\beta V_\lambda), \quad i = 1, \dots, m. \tag{3.10}$$

From well-known properties of principle eigenvalues, we see that $f_i(\beta) = \lambda_1^{D_i}(\beta V_\lambda)$ is a continuous, strictly increasing function of $\beta$, and $f_i(0) = \lambda_1^{D_i}, f_i(\infty) = \infty$. Since $\lambda > \lambda_1^{D_j}$ for $j = 1, \dots, k$, we can find a unique $\beta_j = \beta_j(\lambda) > 0$ such that $f_j(\beta_j) = \lambda$. Therefore,

$$\lambda = \lambda_1^{D_j}(\beta_j V_\lambda), \quad \lambda \geqslant \lambda_1^{D_j}(\beta V_\lambda) \quad \forall \beta \leqslant \beta_j,\ j = 1, \dots, k. \tag{3.11}$$

Comparing (3.11) with (3.10), we immediately obtain the following result.

**Theorem 3.5.** *Suppose $\mu \in (0, \lambda_1^{D_1})$ and $\lambda \in (\lambda_1^{D_k}, \lambda_1^{D_{k+1}})$ for some $1 \leqslant k \leqslant m$. Let $\beta_1, \dots, \beta_k$ be as in* (3.11). *Then* (3.5) *has no positive solution if $0 < \beta \leqslant \max\{\beta_1, \dots, \beta_k\}$.*

The restriction that $\mu \in (0, \lambda_1^{D_1})$ in Theorem 3.5 can be relaxed. Indeed, if we assume $\mu \in (0, \lambda_1^{D_k})$ instead, then by replacing $V_\lambda$ by the minimal positive solution of (3.9) on $\Omega \setminus \left(\bigcup_{j=1}^{k-1} D_j\right)$ with boundary conditions $V_v|_{\partial\Omega} = 0,\ V|_{\bigcup_{j=1}^{k-1} \partial D_j} = \infty$, then it is easy to show that there exists some $\tilde{\beta}_k > 0$ such that (3.5) has no positive solution if $0 < \beta < \tilde{\beta}_k$.

Our next result shows that even if $\lambda > \lambda_1^{D_k}$ for some $k \in \{1, \dots, m\}$, (3.5) can still have a positive solution for every $\beta > 0$ if $\mu$ is large enough; precisely, if $\mu > \max\{\lambda_1^{D_m}, \lambda\}$. Thus, existence of a positive solution is regained when $\mu$ becomes large.

**Theorem 3.6.** *Suppose that* $\mu > \lambda_1^{D_m}$, *then* (3.5) *has a positive solution for every* $\lambda \in (0, \mu)$ *and* $\beta > 0$.

The proof of Theorem 3.6 relies on the following a priori estimates.

**Lemma 3.7.** *Suppose that* $\alpha_n$ *is a sequence in* $C(\bar{\Omega})$ *that converges to* $\alpha$ *in this space, and* $\alpha_n = 0$ *on* $D$. *Let* $\mu > \lambda_1^{D_m}$ *be fixed and* $\lambda_n \in [m, M] \subset (0, \mu)$. *Then there exists a positive constant* $C$ *independent of* $n$ *such that any positive solution* $(u_n, v_n)$ *of* (3.5) *with* $(\lambda, \alpha)$ *replaced by* $(\lambda_n, \alpha_n)$ *satisfies*

$$||u_n||_\infty + ||v_n||_\infty < C.$$

**Proof.** Suppose that the conclusion of the lemma is not true. Then we can find a sequence of positive function pairs $(u_n, v_n)$ satisfying

$$\begin{cases} -\Delta u_n = \lambda_n u_n - \alpha_n(x) u_n^2 - \beta u_n v_n, & x \in \Omega, \\ -\Delta v_n = \mu v_n \left(1 - \dfrac{v_n}{u_n}\right), & x \in \Omega, \\ (u_n)_v = (v_n)_v = 0, & x \in \partial\Omega, \end{cases} \tag{3.12}$$

such that $||u_n||_\infty + ||v_n||_\infty \to \infty$ as $n \to \infty$.

We necessarily have $||u_n||_\infty \to \infty$ since $v_n \leqslant ||u_n||_\infty$. Denote

$$\hat{u}_n = u_n / ||u_n||_\infty, \quad \hat{v}_n = v_n / ||v_n||_\infty.$$

We have

$$-\Delta \hat{u}_n \leqslant M \hat{u}_n, \quad -\Delta \hat{v}_n \leqslant \mu \hat{v}_n.$$

Therefore,

$$\int_\Omega |\nabla \hat{u}_n|^2 + \int_\Omega \hat{u}_n^2 \leqslant (M+1) \int_\Omega \hat{u}_n^2 \leqslant (M+1)|\Omega|.$$

This implies that $\{\hat{u}_n\}$ is a bounded sequence in $H^1(\Omega)$. Therefore, subject to a subsequence, $\hat{u}_n$ converges to some $\hat{u} \in H^1(\Omega)$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Since $\hat{u}_n$ has $L^\infty$-norm 1, we also have $\hat{u}_n \to \hat{u}$ in $L^p(\Omega)$ $\forall p > 1$. Clearly $0 \leqslant \hat{u} \leqslant 1$. We claim that $\hat{u}$ is not the zero function in $H^1(\Omega)$. Assume on the contrary that $\hat{u} = 0$. Then from $\hat{u}_n \to 0$ in $L^p$ for every $p > 1$ we deduce $(-\Delta + I)^{-1} \hat{u}_n \to 0$ in $C^1(\bar{\Omega})$. But from $(-\Delta + I)\hat{u}_n \leqslant (M+1)\hat{u}_n$ we deduce

$$0 \leqslant \hat{u}_n \leqslant (M+1)(-\Delta + I)^{-1} \hat{u}_n.$$

It follows that $\hat{u}_n \to 0$ in $L^\infty(\Omega)$, contradicting the fact that $||\hat{u}_n||_\infty = 1$. Therefore, $\hat{u} \neq 0$.

Similarly, subject to a subsequence, $\hat{v}_n \to \hat{v}$ weakly in $H^1(\Omega)$, strongly in $L^p(\Omega)$ $\forall p > 1$, and $\hat{v} \neq 0$.

By passing to a further subsequence, we have two cases to consider:

(i) $\{||v_n||_\infty\}$ is bounded, (ii) $||v_n||_\infty \to \infty$.

In case (i), we may assume that $\lambda_n \to \lambda^*$ and $\beta||v_n||_\infty \to \xi$. From the equation for $u_n$ we obtain

$$-\Delta \hat{u}_n = \lambda_n \hat{u}_n - \alpha_n u_n \hat{u}_n - \beta||v_n||_\infty \hat{u}_n \hat{v}_n.$$

Multiply this equation by a smooth function $\phi$ whose support is in $D$, and integrate by parts. It results

$$\int_D \nabla \hat{u}_n \cdot \nabla \phi = \lambda_n \int_D \hat{u}_n \phi - \beta||v_n||_\infty \int_D \hat{u}_n \hat{v}_n \phi. \tag{3.13}$$

Letting $n \to \infty$ we obtain

$$\int_D \nabla \hat{u} \cdot \nabla \phi = \int_D (\lambda^* - \xi \hat{v}) \hat{u} \phi.$$

This implies that $\hat{u}$ satisfies in the weak sense

$$-\Delta u = (\lambda^* - \xi \hat{v}) u \quad \text{in } D. \tag{3.14}$$

For all small positive $\sigma$, the set $D_\sigma = \{x \in \Omega : d(x, D) < \sigma\}$ has the same property as $D$, namely, it has $m$ disjoint components each with smooth boundary, and $\bar{D}_\sigma \subset \Omega$. By our assumption, for all large $n$, $\alpha_n > \alpha/2 > 0$ on $\bar{\Omega} \backslash D_\delta$. By [DH], the problem

$$-\Delta u = \lambda u - [\alpha(x)/2] u^2 \quad \text{in } \Omega \backslash \bar{D}_\sigma, \;\; u|_{\partial D_\sigma} = \infty, \;\; u_\nu|_{\partial\Omega} = 0$$

has a minimal positive solution $U_{\lambda,\sigma}$. We need to be more precise here as $\Omega \backslash \overline{D_\sigma}$ might have several components; the number of components must be finite due to the smoothness of the boundary of $D_\sigma$. If this is the case, then the restriction of $U_{\lambda,\sigma}$ on each component is understood to be the minimal positive solution on that component. This case was not stated clearly in [DH] thought it could as well happen there.

