

A DISCRETE-DELAYED MODEL WITH PLASMID-BEARING, PLASMID-FREE COMPETITION IN A CHEMOSTAT

SZE-BI HSU

Department of Mathematics
National Tsing-Hua University
Hsin-Chu 30043, Taiwan.

CHENG-CHE LI

Holistic Education Center
St. John's and St. Mary's Institute of Technology
Tam-Shui 25135, Taiwan.

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ABSTRACT. A discrete-delayed model of plasmid-bearing, plasmid-free organisms competing for a single-limited nutrient in a chemostat is established. Rigorous mathematical analysis of the asymptotic behavior of this model is presented. An interesting method to analyze the local stability of interior equilibrium is developed. The argument is also applicable to a model of plasmid-bearing, plasmid-free organisms competing for two complementary nutrients in a chemostat.

1. Introduction. The chemostat is a basic piece of laboratory apparatus. It plays an important role in many fields, such as ecology, microbiology, chemical engineering, etc. Smith and Waltman had made thorough discussions about the chemostat models in [15]. Genetically altered organisms are used in industry to manufacture a pharmaceutical product (e.g., in the production of insulin). The alternation is accomplished by the introduction of DNA into the cell in the form of a plasmid. Plasmid contains bits of DNA which exist separately from the chromosome and replicates independently. The metabolic load imposed by the production can result in the genetically altered (the plasmid-bearing) organism being a less competitor than the plasmid-free organism. The plasmid can also be lost in the reproductive process, that is, it may not be passed to the daughter cells, producing a plasmid-free organism (the 'wild' type). Since commercial production may take place on a scale of many generations, it is important to understand the asymptotic behavior of these models. A model of competition for a single-limited nutrient between plasmid-bearing and plasmid-free organisms in a chemostat was proposed by Stephanopoulos and Lapidus [17], who give a local analysis of various cases. A global analysis of the behavior of system trajectories was presented by Hsu, Waltman, and Wolkowicz (see [12]).

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The models contain discrete time delays which account for the (species-specific) time period which lapses between uptake of nutrient and the assimilation of nutrient into viable biomass. Golpalsamy [6] and Kuang [13] discussed various aspects of models with discrete time delay. Freedman, So, and Waltman [5] were the first to incorporate time delay in chemostat models. Ellermeyer [4], and Hsu, Waltman, and Ellermeyer [11] analyzed a discrete time delay model with two competitive organisms for a single nutrient in a chemostat.

We develop and analyze a model with discrete time delay and competition between plasmid-bearing, plasmid-free organisms in a chemostat. In section 2, we describe the model and explain the biological meanings of the parameters and the variables. Then we scale the model in non-dimensional form and then consider its limiting system. In section 3, we analyze the local stability of the equilibria. Here an interesting method to analyze the local stability of the interior equilibrium is presented. In section 4, we use the Barbălat's lemma and Fluctuation lemma to prove the global stability of the boundary equilibria. Though we are unable to prove the global stability of the interior equilibrium, we use a perturbation theory to show that the global stability still holds when q is sufficiently close to 0. We also prove that the differential inequality theorem for ODEs is available for some discrete delay equations. Section 5 is the discussion section. We also introduce a model in which the plasmid-bearing, plasmid-free organisms competing for two complementary nutrients instead of one nutrient. At last, we present the computations used in section 3 in Appendix.

2. The Model. In this section, we introduce a discrete time delay model describing plasmid-bearing, plasmid-free organisms competing exploitatively for a single growth-limit nutrient in a well-stirred chemostat as follows

$$\begin{aligned} S'(t) &= (S^{(0)} - S(t))D - \frac{x_1(t)}{y_{S1}} f_1(S(t)) - \frac{x_2(t)}{y_{S2}} f_2(S(t)), \\ x_1'(t) &= -Dx_1(t) + (1 - q)x_1(t - \tau_1)f_1(S(t - \tau_1)), \\ x_2'(t) &= -Dx_2(t) + qx_1(t - \tau_1)f_1(S(t - \tau_1)) + x_2(t - \tau_2)f_2(S(t - \tau_2)), \\ S(\theta) &= \psi_0(\theta), \quad x_1(\theta) = \psi_1(\theta), \quad x_2(\theta) = \psi_2(\theta), \quad -\tau \leq \theta \leq 0, \\ (\psi_0, \psi_1, \psi_2) &\in C_3^+, \end{aligned} \tag{1}$$

