CLASSIFICATION OF POTENTIAL FLOWS UNDER RENORMALIZATION GROUP TRANSFORMATION

SZE-BI HSU, BERNOLD FIEDLER, AND HSIU-HAU LIN

Abstract. Competitions between different interactions in strongly correlated electron systems often lead to exotic phases. Renormalization group is one of the powerful techniques to analyze the competing interactions without presumed bias. It was recently shown that the renormalization group transformations to the one-loop order in many correlated electron systems are described by potential flows. Here we prove several rigorous theorems in the presence of renormalization-group potential and find the complete classification for the potential flows. In addition, we show that the relevant interactions blow up at the maximal scaling exponent of unity, explaining the puzzling power-law Ansatz found in previous studies. The above findings are of great importance in building up the hierarchy for relevant couplings and the complete classification for correlated ground states in the presence of generic interactions.

Keywords: Strongly correlated electron system, RG flow, blow-up in finite time, polar equations, gradient flow, LaSalles' Invariance Principle

1. Introduction

Interactions in many-particle systems often lead to novel collective phenomena, inaccessible by the mean- field theory based on the single-particle picture. Exact solutions for these strongly correlated systems are illuminating and helpful, but scarce. Renormalization groups (RG) ([1]-[3]) are one of the powerful techniques to address these challenging issues in all branches of science. By integrating out the degrees of freedom at the longer length scale, the couplings of the effective Hamiltonian flow are described according to a set of RG equations. This approach is particularly helpful when the quantum fluctuations in the system are strong so that the mean-field description is invalid or hard to justify. For instance, it has been demonstrated that, despite of the repulsive interactions at the short length scale, electrons form pairs with unconventional d-wave symmetry in the low-energy limit in the two-leg ladder or the doped carbon nanotubes. RG analysis predicts that the ground state is a spin liquid beautifully realized in recent experiments.

To be precise, interactions of a correlated system are characterized by a set of couplings g_i. By integrating out the fluctuations at shorter length scale, these couplings are renormalized accordingly and described by a set of first-order differential equations. In general, the perturbative RG equations to the one-loop order take the following form,

$$\frac{dg_i}{dl} = \sum_j \Delta_{ij} g_j + \sum_{jk} A_{ijk} g_j g_k, \tag{1.1}$$

where the last two indices in the coefficient tensor can be made symmetric, $A_{ijk} = A^{(i)}_{jk} = A^{(i)}_{kj} = A_{ikj}$. While the derivation of the RG equations are rather standard and can be found in the literature, interpreting the RG flows obtained in numerics can be subtle and tricky. At the tree-level, classification of relevance for the couplings with non-vanishing scaling dimensions is often straightforward. The fates of the RG flows are dictated by the scaling dimensions obtained by diagonalizing the matrix Δ_{ij} and the phases can be classified rather easily.

However, it is often encountered that $\Delta_{ij} = 0$ for many interesting correlated systems ([4]-[11]) and the RG equations become homogeneously quadratic. Analytic solutions are not known ([12]-[13]) and, at the current stage, numerical integration is the common approach to analyze the RG flows. It is rather subtle to classify the relevant couplings g_i in that case because they often flow to fixed points outside of the scope of perturbative regime. The typical recipe is to integrate the RG equations up to the cutoff length scale l_c where the maximal coupling is of order one. At $l = l_c$, we identify the couplings $g_i(l_c) \sim O(1)$ to be relevant while those couplings $g_i(l_c) \ll 1$ are irrelevant. But, this face-value classification scheme does not always work and some couplings are ambiguous to be classified as either relevant or irrelevant.

A recent breakthrough ([11]) points out the existence of the RG potential and renders a better analytic understanding of the RG flows in these correlated systems. To show the existence of an RG potential, we can apply the rescaling transformation to all couplings, $x_i = \sqrt{\lambda_i} g_i$, and replace the derivative with respective to the logarithmic length l by the derivative with respective to some fictitious time t, denoted as the dot,

$$\dot{x}_i = \sum_{jk} P_{ijk} x_j x_k. \tag{1.2}$$

Here $P_{ijk} = \sqrt{\lambda_i/(\lambda_j \lambda_k)} A^{(i)}_{jk}$ is the coefficient tensor after rescaling. The RG equation can be viewed as a strongly over-damped particle moving in the multidimensional space under the influence of the external force on the right-band side of the equation. The existence of a potential requires the curl of the force to vanish, $\partial_j F_i - \partial_i F_j = 0$. This leads to the following constraint on the rescaled coefficient tensor,

$$\sum_k (P_{ijk} - P_{jik}) x_k = 0. \tag{1.3}$$

Because x_k is arbitrary and $P_{ijk} = P_{jik}$, this requires P_{ijk} to be totally symmetric. It has been shown explicitly how the coefficient tensor P_{ijk} can be made totally symmetric under appropriate rescaling transformation.