By Lemma 2.3 in [DL], we find that $u_n \leqslant U_{M,\sigma}$ on $\Omega \backslash \bar{D}_\sigma$. Therefore, we must have $\hat{u} = 0$ a.e. in $\Omega \backslash \bar{D}_\sigma$. As $\sigma > 0$ can be arbitrarily small, this implies that $\hat{u} = 0$ on $\Omega \backslash \bar{D}$. Since $D$ has smooth boundary, this implies that $\hat{u}|_D \in H_0^1(D)$. Recalling that $\hat{u} \neq 0$ we find that $\hat{u}|_D \in H_0^1(D)$ is a nontrivial nonnegative solution of (3.14). As $(\lambda^* - \xi \hat{v}) \in L^\infty(\Omega)$, by Harnack's inequality we infer that $\hat{u} > 0$ or identically zero on each component of $D$. Therefore, there exists some $D_j$ such that $\hat{u}|_{D_j} > 0$ on $D_j$ and is a weak solution to

$$-\Delta u = (\lambda^* - \xi \hat{v}) u \quad \text{in } D_j, \;\; u|_{\partial D_j} = 0.$$

By standard elliptic regularity, we find that $\hat{u}|_{D_j} \in C^1(\bar{D}_j)$.

Since $\mu > \lambda_1^{D_m} \geqslant \lambda_1^{D_j}$, we can find $\delta > 0$ very small so that $\mu > \lambda_1^{D_j^\delta}$, where

$$D_j^\delta = \{x \in D_j : d(x, \partial D_j) > \delta\}.$$

On $D_j$, $\hat{u}_n$ satisfies

$$-\Delta \hat{u}_n = \lambda_n \hat{u}_n - \beta v_n \hat{u}_n$$

whose right-hand side has an $L^\infty$ bound independent of $n$. By standard interior estimate for elliptic equations, we deduce that $\{\hat{u}_n|_{D_j^\delta}\}$ is a precompact set in $C^1(\overline{D_j^\delta})$. Therefore, $\hat{u}_n|_{D_j^\delta} \to \hat{u}$ in $C^1(\overline{D_j^\delta})$. In particular, for all large $n$, $\hat{u}_n > \hat{u}/2$ on $D_j^\delta$.

Since $\mu > \lambda_1^{D_j^\delta}$, it is well known that the logistic equation

$$-\Delta V = \mu V(1 - (2/\hat{u})V) \quad \text{in } D_j^\delta, \;\; V|_{\partial D_j^\delta} = 0$$

has a unique positive solution $V_0$. From the equation for $v_n$, we find that $v_n/||u_n||_\infty$ satisfies, for all large $n$,

$$-\Delta v = \mu v(1 - v/\hat{u}_n) \geqslant \mu v(1 - (2/\hat{u})v) \quad \text{in } D_j^\delta.$$

Therefore, we can apply Lemma 2.1 in [DM] to conclude that $v_n/||u_n||_\infty \geqslant V_0$ on $D_j^\delta$ for all large $n$. As $||u_n||_\infty \to \infty$, clearly this implies that $||v_n||_\infty \to \infty$, contradicting our assumption that we are in case (i). Therefore, case (i) cannot happen.

Suppose now case (ii) happens. We can still have (3.13). Divide this identity by $\beta||v_n||_\infty$ and let $n \to \infty$. We deduce

$$\int_D \hat{u}\hat{v}\phi = 0.$$

This implies that

$$\hat{u}\hat{v} = 0 \quad \text{on } D. \tag{3.15}$$

Since the problem

$$-\Delta v + \left(\mu \frac{v_n}{u_n}\right) v = \mu v \quad \text{in } \Omega, \;\; v_v|_{\partial\Omega} = 0$$

has a positive solution $v = v_n$, $\mu$ must be the first eigenvalue of the differential operator $(-\Delta + \mu v_n/u_n)$ on $\Omega$ with Neumann boundary conditions. It follows from the variational characterization of the first eigenvalue that,

$$\int_\Omega |\nabla\phi|^2 + (\mu v_n/u_n)\phi^2 \geqslant \mu \int_\Omega \phi^2 \quad \forall \phi \in H^1(\Omega).$$

Taking $\phi = u_n$, we obtain

$$\int_\Omega |\nabla u_n|^2 + \mu u_n v_n \geqslant \mu \int_\Omega u_n^2.$$

Therefore,

$$\int_\Omega |\nabla \hat{u}_n|^2 + (\mu ||v_n||_\infty / ||u_n||_\infty) \int_\Omega \hat{u}_n \hat{v}_n \geqslant \mu \int_\Omega \hat{u}_n^2. \tag{3.16}$$

Recall that we always have $v_n \leqslant ||u_n||_\infty$. Therefore, by (3.15),

$$0 \leqslant (\mu ||v_n||_\infty / ||u_n||_\infty) \int_\Omega \hat{u}_n \hat{v}_n \leqslant \mu \int_\Omega \hat{u}_n \hat{v}_n \to \mu \int_\Omega \hat{u}\hat{v} = \mu \int_{\Omega \setminus D} \hat{u}\hat{v}.$$

But as before we always have $\hat{u} = 0$ on $\Omega \setminus D$. Therefore,

$$(\mu ||v_n||_\infty / ||u_n||_\infty) \int_\Omega \hat{u}_n \hat{v}_n \to 0$$

as $n \to \infty$. We now let $n \to \infty$ in (3.16) and obtain

$$\varliminf_{n\to\infty} \int_\Omega |\nabla \hat{u}|^2 \geqslant \mu \int_\Omega \hat{u}^2.$$

On the other hand, from the equation for $u_n$ in (3.12) and the assumption $\lambda_n \leqslant M < \mu$, we obtain

$$\varlimsup_{n\to\infty} \int_\Omega |\nabla \hat{u}|^2 \leqslant M \int_\Omega \hat{u}^2 < \mu \int_\Omega \hat{u}^2.$$

This contradiction shows case (ii) cannot happen either. This finishes our proof. □

**Proof of Theorem 3.6.** We first prove the result under the extra condition that $\alpha \in C^1(\bar{\Omega})$. Suppose that $(u, v)$ is a positive solution of (3.5) with $\lambda \in [m, M]$, where $0 < m < \lambda_1^{D_1} < M < \mu$. By Lemma 3.7, there exists $C > 0$ independent of $\lambda$ and $(u, v)$ such that $u < C, v < C$ on $\bar{\Omega}$. By standard elliptic regularity, $u, v \in C^2(\bar{\Omega})$. An inspection of the proof of Theorem 3.1 shows that we can use $||v||_\infty \leqslant ||u||_\infty < C$ and the same results of [LNT,LN] to obtain a positive lower bound for $u$ and $v$ that is independent of $\lambda \in [m, M]$ and $(u, v)$, say

$$u(x) > c, \quad v(x) > c \quad \forall x \in \bar{\Omega}.$$

Note that these arguments are not affected by $\alpha(x) = 0$ on $D$.

Now define

$$O = \{(u, v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : c < u < C, c < v < C\}$$

and

$$A(\lambda,u,v)=((-\Delta+I)^{-1}(u+\lambda u-\alpha u^2-\beta uv),(-\Delta+I)^{-1}(v+\mu v-\mu v^2/u)).$$

We find that $\deg(I-A(\lambda,\cdot),O,0)$ is well-defined and independent of $\lambda\in[m,M]$.