The remaining part of the paper is organized as follows. In section 2 we consider the dynamics of the model equations in the presence of a renormalization-group potential and give a sufficient condition on the initial conditions for the blow up behavior of the solutions. In section 3, we derive the polar equation for the governing system. In section 4, we prove that the blow up rate of the solution x_i is $(t - T)^{-1}$ where T is the blow-up time. Section 5 is the discussion section.

2. The Model

Consider the following governing system of equations

$$\frac{dx_i}{dt} = x^T A^{(i)} x, \ \ i = 1, 2, ..., n, \tag{2.1}$$

where $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$, $A^{(i)} = (a^{(i)}_{jk})$ is a symmetric $n \times n$ matrix satisfying the totally symmetric conditions:

$$a^{(i)}_{jk} = a^{(j)}_{ik} = a^{(k)}_{ij} \text{ for all } i, j, k = 1, 2, ..., n. \tag{H}$$

Suppose the initial condition $x(0)$ satisfies

$$x(0) \notin E \tag{2.2}$$

where E is the set of equilibria of (2.1). Let

$$V(x) = -\frac{1}{3}\sum_{i=1}^{n} x_i(x^T A^{(i)} x). \tag{2.3}$$

Then from (H)

$$\begin{aligned}
-3\frac{\partial V}{\partial x_i} &= x^T A^{(i)} x + x_i \frac{\partial}{\partial x_i}(x^T A^{(i)} x) + \sum_{j \neq i} x_j \frac{\partial}{\partial x_i}(x^T A^{(i)} x) \\
&= x^T A^{(i)} x + 2x_i(\sum_{j=1}^{n} a^{(i)}_{ij} x_j) + \sum_{j \neq i} 2x_j \sum_{k=1}^{n} a^{(j)}_{ik} x_k \\
&= x^T A^{(i)} x + 2x_i(\sum_{j=1}^{n} a^{(i)}_{ij} x_j) + 2\sum_{j \neq i} x_j(\sum_{k=1}^{n} a^{(j)}_{jk} x_k) \text{ (from (2.2))} \\
&= 3x^T A^{(i)} x.
\end{aligned}$$

Hence we have

$$-\frac{\partial V}{\partial x_i} = x^T A^{(i)} x \tag{2.4}$$

and (2.1) can be rewritten as

$$\frac{dx}{dt} = -\nabla_x V. \tag{2.5}$$

Obviously (2.1) is a gradient system satisfying

$$\dot{V} = -\nabla_x V \cdot (-\nabla_x V) = -|\nabla V|^2 \leq 0. \tag{2.6}$$

Let $V(t) := V(x(t))$.

Theorem 2.1. *Let $x(t)$ be any solution of the initial value problem* (2.1) *satisfying* (2.2) *and* (H). *If $V(x(0)) < 0$ then the solution $x(t)$ blows up in finite time.*

Proof. Let $g(t) = \sum_{i=1}^{n} x_i^2(t) = |x(t)|_2^2$ and $V(t) := V(x(t))$. Then from (2.3) we have

$$g'(t) = -6V(t). \tag{2.7}$$

Claim:

$$g'(t) \geq C(g(t))^{\delta} \tag{2.8}$$

for all $t > 0$ for some $C > 0$ independent of δ, and for all $1 < \delta \leq 3/2$.
Suppose (2.8) holds. Then

$$\frac{1}{(g(0))^{\delta-1}} - \frac{1}{(g(t))^{\delta-1}} \geq (\delta - 1)Ct,$$

or

$$(g(t))^{\delta-1} \geq \frac{1}{\frac{1}{(g(0))^{\delta-1}} - (\delta - 1)Ct}. \tag{2.9}$$