At $\lambda=m\in(0,\lambda_1^{D_1})$, by [O] (see also [DL]), the degenerate logistic problem

$$-\Delta u=mu-\alpha(x)u^2 \quad \text{in } \Omega,\ \ u_\nu|_{\partial\Omega}=0$$

has a unique positive solution $u_0$. We now notice that the argument in the proof of Theorem 3.1 can be used for our present case once we replace $u_\lambda^*$ there by $u_0$ here. Therefore, there exist $0<m_0<M_0$ such that every possible positive solution of (3.5) with $\lambda=m$ belongs to $O_0$, where

$$O_0=\{(u,v)\in C(\bar\Omega)\times C(\bar\Omega): m_0<u<M_0, m_0<v<M_0\}$$

and $\deg(I-A(m,\cdot),O_0,0)=1$. Since any possible positive solution of (3.5) with $\lambda=m$ belongs to $O\cap O_0$, by the properties of the degree,

$$\deg(I-A(m,\cdot),O,0)=\deg(I-A(m,\cdot),O_0,0).$$

Therefore,

$$\deg(I-A(\lambda,\cdot),O,0)=1 \quad \forall\lambda\in[m,M].$$

This implies that (3.5) has a positive solution in $O$ for every $\lambda\in[m,M]$.

If $\alpha$ is only in $C(\bar\Omega)$, then we can find a sequence of $C^1$ functions $\alpha_n$ such that $\alpha_n\to\alpha$ in $C(\bar\Omega)$ and $\alpha_n=0$ on $D^n=\bigcup_{j=1}^m D_j^n$, $\alpha_n>0$ on $\bar\Omega\backslash\overline{D^n}$, where each $D_j^n$ is a small neighborhood of $D_j$ and $\overline{D_j^n}\cap\overline{D_i^n}=\emptyset$ when $i\neq j$, $\lambda_1^{D_j^n}\to\lambda_1^{D_j}$ as $n\to\infty$. Therefore, for each fixed large $n$ and $\lambda\in[m,M]$, by what has been proved above, (3.5) with $\alpha$ replaced by $\alpha_n$ has a positive solution $(u_n,v_n)$. By Lemma 3.7, $u_n,v_n$ are uniformly bounded from above by some positive constant $C$. Using this and results in [LNT,LN] as before, we can obtain a uniform positive lower bound for $u_n,v_n$, say $u_n,v_n>c$. Then from elliptic regularity we deduce that $(u_n,v_n)$ has a subsequence that converges to a positive solution of (3.5), as in the proof of Theorem 3.1. Therefore, (3.5) has a positive solution for each $\lambda\in[m,M]$. Since $m>0$ can be arbitrarily small and $M$ can be arbitrarily close to $\mu$, this finishes our proof. $\square$

**Remark 3.8.** We do not know whether (3.5) has a positive solution for every $\beta>0$ when $\lambda\geqslant\mu>\lambda_1^{D_m}$.

The nonexistence result, Theorem 3.5, provides us a chance of constructing positive solutions of (3.5) with prescribed patterns. More precisely, if we perturb the degenerate $\alpha(x)$ in (3.5) by $\alpha(x)+\varepsilon$ with small positive $\varepsilon$, then by Theorem 3.1 we know that the perturbed (3.5) has a positive solution $(u_\varepsilon,v_\varepsilon)$; Theorem 3.5 suggests that if $\lambda,\mu$ and $\beta$ are chosen suitably, then as $\varepsilon\to0$, the function pair $(u_\varepsilon,v_\varepsilon)$ has no finite limit and hence may exhibit sharp spatial patterns. This is indeed the case but we are unable to determine the exact location of the pattern in the general case. To overcome this difficulty, we are led to the study of the behavior of $\beta_j(\lambda)$ as $\lambda \to \infty$. Let us recall that for $\mu \in (0, \lambda_1^{D_1})$ and $\lambda > \lambda_1^{D_m}$, $\beta = \beta_j(\lambda)$ is the unique solution to

$$\lambda = \lambda_1^{D_j}(\beta V_\lambda), \quad j = 1, \ldots, m,$$

where $V_\lambda$ is given by (3.9).

Let us fix $\mu \in (0, \lambda_1^{D_1})$ and define $\tilde{V}_\lambda = \lambda^{-1} V_\lambda$, $\tilde{U}_\lambda = \lambda^{-1} U_\lambda$, where $U_\lambda$ is given by (3.8) with $k = m$. We easily see that $\tilde{U}_\lambda$ satisfies

$$-\Delta U = \lambda U(1 - \alpha(x)U) \quad \text{in } \Omega \backslash \bar{D}, \;\; U|_{\partial D} = \infty, \;\; U_\nu|_{\partial \Omega} = 0, \tag{3.17}$$

and $\tilde{V}_\lambda$ satisfies

$$-\Delta V = \mu V(1 - \tilde{\alpha}_\lambda(x)V) \quad \text{in } \Omega, \;\; V_\nu|_{\partial\Omega} = 0, \tag{3.18}$$

where

$$\tilde{\alpha}_\lambda(x) = \begin{cases} 0, & x \in \bar{D}, \\ 1/\tilde{U}_\lambda(x), & x \in \bar{\Omega} \backslash \bar{D}. \end{cases}$$

**Lemma 3.9.** $\lim_{\lambda \to \infty} \tilde{U}_\lambda(x) = 1/\alpha(x)$ *uniformly on any compact subset of* $\Omega \backslash \bar{D}$.

**Proof.** For arbitrary $\varepsilon > 0$ and $x_0 \in \Omega \backslash \bar{D}$, we can find a small open ball $B_\delta(x_0)$ of radius $\delta$ and center $x_0$ such that $\bar{B}_\delta(x_0) \subset \Omega \backslash \bar{D}$, and

$$|\alpha(x)^{-1} - \alpha(x_0)^{-1}| \leqslant \varepsilon \quad \forall x \in B_\delta(x_0).$$

Denote by $\alpha^*$ and $\alpha_*$, respectively, the maximum and minimum of $\alpha(x)$ on the closure of $B_\delta(x_0)$ and consider the auxiliary problems,

$$-\Delta w = \lambda w(1 - \alpha^* w) \quad \text{in } B_\delta(x_0), \;\; w|_{\partial B_\delta(x_0)} = 0 \tag{3.19}$$

and

$$-\Delta z = \lambda z(1 - \alpha_* z) \quad \text{in } B_\delta(x_0), \;\; z|_{\partial B_\delta(x_0)} = \infty. \tag{3.20}$$

By Lemmas 2.2 and 2.3 of [DM], we find that (3.19) has a unique positive solution $w_\lambda$ for all large $\lambda$ and $w_\lambda(x) \to 1/\alpha^*$ as $\lambda \to \infty$ uniformly on $B_{\delta/2}(x_0)$; (3.20) has a unique positive solution $z_\lambda$ for every $\lambda$ and $z_\lambda(x) \to 1/\alpha_*$ as $\lambda \to \infty$ uniformly on $B_{\delta/2}(x_0)$. By Lemma 2.1 of [DM] (which is valid for $C^1$ functions), we deduce

$$w_\lambda \leqslant \tilde{U}_\lambda \leqslant z_\lambda \quad \text{in } B_\delta(x_0).$$

Therefore, we can find $\lambda_\varepsilon > 0$ sufficiently large so that

$$1/\alpha^* - \varepsilon \leqslant \tilde{U}_\lambda(x) \leqslant 1/\alpha_* + \varepsilon \quad \forall \lambda > \lambda_\varepsilon \ \forall x \in B_{\delta/2}(x_0).$$

This implies that

$$|\tilde{U}_\lambda(x) - 1/\alpha(x)| \leqslant 2\varepsilon \quad \forall \lambda > \lambda_\varepsilon \ \forall x \in B_{\delta/2}(x_0).$$

Hence $\tilde{U}_\lambda(x) \to 1/\alpha(x)$ as $\lambda \to \infty$ uniformly in $B_{\delta/2}(x_0)$. By a standard finite covering argument, this implies that $\tilde{U}_\lambda(x) \to 1/\alpha(x)$ as $\lambda \to \infty$ uniformly on any compact subset of $\Omega \backslash \bar{D}$. $\square$

**Lemma 3.10.** *As $\lambda \to \infty$, $\tilde{\alpha}_\lambda \to \alpha$ uniformly on $\bar{\Omega}$.*

**Proof.** For small $\delta > 0$, let us denote

$$A_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}, \quad B_\delta = \{x \in \Omega \backslash \bar{D} : d(x, \partial D) < \delta\}.$$

Since $\tilde{\alpha}_\lambda = \alpha = 0$ on $\bar{D}$ and by Lemma 3.9, $\tilde{\alpha} \to \alpha$ uniformly on compact subsets of $\Omega \backslash \bar{D}$ as $\lambda \to \infty$, it suffices to show the uniform convergence on $\bar{A}_{\delta_0}$ and $\bar{B}_{\delta_0}$ for some small positive $\delta_0$.

Let us now fix $\delta_0$ small. For any given $\varepsilon > 0$, we can find $\delta \in (0, \delta_0)$ such that $\alpha_\delta := \max_{\bar{B}_\delta} \alpha < \varepsilon/3$. By Lemma 3.9, we can find $\Lambda_1 > 0$ large enough such that, for $\lambda > \Lambda_1$,

$$1/\tilde{U}_\lambda(x) < 2\alpha(x) \leqslant 2\alpha_\delta \quad \forall x \in \partial^1 B_\delta := \{x \in \Omega \backslash \bar{D} : d(x, \partial D) = \delta\}.$$

Therefore, $\tilde{U}_\lambda$ is an upper solution to the problem,

$$-\Delta u = \lambda u(1 - 2\alpha_\delta u) \quad \text{in } B_\delta, \ \ u|_{\partial B_\delta} = 1/(2\alpha_\delta),$$

which has a unique solution $u \equiv 1/(2\alpha_\delta)$. By Lemma 2.1 of [DM], we deduce $\tilde{U}_\lambda \geqslant 1/(2\alpha_\delta)$ in $B_\delta$. Therefore, for $\lambda > \Lambda_1$,

$$|\tilde{\alpha}_\lambda(x) - \alpha(x)| \leqslant \tilde{\alpha}_\lambda(x) + \alpha(x) \leqslant 3\alpha_\delta \leqslant \varepsilon \quad \forall x \in B_\delta.$$

By Lemma 3.9, we can find $\Lambda_2 \geqslant \Lambda_1$ sufficiently large such that,

$$|\tilde{\alpha}_\lambda(x) - \alpha(x)| \leqslant \varepsilon \quad \forall \lambda > \Lambda_2 \ \forall x \in \bar{B}_{\delta_0} \backslash B_\delta.$$

Therefore, $\tilde{\alpha}_\lambda \to \alpha$ as $\lambda \to \infty$ uniformly on $\bar{B}_{\delta_0}$.

It remains to prove the uniform convergence on $\bar{A}_{\delta_0}$. We argue indirectly. Suppose there exist a sequence $x_n \in \bar{A}_{\delta_0}$ and a sequence of increasing numbers $\lambda_n \to \infty$ such that

$$|\tilde{\alpha}_{\lambda_n}(x_n) - \alpha(x_n)| \geqslant \varepsilon_0 > 0.$$

By Lemma 3.9, on $\partial^1 A_{\delta_0} := \partial A_{\delta_0} \backslash \partial\Omega$, $\tilde{U}_{\lambda_n}(x) \to 1/\alpha(x)$ uniformly. If we denote $\alpha_0 = \inf_{A_{\delta_0}} \alpha$ and $\alpha^0 = \sup_{A_{\delta_0}} \alpha$, then we easily see that for all large $\lambda$, $\tilde{U}_\lambda$ is a lower solution to

$$-\Delta u = \lambda u(1 - (\alpha_0/2)u) \quad \text{in } A_{\delta_0}, \ \ u|_{\partial^1 A_{\delta_0}} = 2/\alpha_0, \ \ u_\nu|_{\partial\Omega} = 0,$$

which has a unique solution $u \equiv 2/\alpha_0$. Therefore we can apply Lemma 2.3 of [DL] to conclude that $\tilde{U}_{\lambda_n} \leqslant 2/\alpha_0$ on $\bar{A}_{\delta_0}$ for all large $n$.

A parallel consideration shows $\tilde{U}_{\lambda_n} \geqslant 1/(2\alpha^0)$ on $\bar{A}_{\delta_0}$ for all large $n$. Therefore $\{\tilde{\alpha}_{\lambda_n}(x_n)\}$ is a bounded sequence that has a subsequence converging to a positive constant. Without loss of generality we assume

$$\tilde{\alpha}_{\lambda_n}(x_n) \to \tilde{\alpha}_*, \quad x_n \to x_*, \quad \alpha(x_n) \to \alpha_* = \alpha(x_*).$$

By Lemma 3.9, we necessarily have $x_* \in \partial\Omega$. Our assumption implies that $|\tilde{\alpha}_* - \alpha_*| \geqslant \varepsilon_0 > 0$.

We now define $W_n(x) = \tilde{U}_{\lambda_n}(x_n + \lambda_n^{-1/2} x)$ and apply a standard blowing-up argument. We find that $W_n \to W$ uniformly on any bounded subset of $T$, where $T$ is the entire $R^N$ or a half space of $R^N$, and $W$ satisfies

$$-\Delta W = W - \alpha_* W^2, \ \ 1/(2\alpha^0) \leqslant W \leqslant 2/\alpha_0 \quad \text{in } T, \ \ W(0) = 1/\tilde{\alpha}_* \tag{3.21}$$

and in case $T$ is a half-space, $W_\nu|_{\partial T} = 0$, where $\nu$ is the outward unit normal of $\partial T$.

However, by Theorem 1.2 of [DM], the only positive solution of

$$-\Delta w = w - \alpha_* w^2 \quad \text{in } R^N \tag{3.22}$$

is $w \equiv 1/\alpha_*$ which does not satisfy the last part of (3.21). Therefore $T$ must be a half-space. But the boundary condition on $\partial T$ implies that the even extension of $W$ across $\partial T$, which we denote by $\tilde{W}$, is a positive solution of (3.22). Therefore we must have $\tilde{W} \equiv 1/\alpha_*$ which yields a contradiction to the last part of (3.21) too. This shows that we must have $\tilde{\alpha}_n \to \alpha$ uniformly on $\bar{A}_{\delta_0}$ as $\lambda \to \infty$, as we wanted. The proof is now complete. □

**Lemma 3.11.** *As $\lambda \to \infty$, $\tilde{V}_\lambda \to \tilde{V}$ in $C^1(\bar{\Omega})$, where $\tilde{V}$ is the unique positive solution of*

$$-\Delta V = \mu V(1 - \alpha(x)V) \quad \text{in } \Omega, \ \ V_\nu|_{\partial\Omega} = 0. \tag{3.23}$$

**Proof.** Let $\{\lambda_n\}$ be an arbitrary sequence that converges to $\infty$, and denote $\tilde{V}_n = \tilde{V}_{\lambda_n}$. Since $\mu < \lambda_1^{D_1}$, we can find a small $\delta > 0$ such that $D_j^\delta := \{x \in R^N : d(x, D_j) < \delta\}$ has smooth boundary, $\overline{D_j^\delta} \cap \overline{D_i^\delta} = \emptyset$ for $i \neq j$, and $\mu < \lambda_1^{D_j^\delta}$ for $j = 1, \ldots, m$.

Let $\alpha^\delta(x)$ be a continuous function such that $\alpha^\delta(x)=0$ on $D^\delta := \bigcup_{j=1}^m D_j^\delta$, and $0<\alpha^\delta(x)<\alpha(x)/2$ on $\bar{\Omega}\backslash\overline{D^\delta}$. By our choice of $\delta$ and the main result of [O], the problem

$$-\Delta V = \mu V(1-\alpha^\delta(x)V) \quad \text{in } \Omega, \quad V_\nu|_{\partial\Omega}=0 \tag{3.24}$$

has a unique positive solution $V^\delta$. From Lemma 3.10, we see that $\tilde{\alpha}_{\lambda_n} \geqslant \alpha^\delta$ on $\Omega$ for all large $n$. Therefore, for such $n$, $\tilde{V}_n$ is a lower solution to (3.24). Clearly, for large $M>1$, $MV^\delta$ is an upper solution of (3.24) and $MV^\delta>\tilde{V}_n$. Therefore, the unique positive solution $V^\delta$ must satisfy $\tilde{V}_n \leqslant V^\delta \leqslant MV^\delta$ in $\Omega$ for all large $n$. Hence $\{||\tilde{V}_n||_\infty\}$ is a bounded sequence. It follows now from the equation for $\tilde{V}_n$ and standard elliptic estimates that $\{\tilde{V}_n\}$ is bounded in $W^{2,p}(\Omega)$ for any $p>1$. Therefore, by passing to a subsequence, $\tilde{V}_n \to \tilde{V}$ in $C^1(\bar{\Omega})$ and $\tilde{V}$ is a nonnegative solution of (3.23). Let $M>0$ be a large constant such that $||\tilde{\alpha}_{\lambda_n}||_\infty \leqslant M$ for all $n$. Then $\tilde{V}_n$ is an upper solution to