Then $g(t) \to \infty$ as $t \to (t^*)^-$, $t^* = t^*(\delta) = \frac{1}{(\delta-1)C} \frac{1}{(g(0))^{\delta-1}}$. We note that from (2.6) and (2.7), $g'(t) \geq -6V(0) > 0$. Either $g(t)$ bolws up in finite time or $g(t) \geq (-6V(0))t$ for all $t > 0$. In either case without loss of generality we may assume $g(0) > 1$. Then $t^*(\delta)$ is decreasing strictly in δ. Hence $t^*(\delta) > t^*(\frac{3}{2}) = T$ for all $1 < \delta < \frac{3}{2}$. Let $\delta = \frac{3}{2}$, from (2.9) it follows that

$$|x(t)| \geq \frac{2}{C}\Big(\frac{1}{T - t}\Big), \ 0 < t < T.$$

Now we prove that (2.8) holds. Consider $1 < \delta \leq 3/2$, C sufficiently small such that

$$-6V(0) - C(g(0))^{\delta} > 0,$$

or

$$0 < C < \min_{1 \leq \delta \leq \frac{3}{2}} \frac{-6V(0)}{(g(0))^{\delta}}$$

From (2.7), (2.8) is equivalent to

$$-6V(t) \geq C(g(t))^{\delta} \text{ for all } t \geq 0.$$

Let

$$h(t) = -6V(t) - C(g(t))^{\delta}.$$

Then $h(0) > 0$. From (2.6)

$$\begin{aligned} h'(t) &= -6V'(t) - C\delta(g(t))^{\delta-1}g'(t) \\ &= 6\sum_{i=1}^{n}(x^T A^{(i)} x)^2 - C\delta(g(t))^{\delta-1}(-6V) \\ &= 6\Big[\sum_{i=1}^{n}(x^T A^{(i)} x)^2 + C\delta(g(t))^{\delta-1}V\Big] \end{aligned}$$

By the Cauchy-Schwarz inequality and (2.3), it follows that

$$9V^2(t) = \Big(\sum_{i=1}^{n} x_i(x^T A^{(i)} x)\Big)^2 \leq \Big(\sum_{i=1}^{n} x_i^2\Big)\Big(\sum_{i=1}^{n}(x^T A^{(i)} x)^2\Big).$$

Then

$$\begin{aligned} h'(t) &\geq 6\Big[\frac{9V^2(t)}{g(t)} + C\delta(g(t))^{\delta-1}V\Big] \\ &= \frac{6V(t)}{g(t)}\Big[9V(t) + C\delta(g(t))^{\delta}\Big] \\ &= \frac{6V(t)}{g(t)}\Big[9V(t) + \delta\Big(-6V(t) - h(t)\Big)\Big] \\ &= \frac{6V(t)}{g(t)}\Big[3V(t)(3-2\delta) - \delta h(t)\Big]. \end{aligned} \tag{2.10}$$

From (2.6) and $V(0) < 0$, we have $V(t) < 0$ for all $t \geq 0$. Since $h(0) > 0,\ 1 < \delta < 3/2$, then from (2.10), $h'(0) > 0$ and $h(t) > 0$ for all $t > 0$. Thus we complete the proof of (2.8). □

Corollary 2.1. *If $V(x(0)) = 0$ and $x(0)$ is not an equilibrium, then $\dot{V}(x(0)) < 0$ and the conclusion of Theorem 2.1 holds.*

Proof. From (2.6), $\dot{V}(x(0)) \leq 0$. If $\dot{V}(x(0)) = 0$ then $x(0)$ is an equilibrium of the system (2.1), a contradiction to the assumption on the initial condition. □

Corollary 2.2. *If $V(x(0)) > 0$ then either*

(i) *there exists $t_1 > 0$ such that $V(x(t_1)) = 0$, $\dot{V}(x(t_1)) < 0$ and the conclusion of Theorem 2.1 holds, or*

(ii) *$V(x(t)) > 0$ for all $t \geq 0$ and $\mathrm{dist}(x(t), E) \to 0$ as $t \to \infty$, where E is the set of equilibria of (2.1).*

Proof. If $V(x(t_1)) = 0$ then $\dot{V}(x(t_1)) < 0$. Otherwise $x(t_1)$ is an equilibrium, a contradiction.