$$-\Delta V = \mu V(1-MV) \quad \text{in } \Omega, \quad V_\nu|_{\partial\Omega}=0,$$

which has a unique positive solution $V\equiv 1/M$. Since any small positive constant $\sigma_0$ is a lower solution of this equation, we must have $\tilde{V}_n \geqslant 1/M \geqslant \sigma_0$ in $\Omega$. Therefore $\tilde{V} \geqslant 1/M$ in $\Omega$ and $\tilde{V}$ is the unique positive solution of (3.23). Since $\lambda_n$ is an arbitrary sequence converging to $\infty$, we can conclude that $\tilde{V}_\lambda \to \tilde{V}$ in $C^1(\bar{\Omega})$ as $\lambda\to\infty$. $\square$

**Theorem 3.12.** *Let $\tilde{V}$ be as in Lemma* 3.11. *Then*

$$\lim_{\lambda\to\infty} \beta_j(\lambda) = \beta_j^\infty := \left(\min_{\bar{D}_j} \tilde{V}\right)^{-1}.$$

**Proof.** For fixed $\beta>0$, denote $\mu_1^{D_j}(\lambda)=\lambda_1^{D_j}(\beta\lambda\tilde{V})/\lambda$. We easily see that $\mu_1^{D_j}(\lambda)$ is the first eigenvalue of the operator $-\lambda^{-1}\Delta+\beta\tilde{V}$ over $D_j$ under Dirichlet boundary conditions. Therefore, by its variational characterization,

$$\mu_1^{D_j}(\lambda) = \inf_{\phi\in H_0^1(D_j)\backslash\{0\}} \frac{\lambda^{-1}\int_{D_j}|\nabla\phi|^2+\int_{D_j}\beta\tilde{V}\phi^2}{\int_{D_j}\phi^2} \geqslant \beta\min_{\bar{D}_j}\tilde{V}. \tag{3.25}$$

On the other hand, for any $m>\min_{\bar{D}_j}\tilde{V}$, we can find a small ball $B_0\subset D_j$ such that $\tilde{V}\leqslant m$ on $B_0$. We now let $\phi_0$ be a smooth nonnegative function with support in $B_0$ and satisfying $\int_{B_0}\phi_0^2=1$. Then we find

$$\mu_1^{D_j}(\lambda) \leqslant \lambda^{-1}\int_{D_j}|\nabla\phi_0|^2+\beta m \to \beta m \quad \text{as } \lambda\to\infty.$$

Together with (3.25), this implies that

$$\lim_{\lambda \to \infty} \mu_1^{D_j}(\lambda) = \beta \min_{\bar{D}_j} \tilde{V}.$$

By Lemma 3.11, for any given $\varepsilon > 0$, we can find $\Lambda_\varepsilon > 0$ large enough such that

$$\tilde{V}(x) - \varepsilon \leqslant \tilde{V}_\lambda(x) \leqslant \tilde{V}(x) + \varepsilon \quad \forall x \in \Omega \ \forall \lambda \geqslant \Lambda_\varepsilon.$$

Therefore,

$$\begin{aligned} \frac{\lambda_1^{D_j}(\beta V_\lambda)}{\lambda} &= \frac{\lambda_1^{D_j}(\beta \lambda \tilde{V}_\lambda)}{\lambda} \in \left( \frac{\lambda_1^{D_j}(\beta \lambda \tilde{V} - \varepsilon \beta \lambda)}{\lambda}, \frac{\lambda_1^{D_j}(\beta \lambda \tilde{V} + \varepsilon \beta \lambda)}{\lambda} \right) \\ &= (\mu_1^{D_j}(\lambda) - \varepsilon \beta, \mu_1^{D_j}(\lambda) + \varepsilon \beta). \end{aligned}$$

Since $\varepsilon$ is arbitrary, it follows that,

$$\lim_{\lambda \to \infty} \frac{\lambda_1^{D_j}(\beta V_\lambda)}{\lambda} = \lim_{\lambda \to \infty} \mu_1^{D_j}(\lambda) = \beta \min_{\bar{D}_j} \tilde{V}. \tag{3.26}$$

Since

$$\frac{\lambda_1^{D_j}(\beta_j(\lambda) V_\lambda)}{\lambda} \equiv 1,$$

we easily see from (3.26) that

$$\lim_{\lambda \to \infty} \beta_j(\lambda) = \left( \min_{\bar{D}_j} \tilde{V} \right)^{-1}.$$

This finishes our proof. $\square$

**Corollary 3.13.** *Suppose that* $\mu \in (0, \lambda_1^{D_1})$ *and* $\beta \in (0, \max\{\beta_1^\infty, \ldots, \beta_m^\infty\})$. *Then there exists* $\Lambda > 0$ *such that* (3.5) *has no positive solution for* $\lambda > \Lambda$.

**Proof.** By our assumption, $\beta < \beta_j^\infty$ for some $1 \leqslant j \leqslant m$. By Theorem 3.12, we can find $\Lambda > 0$ such that $\beta_j(\lambda) > \beta$ for $\lambda > \Lambda$. Therefore $\lambda > \lambda_1^{D_j}(\beta V_\lambda)$ for $\lambda > \Lambda$. Due to (3.10), this implies that (3.5) has no positive solution for $\lambda > \Lambda$. $\square$

**Remark 3.14.** Our results in this subsection provide interesting contrast to those in Section 2 of [DD], where the Lotka–Volterra model

$$\begin{cases} -\Delta u = \lambda u - \alpha(x) u^2 - \beta u v, & x \in \Omega, \\ -\Delta v = \mu v - v^2 + d u v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{3.27}$$

was considered. Here we have modified the notations in [DD] to match the notation for (3.5). Assume that $\lambda, \beta, d$ are positive constants, $\mu$ is allowed to be negative, and $\alpha(x)$ is continuous and positive on $\bar{\Omega} \backslash \bar{D}$, and is identically zero on $D = D_1$. It is shown in [DD] that (3.27) behaves as if $\alpha(x)$ is a positive constant when $\lambda < \lambda_1^D$ while essential changes occur once $\lambda > \lambda_1^D$, namely, the range of the other parameters for the existence of positive solutions is greatly *enlarged* when $\lambda > \lambda_1^D$. For (3.5), under such conditions for $\alpha(x)$, by results of this subsection, when $\lambda < \lambda_1^D$, similar to (3.27), it behaves as if $\alpha$ is a positive constant, but for the case $\lambda > \lambda_1^D$, in sharp contrast to (3.27), the behavior of (3.5) has an essential change in that the range of the other parameters for the existence of positive solutions is greatly *reduced*. This difference has important consequences in terms of existence of patterned solutions (see Remark 3.20 later). We wish to point out that these different effects of $\alpha(x)$ on the two models are not due to the Dirichlet boundary conditions in (3.27); similar results can be proved for (3.27) under Neumann boundary conditions.

### 3.3. *Positive solutions with prescribed patterns*

Throughout this subsection, we assume that $\alpha(x)$ is as in Section 3.2, that is, $\alpha(x)$ is continuous over $\bar{\Omega}$, is positive on $\bar{\Omega} \backslash \bar{D}$, and $\alpha(x) \equiv 0$ on $D = \bigcup_{j=1}^m D_j$. Moreover, we assume that,

$$\mu \in (0, \lambda_1^{D_1}), \quad \beta \in (0, \min\{\beta_1^\infty, \ldots, \beta_m^\infty\}).$$

Therefore, in view of Theorem 3.12, there exists $\Lambda > 0$ such that

$$\beta < \beta_j(\lambda) \quad \forall \lambda > \Lambda, \ j = 1, 2, \ldots, m.$$

We now fix $\lambda > \Lambda$ and consider the following perturbation of (3.5):

$$\begin{cases} -\Delta u = \lambda u - [\alpha(x) + \varepsilon]u^2 - \beta uv, & x \in \Omega, \\ -\Delta v = \mu v\left(1 - \dfrac{v}{u}\right), & x \in \Omega, \\ u_\nu = v_\nu = 0, & x \in \partial\Omega, \end{cases} \tag{3.28}$$

where $\varepsilon > 0$ is a positive constant. By Theorem 3.1, (3.28) always has a positive solution. Denote by $(u_\varepsilon, v_\varepsilon)$ an arbitrary positive solution of (3.28), we want to show that as $\varepsilon \to 0$, $(u_\varepsilon, v_\varepsilon)$ exhibits a clear spatial pattern. To this end, let $\{\varepsilon_n\}$ be an arbitrary sequence of positive numbers decreasing to 0 as $n \to \infty$, and denote $(u_n, v_n) = (u_{\varepsilon_n}, v_{\varepsilon_n})$.

**Lemma 3.15.** *As $n \to \infty$, $u_n(x) \to \infty$ uniformly on $\bar{D}$.*

**Proof.** Let $\alpha_\lambda(x)$ be defined as in (3.9). We easily see that

$$-\Delta v_n \leqslant \mu v_n(1-\alpha_\lambda(x)v_n) \quad \text{in } \Omega.$$

Therefore, $v_n$ is a lower solution to (3.9), which has a unique positive solution $V_\lambda$. On the other hand, for any constant $M>1$, $MV_\lambda$ is an upper solution of (3.9) and $MV_\lambda > v_n$ if $M$ is large enough. It follows that $v_n \leqslant V_\lambda \leqslant MV_\lambda$ in $\Omega$.

By our choice of $\lambda$ and $\beta$, we have

$$\lambda > \lambda_1^{D_j}(\beta V_\lambda), \quad j=1,2,\dots,m. \tag{3.29}$$

Let $U_n$ be the unique positive solution of

$$-\Delta u + \beta V_\lambda u = \lambda u - [\alpha(x)+\varepsilon_n]u^2 \quad \text{in } \Omega, \; u_\nu|_{\partial\Omega}=0.$$

By Theorem 2.2(iii) of [DL], (3.29) implies that $U_n \to \infty$ uniformly on $\bar{D}$ as $n \to \infty$. (To avoid confusion, let us note that the $D_j$'s in [DL] are closed sets, and therefore correspond to $\bar{D}_j$ here.)

Since $v_n \leqslant V_\lambda$, we deduce from the equation for $u_n$ that,

$$-\Delta u_n + \beta V_\lambda u_n \geqslant \lambda u_n - [\alpha(x)+\varepsilon_n]u_n^2 \quad \text{in } \Omega.$$

Hence, by a simple comparison argument similar to that leading to $v_n \leqslant V_\lambda$ above, $u_n \geqslant U_n \geqslant U_{n-1} \geqslant \cdots \geqslant U_1$ in $\Omega$. It follows that $u_n \to \infty$ uniformly on $\bar{D}$ as $n \to \infty$. $\square$

**Theorem 3.16.** *$\{(u_n,v_n)\}$ has a subsequence, still denoted by $(u_n,v_n)$, such that,*

$$u_n \to \tilde{u} \text{ in } C^1(\bar{\omega}) \text{ for any subdomain } \omega \text{ satisfying } \bar{\omega} \subset \bar{\Omega}\backslash\bar{D},$$

$$u_n \to \infty \text{ uniformly on } \bar{D}, \quad v_n \to \tilde{v} \quad \text{in } C^1(\bar{\Omega}),$$

*where $\tilde{u}$ is a positive solution to*

$$-\Delta\tilde{u} = \lambda\tilde{u} - \alpha(x)\tilde{u}^2 - \beta\tilde{u}\tilde{v} \quad \text{in } \Omega\backslash\bar{D}, \; \tilde{u}|_{\partial D} = \infty, \; \tilde{u}_\nu|_{\partial\Omega}=0 \tag{3.30}$$

*and $\tilde{v}$ is a positive solution to*

$$-\Delta\tilde{v} = \mu\tilde{v}(1-\tilde{\alpha}(x)\tilde{v}) \quad \text{in } \Omega, \; \tilde{v}_\nu|_{\partial\Omega}=0, \tag{3.31}$$

*where*

$$\tilde{\alpha}(x) = \begin{cases} 0, & x \in \bar{D}, \\ 1/\tilde{u}(x), & x \in \bar{\Omega}\backslash\bar{D}. \end{cases}$$

*Moreover*,

$$\lambda_1^{D_j}(\beta\tilde{v}) = \lambda, \quad j = 1, 2, \dots, m. \tag{3.32}$$

**Remark 3.17.** Note that $(\tilde{u}, \tilde{v})$ is determined by (3.30)–(3.32) altogether, and Theorem 3.16 implies that there is at least one positive solution $(\tilde{u}, \tilde{v})$ to (3.30)–(3.32), provided that $\mu \in (0, \lambda_1^{D_1})$, $\beta \in (0, \min\{\beta_1^\infty, \dots, \beta_m^\infty\})$ and $\lambda > \Lambda$.

**Proof of Theorem 3.16.** By the proof of Lemma 3.15, we find that,

$$0 < v_n \leqslant V_\lambda, \quad 0 < 1/u_n \leqslant 1/U_n \leqslant 1/U_1 \quad \text{in } \Omega \quad \forall n \geqslant 1.$$

Therefore, $\{-\Delta v_n\}$ and $\{v_n\}$ are both bounded sets in $L^\infty(\Omega)$. By standard elliptic estimates, it follows that $\{v_n\}$ is bounded in $W^{2,p}(\Omega)$ for all $p > 1$. Hence, subject to a subsequence, $v_n \to \tilde{v}$ in $C^1(\bar{\Omega})$.

Let $U_\lambda$ denote the minimal positive solution of

$$-\Delta u = \lambda u - \alpha(x)u^2 \quad \text{in } \Omega \backslash \bar{D}, \ u|_{\partial D} = \infty, \ u_\nu|_{\partial\Omega} = 0.$$

By Lemma 2.3 in [DL], we obtain

$$u_n(x) \leqslant U_\lambda(x) \quad \forall x \in \Omega \backslash \bar{D}, \ n = 1, 2, \dots \ . \tag{3.33}$$

For small $\delta > 0$, let us denote $D_\delta = \{x \in \Omega : d(x, D) < \delta\}$. Then, by (3.33) we find that $\{u_n|_{\bar{\Omega} \backslash D_\delta}\}$ is a bounded sequence in $L^\infty(\Omega \backslash D_\delta)$. Therefore we can apply standard elliptic regularity results (up to $\partial\Omega$ but away from $\partial D_\delta$) to conclude that $\{u_n|_{\bar{\Omega} \backslash D_{2\delta}}\}$ is compact in $C^1(\bar{\Omega} \backslash D_{2\delta})$.

We now use a diagonal process to extract a subsequence of $\{u_n\}$ that converges in $\bar{\Omega} \backslash \bar{D}$. Let $\{\delta_n\}$ be a sequence of small positive numbers decreasing to 0 as $n \to \infty$. Then we can find a subsequence $\{u_n^1\}$ of $\{u_n\}$ that converges to some $u^1$ in $C^1(\bar{\Omega} \backslash D_{2\delta_1})$. From $\{u_n^1\}$ we can find a further subsequence $\{u_n^2\}$ that converges to some $u^2$ in $C^1(\bar{\Omega} \backslash D_{2\delta_2})$. In general, for $k = 1, 2, \dots$, we have a subsequence $\{u_n^{k+1}\}$ of $\{u_n^k\}$ that converges to some $u^{k+1}$ in $C^1(\bar{\Omega} \backslash D_{2\delta_{k+1}})$. Clearly we must have $u^k = u^{k+1}$ on $\bar{\Omega} \backslash D_{2\delta_k}$. Therefore, if we define $\tilde{u}(x) = u^k(x)$ for $x \in \bar{\Omega} \backslash D_{2\delta_k}$, $k = 1, 2, \dots$, then $\tilde{u}$ is well-defined in $\bar{\Omega} \backslash \bar{D}$, and $u_n^n \to \tilde{u}$ in $C^1(\bar{\Omega} \backslash D_\delta)$ for every small $\delta > 0$. In other words, $\{u_n\}$ has a subsequence which we still denote by $u_n$ that converges to $\tilde{u}$ in $C^1(\bar{\Omega} \backslash D_\delta)$ for any small $\delta > 0$.