If $V(x(t)) > 0$ for all $t \geq 0$, then from (2.7), $x(t)$ is bounded for all $t \geq 0$. From LaSalles invariance principle ([14]-[15]), $\mathrm{dist}(x(t), E) \to 0$ as $t \to \infty$ where $E = \{x : \dot{V}(x) = 0\}$. □

3. Polar coordinates

In this section we derive the polar equation for the gradient system (2.5).

Let $x = r\phi$ where $r = |x|_2,\ \phi = \frac{x}{r} \in S^{n-1}$. Then

$$x' = r'\phi + r\phi' = -\nabla_x V(x). \tag{3.1}$$

We note that $\phi^T\phi = 1$ and $\phi^T\phi' = 0$. Let

$$P^{\phi} = Id - \phi\phi^T$$

be the orthogonal projection onto the tangent space $T_{\phi}S^{n-1} = \langle\phi\rangle^T$ to the unit sphere S^{n-1} at ϕ. Then from (3.1)

$$\phi^T(r'\phi + r\phi') = \phi^T(-\nabla_x V(x)).$$

Thus we obtain the radical part

$$r' = -\phi^T\nabla_x V(x). \tag{3.2}$$

Similarly applying P^ϕ to both sides of (3.1) and using $P^\phi \phi' = 0$ yields the tangential part

$$r\phi' = -P^\phi \nabla_x V(x). \tag{3.3}$$

We now express $\nabla_x V(x)$ in polar coordinates.

From (2.3), $V(x)$ is a homogeneous polynomial with degree $p = 3$. Therefore

$$V(x) = V(r\phi) = r^p V(\phi). \tag{3.4}$$

Differentiating both sides of (3.4) with respect to x_i yields

$$\frac{\partial V}{\partial x_i} = pr^{p-1}\frac{\partial r}{\partial x_i}V(\phi) + r^p \frac{\partial}{\partial x_i}V(\frac{x_1}{r}, \frac{x_2}{r}, ..., \frac{x_n}{r}). \tag{3.5}$$

Since $\frac{\partial r}{\partial x_i} = \phi_i$ and $\frac{\partial}{\partial x_i}(\frac{1}{r}) = -\frac{\phi_i}{r^2}$, it follows that

$$\begin{aligned}
\frac{\partial}{\partial x_i}V(\frac{x_1}{r}, ..., \frac{x_n}{r}) &= \frac{\partial V}{\partial \phi_i}\frac{\partial}{\partial x_i}(\frac{x_i}{r}) + \sum_{k\neq i}\frac{\partial V}{\partial \phi_k}x_k \frac{\partial}{\partial x_i}(\frac{1}{r}) \\
&= \frac{\partial V}{\partial \phi_i}(\frac{1}{r} + x_i \frac{-\phi_i}{r^2}) + \sum_{k\neq i}\frac{\partial V}{\partial \phi_k}x_k(\frac{-\phi_i}{r^2}) \\
&= \frac{1}{r}\frac{\partial V}{\partial \phi_i} - \frac{\partial V}{\partial \phi_i}\frac{\phi_i^2}{r} + \sum_{k\neq i}\frac{\partial V}{\partial \phi_k}(-1)\frac{\phi_k\phi_i}{r} \\
&= \frac{1}{r}\frac{\partial V}{\partial \phi_i} + \frac{-1}{r}\Big(\sum_{k=1}^{n}\frac{\partial V}{\partial \phi_k}\phi_k\Big)\phi_i
\end{aligned}$$

Hence we have

$$\nabla_x V(\phi_1, ..., \phi_n) = -\frac{1}{r}(\phi^T \nabla_\phi V)\phi + \frac{1}{r}\nabla_\phi V. \tag{3.6}$$

Differentiating (3.4) with respect to r, we have

$$\phi^T \nabla_x V(r\phi) = pr^{p-1}V(\phi).$$

Hence (3.2) implies

$$r' = -pr^{p-1}V(\phi).$$

From (3.3), (3.5), (3.6) and $P^\phi \phi = 0$, we have

$$\begin{aligned}
r\phi' &= -P^\phi \nabla_x V(x) \\
&= -P^\phi\Big[pr^{p-1}V(\phi)\phi + r^p\Big(-\frac{1}{r}(\phi^T\nabla_\phi V)\phi + \frac{1}{r}\nabla_\phi V\Big)\Big] \\
&= -r^{p-1}P^\phi \nabla_\phi V
\end{aligned}$$

Thus with $p = 3$ the polar equation of (2.5) is

$$\frac{dr}{dt} = -3r^2 V(\phi) \tag{3.7}$$

$$\frac{d\phi}{dt} = -rP^\phi \nabla_\phi V \tag{3.8}$$

We discuss the resulting dynamics of system (3.7), (3.8) in the next section.

4. Blow up rate

In this section we discuss the global dynamics of system (3.7), (3.8). In particular we summarize blow up rates versus convergence behavior in Theorem 4.1.

The trivial equilibrium $x \equiv 0$ of (2.1), alias $r \equiv 0$ of (3.7), (3.8) implies that the set $\{r > 0\}$ of positive radii $r = |x|_2$ is forward and backward invariant. Moreover, systems (2.1) and (3.7), (3.8) are time reversible: $x(t)$, $r(t)$, $\phi(t)$ solve their respective equations, if and only if $-x(-t)$, $r(-t)$, $-\phi(-t)$ do, with respective odd Lyapunov functions $V(-x) = -V(x)$ and $V(-\phi) = -V(\phi)$. Therefore we restrict attention to forward solutions $0 \le t < T \le \infty$ on their maximal time interval of existence up to $T = T(r(0), \phi(0))$. Finite time blow up occurs if and only if $T < \infty$, i.e.

$$\lim_{t \uparrow T} r(t) = +\infty. \tag{4.1}$$

Here we used that $\phi \in S^{n-1}$ is compact, and hence cannot blow up. Common terminology speaks of grow up if (4.1) holds with $T = \infty$, i.e. in case of global existence of solutions $(r(t), \phi(t))$.

To study the global behavior of system (3.7), (3.8), it is useful to rescale time t to a new time τ. Writing $t = t(\tau)$ for the inverse transformation, introducing the new variables $R(\tau) := r(t)$, $\Phi(\tau) := \phi(t)$ we rescale system (3.7), (3.8) to become

$$\frac{dt}{d\tau} = \frac{1}{R} > 0, \tag{4.2}$$

$$\frac{dR}{d\tau} = -3RV(\Phi), \tag{4.3}$$

$$\frac{d\Phi}{d\tau} = -P^{\phi}\nabla_{\phi}V(\Phi), \tag{4.4}$$

with initial condition $t = 0$ at $\tau = 0$ and $R = r$, $\Phi = \phi$ there. Because $\Phi \in S^{n-1}$ is compact and V is bounded on S^{n-1}, all solutions of (4.3)-(4.4) exist globally for all times $\tau \in \mathbb{R}$. Moreover $t = t(\tau)$ is strictly increasing and

$$0 < T = \lim_{\tau \to \infty} t(\tau) = \int_0^{\infty} \frac{1}{R(\tau)} d\tau \le \infty, \tag{4.5}$$

is the maximal time of existence in the original time variable $0 \le t < T$. In particular, finite time blow-up $T < \infty$ is equivalent to a finite integral in (4.5).

We proceed with care and study the autonomous equation (4.4) on $\Phi \in S^{n-1}$ first.

Lemma 4.1. *The system* (4.4) *is autonomous in* Φ *and leaves* S^{n-1} *invariant. Furthermore* $V(\Phi)$ *is a Lyapunov function for the system* (4.4)*. Hence* $\mathrm{dist}(\Phi(\tau), \widetilde{E}) \to 0$ *as* $\tau \to \infty$*, where* $\widetilde{E} \subseteq S^{n-1}$ *denotes the set of equilibria of* (4.4).

Proof. Since P^{ϕ} projects orthogonally onto the tangent space $T_{\phi}S^{n-1}$, the system (4.4) leaves S^{n-1} invariant in forward and backward time τ.