Now from the equation for $u_n$ and the fact that $v_n \to \tilde{v}$ in $C^1(\bar{\Omega})$, we find

$$-\Delta\tilde{u} = \lambda\tilde{u} - \alpha(x)\tilde{u}^2 - \beta\tilde{u}\tilde{v} \quad \text{in } \Omega \backslash \bar{D}, \ \tilde{u}_\nu|_{\partial\Omega} = 0.$$

By the proof of Lemma 3.15, we have $u_n \geqslant U_n$. But we know from Theorem 2.2 of [DL] that, as $n \to \infty$, $U_n \to U_\lambda$ uniformly on $\bar{\Omega} \backslash D_\delta$ for any $\delta > 0$. It follows that $\tilde{u} \geqslant U_\lambda$, and hence $\tilde{u}|_{\partial D} = \infty$. Thus we have proved that $\tilde{u}$ satisfies (3.30).

Consider now $1/u_n$. From Lemma 3.15 we easily see that $1/u_n \to 0$ uniformly on $\bar{D}$. By our above discussion, we have $1/u_n \to 1/\tilde{u}$ uniformly over $\bar{\Omega} \backslash D_\delta$ for every small $\delta > 0$. Since $u_n \geqslant U_n \geqslant U_1$ for all $n \geqslant 1$, we have $1/u_n \leqslant 1/U_1$ in $\bar{\Omega} \backslash \bar{D}$. Therefore, by the dominate convergence theorem, we have $1/u_n \to \tilde{\alpha}$ in $L^p(\Omega)$ for all $p > 1$. We now easily see from the equation for $v_n$ that,

$$-\Delta \tilde{v} = \mu \tilde{v}(1 - \tilde{\alpha}(x)\tilde{v}) \quad \text{in } \Omega, \;\; \tilde{v}_v|_{\partial\Omega} = 0.$$

This verifies that $\tilde{v}$ satisfies (3.31).

It remains to prove (3.32). To this end, we denote $\hat{u}_n = u_n / ||u_n||_\infty$. Then

$$-\Delta \hat{u}_n \leqslant \lambda \hat{u}_n \quad \text{in } \Omega,$$

and as in the proof of Lemma 3.7, this implies that, subject to a subsequence, $\hat{u}_n \to \hat{u}$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for every $p > 1$, and $\hat{u} \not\equiv 0$. From (3.33) we find that $\hat{u} \equiv 0$ on $\Omega \backslash D$. Multiplying the equation for $u_n$ by $\phi_j / ||u_n||_\infty$, where $\phi_j \in C^\infty(D_j)$ and has support contained in $D_j$, integrating by parts over $D_j$ and then letting $n \to \infty$, we obtain

$$\int_{D_j} \nabla \hat{u} \cdot \nabla \phi_j = \int_{D_j} (\lambda - \beta \tilde{v}) \hat{u} \phi_j.$$

This implies that $\hat{u}|_{D_j}$ is a nonnegative solution to

$$-\Delta u = (\lambda - \beta \tilde{v}) u \quad \text{in } D_j, \;\; u|_{\partial D_j} = 0. \tag{3.34}$$

Clearly (3.32) follows from (3.34) if we can show that $\hat{u}|_{D_j} \not\equiv 0$.

Since $\lambda > \lambda_1^{D_j}(\beta V_\lambda)$, the problem

$$-\Delta z + \beta V_\lambda z = \lambda z - z^2 \quad \text{in } D_j, \;\; z|_{\partial D_j} = 0 \tag{3.35}$$

has a unique positive solution $\theta_j$. On the other hand, it is easily seen that $z_n := \varepsilon_n u_n$ satisfies

$$-\Delta z_n = \lambda z_n - z_n^2 - \beta v_n z_n \geqslant \lambda z_n - z_n^2 - \beta V_\lambda z_n \quad \text{in } D_j, \;\; z_n|_{\partial D_j} > 0.$$

Therefore, by Lemma 2.1 of [DM], we deduce

$$\varepsilon_n u_n \geqslant \theta_j \quad \text{in } D_j \;\; \forall n \geqslant 1. \tag{3.36}$$

On the other hand,

$$-\Delta z_n = \lambda z_n - [\varepsilon_n^{-1} \alpha(x) + 1] z_n^2 - \beta v_n z_n \leqslant \lambda z_n - z_n^2 \quad \text{in } \Omega.$$

Therefore by a simple comparison consideration, $z_n$ is not larger than the unique positive solution $z \equiv \lambda$ of the problem

$$-\Delta z = \lambda z - z^2 \quad \text{in } \Omega, \ z_\nu|_{\partial\Omega} = 0,$$

that is, $\varepsilon_n u_n \leqslant \lambda$. Hence $||u_n||_\infty \leqslant \lambda/\varepsilon_n$. Using (3.36), we obtain

$$u_n/||u_n||_\infty \geqslant \varepsilon_n u_n/\lambda \geqslant \theta_j/\lambda \quad \text{in } D_j.$$

It follows that $\hat{u} \geqslant \theta_j/\lambda$ in $D_j$. Thus we have $\lambda = \lambda_1^{D_j}(\beta\tilde{v})$. The proof is complete. $\square$

Theorem 3.16 implies that for all large $n$, $(u_n, v_n)$ is close to a function $(u^*, v^*)$ of the form $u^*(x) = \infty$ on $\bar{D}$, $u^*(x) = \tilde{u}(x)$ on $\bar{\Omega}\backslash\bar{D}$, $v^* = \tilde{v}$ in $\Omega$, where $(\tilde{u}, \tilde{v})$ solves (3.30)–(3.32). Clearly $u_n$ develops a sharp pattern over $\Omega$: its value over $D$ is much bigger than that over the rest of $\Omega$. However, $v_n$ does not develop into a sharp pattern. The following result further describes the profile of $u_n$ for large $n$.

**Theorem 3.18.** *Suppose that* $(u_n, v_n)$ *converges to* $(\tilde{u}, \tilde{v})$ *as in Theorem* 3.16. *Then* $\varepsilon_n u_n \to w$ *in* $C(\bar{\Omega})$, *where* $w = 0$ *on* $\bar{\Omega}\backslash D$, *and on each* $D_j$, $j = 1, \dots, m$, $w$ *is the unique positive solution of*

$$-\Delta w = \lambda w - w^2 - \beta\tilde{v}w \quad \textit{in } D_j, \ w|_{\partial D_j} = 0.$$

**Proof.** From the proof of Theorem 3.16, we find that,

$$\theta_j(x) \leqslant \varepsilon_n u_n(x) \quad \forall x \in D_j \ \forall n \geqslant 1,$$

where $\theta_j$ denotes the unique positive solution of (3.35). We also find from there that $\varepsilon_n u_n \leqslant \lambda$. Hence $c_0 \leqslant \varepsilon_n ||u_n||_\infty \leqslant C_0$ for some positive constants $c_0$, $C_0$ and all $n$. We may assume that $\varepsilon_n ||u_n||_\infty \to \xi \in [c_0, C_0]$ as $n \to \infty$. Therefore, by the proof of Theorem 3.16, $\varepsilon_n u_n \to \xi\hat{u}$ in $L^p(\Omega)$ for any $p > 1$. When restricted on $D$, $z_n := \varepsilon_n u_n$ satisfies

$$-\Delta z_n = (\lambda - \beta v_n) z_n - z_n^2,$$

whose right-hand side has a bound in $L^\infty$ that is independent in $n$. Therefore, by standard interior elliptic estimates, we can conclude that $z_n \to \xi\hat{u}$ in the $C^1$ norm over any compact subset of $D$. By (3.33), we find that $z_n \to 0$ uniformly on any compact subset of $\bar{\Omega}\backslash\bar{D}$.