Next we show that (4.4) is a gradient system. By Cauchy Schwarz inequality we have

$$\begin{aligned}\frac{d}{d\tau}V(\Phi(\tau)) =&\dot{V}(\Phi) = (\nabla_\phi V)^T(-P^\phi \nabla_\phi V)\\ =&-(\nabla_\phi V)^T(I-\Phi\Phi^T)\nabla_\phi V\\ =&-|\nabla_\phi V|^2 + |\Phi^T\nabla_\phi V|^2 \leq 0.\end{aligned} \tag{4.6}$$

Let $M = \{\Phi \in S^{n-1} : \dot{V}(\Phi) = 0\}$. Equality in (4.6) holds if and only if $\nabla_\phi V(\Phi) = \alpha\Phi$ for some $\alpha \in \mathbb{R}$. It is easy to verify that $M = \widetilde{E} = \{\Phi \in S^{n-1} : \Phi$ is an equilibrium of (4.4)$\}$. By LaSalle's invariance principle, it follows that $\mathrm{dist}(\Phi(\tau), \widetilde{E}) \to 0$ as $\tau \to \infty$. □

Since $V(\Phi)$ is a Lyapunov function of (4.4) on the compact sphere $\Phi \in S^{n-1}$, we have a finite monotone limit

$$V_* = \lim_{\tau\to\infty} V(\Phi(\tau)) \in \mathbb{R}. \tag{4.7}$$

We can now formulate our main blow up and convergence result for system (3.7), (3.8) alias (2.1).

Theorem 4.1. *Let $x(0) \neq 0$, alias, $r(0) = |x(0)|_2 > 0$, and define the angular limit V_* by (4.5) above. Then two cases arise for the solutions $(r(t), \phi(t))$ of (3.7), (3.8) and their maximal time interval of existence $0 \leq t < T \leq \infty$.*

Case 1: $V_* \geq 0$.
Then $T = +\infty$ and $r(t) \downarrow r_*$ decreases monotonically to r_* for $t \to \infty$. The limit r_* is strictly positive if and only if

$$\int_0^\infty V(\Phi(\tau))d\tau < \infty. \tag{4.8}$$

If the integral is infinite, however, then $r_* = 0$ and $x(t)$ converges to the trivial equilibrium for $t \to \infty$. The latter case occurs, in particular, whenever $V_* > 0$.

Case 2: $V_* < 0$.
Then $T < \infty$ and $r(t)$ blows up in finite time, as in (4.1). The blow up rate is $(T-t)^{-1}$; more pricisely

$$\lim_{t\uparrow T}\ (T-t)r(t) = (3|V_*|)^{-1}. \tag{4.9}$$

In either case, the angular direction $\phi(t) = \frac{x(t)}{|x(t)|_2} = \Phi(\tau)$ of blow up or of convergence converges to the equilibrium set $\widetilde{E}$ of (4.4) for $t \uparrow T \leq \infty$.

Proof. We address the case $V_* \geq 0$, first. Since $V(\phi(t)) = V(\Phi(\tau)) \geq V_* \geq 0$, the variable $r(t) > 0$ indeed decreases monotonically, by (3.7), and hence remains bounded. Therefore $T = \infty$, and solutions exist globally in t. Moreover the monotone limit $r_* = \lim_{t\to\infty} r(t) \geq 0$ exists. Note $r_* = R_* = \lim_{\tau\to\infty} R(\tau)$. In particular

(4.3) implies

$$-\ln R_* + \ln R(0) = -\int_0^\infty \frac{d}{d\tau}\ln(R(\tau))d\tau = -\int_0^\infty (R(\tau))^{-1}\frac{d}{d\tau}R(\tau)d\tau$$
$$= 3\int_0^\infty V(\Phi(\tau))d\tau.$$

This proves the remaining claims of Case 1; see in particular (4.8).

We address the case $V_* < 0$, next. Let $\rho(t) = \frac{1}{r(t)}$. Then (3.7) implies

$$\frac{d\rho}{dt} = 3V(\phi(t)). \tag{4.10}$$

Choose $\tau_0 > 0$ such that $V(\Phi(\tau_0)) < 0$ and let $t_0 = t(\tau_0) > 0$ be defined by the time rescaling (4.2). Since $V(\phi(t)) = V(\Phi(\tau))$ is monotonically decreasing, $V(\phi(t)) \leq V(\Phi(\tau_0)) < 0$ for $t \geq t_0$. Therefore (4.10) implies that $\rho(t)$ is decreasing monotonically and $\rho(T) = 0$ at some finite time T, where

$$\frac{1}{r(t)} = \rho(t) = \rho(t) - \rho(T) = 3\int_t^T (-V(\phi(s)))ds$$
$$=3\int_t^T (-V_* + o(1))ds$$
$$=3(T-t)(-V_* + o(1)). \tag{4.11}$$

Here $o(1) = V_* - V(\phi(t))$ denotes a correction which converges to zero for $t \uparrow T$. The reciprocal of (4.11) implies blow up of $r(t)$ with rate $(T-t)^{-1}$ and positive coefficient $-V_*$ as claimed in (4.9). This proves Theorem 4.1. $\square$

We remark that dichotomy of the two cases of Theorem 4.1 does not allow for infinite time grow up $r(t) \to \infty$ for $t \to T = \infty$. The case $V_* = 0$ of an equilibrium $\phi = \Phi$ of the rescaled angular equation (4.4) is of course degenerate: by homogeneity of V, it implies $\nabla_x V = 0$ at $x = r\phi$ for all r. Thus we may consider convergence to finite $r_* > 0$ as exceptional nongeneric or degenerate , and we arrive at the dichotomy that either $r(t) \to 0$ or else $r(t) \to \infty$ in forward time $t \to +\infty$. By reversibility, the same holds true in backward time. Trajectories which start at the zero level set $\{V = 0\}$ with $r(0) > 0$, then blow up in, both, forward and backward finite time $t = T_\pm$, for some maximal time interval of existence $T_- < t < T_+$. Trajectories which do not cross $\{V = 0\}$ blow up in precisely one time direction, depending on their sign of V, and converge to zero in the opposite direction.

In case the rescaled angular equation (4.4) possesses only a discrete, and hence finite, set of equilibria, the asymptotic direction $\Phi = \Phi_*$ of blow up or convergence is also well-defined and

$$\Phi_* = \lim_{t\to\infty} x(t)/|x(t)|_2 \tag{4.12}$$

exists, for $t \to T$. If the equilibrium Φ_* of (4.4) is a nondegenerate critical point of V on S^{n-1}, i.e. hyperbolic, then the convergence of $\Phi(\tau)$ to Φ_*, and of $V(\Phi(\tau))$ to V_*, is exponential. This further refines the blow up result (4.9). We neither pursue this equation any further here, nor do we address the general case of p-homogeneous potentials V, $p > 3$ which can be treated in a similar spirit.

5. Discussion

The rigorous results in the previous sections make a fundamental advance in renormalization group analysis. One common challenge in studying strongly correlated electron systems is the huge dimension of the couplings x_i. For instance, renormalization-group analysis ([16]-[18]) for iron-based superconductors, easily involves hundreds or even thousands of couplings. With modern computation powers, these coupled differential equations can be solved numerically within negligible errors. However, because a huge number of couplings diverges under RG transformations, the interpretation of the numerical outcome remains a difficult task.

Our theorems provide a systematic classification for the ultimate fates of the RG flows. Depending on the sign of the RG potential V, the flows can be classified into three regimes: divergent, backward-orbit and stable. One can introduce the distance to the trivial fixed point $x = 0$ as $r(t) = |x|_2$. In the divergent regime, $r(t)$ grows monotonically and the ground state is driven away from the trivial fixed point under RG transformations. On the other hand, in the stable regime, $r(t)$ decreases monotonically and all interactions become irrelevant, i.e. no instability develops within the stable basin. The most interesting case is the backward-orbit regime. The distance $r(t)$ decreases initially and starts to grow upon hitting the zero-potential surface. Because the fictitious time t corresponds to cooling down the temperature (logarithmically) in realistic situations, the backward-orbit regime implies a nontrivial crossover from the trivial fixed point (Fermi liquid) to the correlated ground state. This peculiar crossover behavior in unconventional superconductivity is the realization of the backward-orbit regime. The complete classification presented here answers the long-standing puzzle why limit cycles, chaos and so on are rarely seen in RG transformations while they often occur in generic coupled differential equations.

The presence of the RG potential enables the complete classification of the phase diagram. In addition, it also provides a powerful scheme to quantify the relevance of runaway couplings. Because the blow-up rate for the radius is $r(t) \sim (t - T)^{-1}$, where T is the blow-up time, it implies the couplings can at most diverge at the same exponent of unity. Linear analysis near the blow-up regime shows that the relevant couplings scales as $x_i \sim (t-T)^{-\gamma_i}$, with exponents $0 \leq \gamma_i \leq 1$. This inverse power law was proposed before in previous studies, but can now be derived from Theorem 4.2. The robustness of these exponents renders the unique way to build up the hierarchy of all relevant couplings and will be extremely helpful to identify the dominant interactions in numerical RG studies.