**Claim 1.** *For each* $j = 1, \dots, m$, $\eta_n = \eta_n^j := \max_{\partial D_j} z_n \to 0$ *as* $n \to \infty$.

Arguing indirectly we assume that there exists $1 \leqslant j \leqslant m$, a sequence $x_n \in \partial D_j$ and some $\delta_0 > 0$ such that $z_n(x_n) \geqslant \delta_0$ for all $n \geqslant 1$.

For each small $\delta>0$, let $D_j^\delta$ denote the $\delta$-neighborhood of $D_j$ and consider the problem

$$-\Delta u = \lambda u - u^2 - \beta(\tilde{v}-\delta)u \quad \text{in } D_j^\delta,\ u|_{\partial D_j^\delta} = 0. \tag{3.37}$$

Since $\tilde{v} \leqslant V_\lambda$ and $\lambda > \lambda_1^{D_j}(\beta V_\lambda)$, we find that for all small $\delta>0$, $\lambda > \lambda_1^{D_j^\delta}(\beta(\tilde{v}-\delta))$. Hence (3.37) has a unique positive solution $u_\delta$. By the uniqueness and a standard regularity and compactness argument, we can easily show that

$$u_\delta|_{D_j} \to w_j \quad \text{in } C^2(\bar{D}_j) \text{ as } \delta \to 0, \tag{3.38}$$

where $w_j$ is the unique positive solution of (3.37) with $\delta = 0$. For each fixed $\delta>0$, by our discussion above, $z_n \to 0$ uniformly on $\partial D_j^{\delta/2}$. Hence $z_n \leqslant u_\delta$ on $\partial D_j^{\delta/2}$ for all large $n$. Moreover, in $D_j^{\delta/2}$, $z_n$ satisfies

$$-\Delta z_n = \lambda z_n - z_n^2 - \beta v_n z_n \leqslant \lambda z_n - z_n^2 - \beta(\tilde{v}-\delta) z_n,$$

provided that $n$ is large enough, since $v_n \to \tilde{v}$ uniformly on $\bar{\Omega}$. Therefore we can apply Lemma 2.1 in [DM] to conclude that $z_n \leqslant u_\delta$ in $D_j^{\delta/2}$ for all large $n$. In particular, for all large $n$,

$$u_\delta(x_n) \geqslant z_n(x_n) \geq \delta_0.$$

This implies that $\max_{\partial D_j} u_\delta \geqslant \delta_0$ for all small $\delta>0$. Clearly this contradicts (3.38), and the proof of Claim 1 is complete.

**Claim 2.** *$z_n \to w_j$ uniformly on $\bar{D}_j$ for $j = 1, 2, \ldots, m$, and $z_n \to 0$ uniformly on $\bar{\Omega} \backslash D$.*

Let $\delta_n = \beta \|v_n - \tilde{v}\|_\infty$. Then $\delta_n \to 0$ as $n \to \infty$. A simple upper and lower solution consideration shows that the problem

$$-\Delta w = (\lambda + \delta_n) w - w^2 - \beta \tilde{v} w \quad \text{in } D_j,\ w|_{\partial D_j} = \eta_n^j$$

has a positive solution $w^n$. By Lemma 2.1 of [DM], the solution is unique. It is then easily shown that $w^n \to w_j$ uniformly on $\bar{D}_j$ as $n \to \infty$.

One easily checks that, for all large $n$,

$$-\Delta z_n \leqslant (\lambda + \delta_n) z_n - z_n^2 - \beta \tilde{v} z_n \quad \text{in } D_j,\ z_n \leqslant w^n \text{ on } \partial D_j.$$

By Lemma 2.1 of [DM], it follows that

$$z_n \leqslant w^n \quad \text{in } D_j.$$

On the other hand, the problem

$$-\Delta w = (\lambda - \delta_n)w - w^2 - \beta\tilde{v}w \quad \text{in } D_j, \ \ w|_{\partial D_j} = 0$$

has a unique positive solution $\underline{w}^n$ and $\underline{w}^n \to w_j$ uniformly on $\bar{D}_j$ as $n \to \infty$. Since

$$-\Delta z_n \geqslant (\lambda - \delta_n)z_n - z_n^2 - \beta\tilde{v}z_n \quad \text{in } D_j, \ \ z_n|_{\partial D_j} > 0,$$

we can apply Lemma 2.1 of [DM] to obtain $z_n \geqslant \underline{w}^n$ in $D_j$. Therefore,

$$\underline{w}^n \leqslant z_n \leqslant w^n \quad \text{in } D_j \text{ for all large } n.$$

It follows that $z_n \to w_j$ uniformly on $\bar{D}_j$ as $n \to \infty$. This proves the first half of Claim 2.

To prove the second half of Claim 2, we consider the problem,

$$-\Delta u = (\lambda + \delta_n)u - u^2 - \beta\tilde{v}u \quad \text{in } \Omega \backslash \bar{D}, \ \ u|_{\partial D} = \xi_n, \ \ u_\nu|_{\partial\Omega} = 0, \tag{3.39}$$

where $\xi_n = \max_{\partial D} z_n$. By what has just been proved, we know that $\xi_n \to 0$ as $n \to \infty$. It is easily seen by a lower and upper solution argument that (3.39) has a unique positive solution $z^n$. A standard regularity and compactness argument shows that $z^n$ converges uniformly to $z^\infty$ which is the maximal nonnegative solution of the problem

$$-\Delta u = \lambda u - u^2 - \beta\tilde{v}u \quad \text{in } \Omega \backslash \bar{D}, \quad u|_{\partial D} = 0, \ \ u_\nu|_{\partial\Omega} = 0. \tag{3.40}$$

It is well known that $z^\infty = 0$ when $\lambda \leqslant \lambda_1^{\Omega \backslash D}(\beta\tilde{v})$ and $z^\infty$ is the unique positive solution of (3.40) in the remaining case.

Applying Lemma 2.3 of [DL] we find that $z_n \leqslant z^n$ in $\Omega \backslash D$. Therefore, for any sequence $\{x_n\} \subset \Omega \backslash D$ satisfying $d(x_n, D) \to 0$, we have, as $n \to \infty$,

$$z_n(x_n) \leqslant z^n(x_n) \leqslant ||z^n - z^\infty||_{L^\infty(\Omega \backslash D)} + z^\infty(x_n) \to 0. \tag{3.41}$$

We have already proved that $z_n \to 0$ uniformly on any compact subset of $\bar{\Omega} \backslash \bar{D}$. Combined with (3.41), we find that $z_n \to 0$ uniformly on $\bar{\Omega} \backslash D$. This finishes the proof for the second half of Claim 2 and hence the proof of the theorem. □

It seems worthwhile to point out that if $(u_\varepsilon, v_\varepsilon)$ is a positive solution to (3.28), then $(z_\varepsilon, v_\varepsilon)$, with $z_\varepsilon = \varepsilon u_\varepsilon$, is a positive solution to the predator–prey model

$$\begin{cases} -\Delta z = \lambda z - [\varepsilon^{-1}\alpha(x) + 1]z^2 - \beta z v, & x \in \Omega, \\ -\Delta v = \mu v\left(1 - \varepsilon \dfrac{v}{z}\right), & x \in \Omega, \\ z_\nu = v_\nu = 0, & x \in \partial\Omega. \end{cases}$$

**Remark 3.19.** Theorem 3.18 implies that for small $\varepsilon > 0$, $z_\varepsilon$ exhibits a sharp pattern over $\Omega$: it is close to 0 over $\bar{\Omega} \backslash D$, and is close to a continuous positive function over $D$. Note that $v_\varepsilon$ is close to a continuous positive function over the entire $\Omega$. By choosing $D$ suitably, we see that rather arbitrary patterns can be realized by $z_\varepsilon$.

**Remark 3.20.** It is easy to check that if we perturb the classical Lotka–Volterra model (3.27) by replacing a degenerate $\alpha(x)$ with $\alpha(x) + \varepsilon$, then no positive solution $(u_\varepsilon, v_\varepsilon)$ of the perturbed (3.27) develops a sharp pattern as $\varepsilon \to 0$. In fact, it is easy to show that $(u_\varepsilon, v_\varepsilon)$ is close to a positive solution of the unperturbed (3.27) when $\varepsilon$ is small.

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