In summary, the presence of a homogeneous renormalization-group potential V allows us to prove several rigorous theorems and, in consequence, find the complete classification for the potential flows. Meanwhile, the relevant couplings follow the scaling form described by a set of RG exponents. The above findings are of great importance in interpreting numerical RG studies and also revealing the generic structure of the phase diagrams for strongly correlated electron systems.

References

[1] R. Shankar, *Renormalization-group approach to interacting fermions*, Rev. Mod. Phys. **66**, 129-192 (1994).

[2] M. Salmhofer and C. Honerkamp, *Fermionic renormalization group flows – technique and theory*, Progress of Theoretical Physics **105**, 1-35 (2001).

[3] W. Metzner, M. Salmhofer, C. Honerkamp, V. Meden and K. Schönhammer, *Functional renormalization group approach to correlated fermion systems*, Rev. Mod. Phys. 84, 299-352 (2012).

[4] M. Fabrizio, *Role of transverse hopping in a two-coupled-chains model*, Phys. Rev. B **48**, 15838-15860 (1993).

[5] L. Balents and M. P. A. Fisher, *Weak-coupling phase diagram of the two-chain Hubbard model*, Phys. Rev. B **53**, 12133 (1996).

[6] H. J. Schulz, *Phases of two coupled Luttinger liquids*, Phys. Rev. B **53**, R2959-R2962 (1996).

[7] H.-H. Lin, L. Balents and M. P. A. Fisher, *N-chain Hubbard model in weak coupling*, Phys. Rev. B **56**, 6569-6593 (1997).

[8] H.-H. Lin, L. Balents and M. P. A. Fisher, *Exact SO(8) symmetry in the weakly-interacting two-leg ladder*, Phys. Rev. B **58**, 1794-1825 (1998).

[9] M.-H. Chang, W. Chen and H.-H. Lin, *Renormalization group potential for quasi-one-dimensional correlated systems*, Prog. Theor. Phys. Suppl. **160**, 79-113 (2005).

[10] E. Szirmai and J. Solyom, *Possible phases of two coupled n-component fermionic chains determined using an analytic renormalization group method*, Phys. Rev. B **74**, 155110 (2006).

[11] H.-Y. Shih, W.-M. Huang, S.-B. Hsu and H.-H. Lin, *Hierarchy of relevant couplings in perturbative renormalization group transformations*, Phys. Rev. B **81**, 121107(R) (2010).

[12] A. Goriely and C. Hyde, *Finite time blow-up in dynamical systems*, Phys. Lett. A **250**, 311-318 (1998)

[13] A. Goriely and C. Hyde, *Necessary and sufficient conditions for finite time singularity in ordinary differential equations*, J. of diff. eq. **161**, 422-448 (2000).

[14] S. B. Hsu, *Ordinary Differential Equations* (second edition), World Scientific Press (2013).

[15] J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience (1969).

[16] A. V. Chubukov, D. V. Efremov and I. Eremin, *Magnetism, superconductivity, and pairing symmetry in iron-based superconductors*, Phys. Rev. B **78**, 134512 (2008).

[17] F. Wang, H. Zhai, Y. Ran, A. Vishwanath and D.-H. Lee, *Functional renormalization-group study of the pairing symmetry and pairing mechanism of the FeAs-based high-temperature superconductor*, Phys. Rev. Lett. **102**, 047005 (2009).

[18] F. Wang and D.-H. Lee, *The electron-pairing mechanism of iron-based superconductors*, Science **332**, 200-204 (2011).

Department of Mathematics and The National Center for Theoretical Science, National Tsing-Hua University, Hsinchu 300, Taiwan

E-mail address: `sbhsu@math.nthu.edu.tw`

Institute of Mathematics, Free University Berlin, Arnimallee 3, D-14195 Berlin, Germany

E-mail address: `Bernold.Fiedler@gmail.com`

Department of Physics and The National Center for Theoretical Science, National Tsing Hua University, Hsinchu 30013, Taiwan

E-mail address: `hsiuhau@phys.nthu.edu.tw`