where $C_3^+ = \{\varphi = (\varphi_0, \varphi_1, \varphi_2) : \varphi_i(\theta) \geq 0 \text{ and continuous for all } \theta \in [-\tau, 0], i = 0, 1, 2, \text{ where } \tau = \max\{\tau_1, \tau_2\}\}$. Here $S(t)$ denotes the concentration of the nutrient in the chemostat at time t ; for $i = 1, 2$, $x_i(t)$ denotes, at time t , the biomass of the plasmid-bearing organisms and plasmid-free organisms, respectively; the growth rates and the consumption rates of plasmid-bearing and plasmid-free organisms are f_1 , f_2 , f_1/y_{S1} , and f_2/y_{S2} , respectively, where y_{Si} , $i = 1, 2$, are the yield constants; the constant q , $0 < q < 1$, represents the probability that plasmid-bearing species lost its plasmid during duplication; the operating parameters are $S^{(0)}$, the input concentration of the nutrient and D , the washout rate of the chemostat; the constants τ_1 and τ_2 stand for the time delays in conversion of nutrient to viable biomass for plasmid-bearing and plasmid-free organisms, respectively.

By using the method of steps, it can be shown that for each $\varphi \in C_3^+$, there is a unique solution of (1), $\pi(\varphi; t) = (S(\varphi; t), x_1(\varphi; t), x_2(\varphi; t)) \in R_3^+$ through φ , that is well-defined for all $t \geq 0$ and satisfies $\pi(\varphi; \cdot)|_{[-\tau, 0]} = \varphi$.

For $i = 1, 2$, we assume the growth rates $f_i(S)$ satisfy the following conditions: $f_i(S)$ is continuously differentiable, $f_i(0) = 0$, and $f'_i(S) > 0$ for all $0 < S \leq S^{(0)}$. The most commonly used growth rate is the Michaelis-Menten type,

$$\frac{\mu_S S}{K_S + S}.$$

There are some other types of f_i discussed in Hsu, Waltman, and Wolkowicz [12]. Furthermore, we assume that the yield constants are the same for both populations, i.e., $y_{S1} = y_{S2} = y_S$, and that the time delays for both organisms are the same, i.e., $\tau_1 = \tau_2 = \tau$.

By measuring concentrations of nutrient in units of $S^{(0)}$, time in units of $1/D$, and x_i in units of $y_S S^{(0)}$, the number of parameters can be reduced and the equations take the non-dimensional form

$$\begin{aligned} S'(t) &= 1 - S(t) - x_1(t)f_1(S(t)) - x_2(t)f_2(S(t)), \\ x'_1(t) &= -x_1(t) + (1 - q)x_1(t - \tau)f_1(S(t - \tau)), \\ x'_2(t) &= -x_2(t) + qx_1(t - \tau)f_1(S(t - \tau)) + x_2(t - \tau)f_2(S(t - \tau)), \\ S(\theta) &= \psi_0(\theta), \quad x_1(\theta) = \psi_1(\theta), \quad x_2(\theta) = \psi_2(\theta), \quad -\tau \leq \theta \leq 0, \\ (\psi_0, \psi_1, \psi_2) &\in C_3^+. \end{aligned} \tag{2}$$

Now we simplify system (2) via taking its limiting system. Let $\Sigma(t) = 1 - S(t) - x_1(t + \tau) - x_2(t + \tau)$, then it follows from system (2) that $\Sigma'(t) = -\Sigma(t)$ for all $t \geq 0$ and consequently,

$$S(t) + x_1(t + \tau) + x_2(t + \tau) = 1 + \epsilon(t), \quad t \geq 0, \tag{3}$$

where $\epsilon(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$.

It is easy to see, for every $\varphi \in C_3^+$, the solution $\pi(\varphi; t)$ remains positive and bounded for $t > 0$ by the following arguments. If $S(\xi) = 0$ for some $\xi \geq 0$, then $S'(\xi) > 0$. This implies that $S(t) > 0$ for all $t > 0$. That $x_1(t), x_2(t) > 0$ follows directly by

$$\begin{aligned} x_1(t) &= \varphi_1(0)e^{-t} + \int_0^t e^{-(t-\theta)}(1 - q)f_1(S(\theta - \tau))x_1(\theta - \tau)d\theta \\ x_2(t) &= \varphi_2(0)e^{-t} + \int_0^t e^{-(t-\theta)}\left(qf_1(S(\theta - \tau))x_1(\theta - \tau) + f_2(S(\theta - \tau))x_2(\theta - \tau)\right)d\theta \end{aligned}$$

for $t > 0$. That $\pi(\varphi; t)$ is bounded follows immediately from (3).

Clearly $\lim_{t \rightarrow \infty} \Sigma(t) = 0$. Since what we concern is the asymptotic behavior of this model, we consider its limiting system

$$\begin{aligned} x'_1 &= -x_1 + (1 - q)x_1(t - \tau)f_1(1 - x_1 - x_2), \\ x'_2 &= -x_2 + qx_1(t - \tau)f_1(1 - x_1 - x_2) + x_2(t - \tau)f_2(1 - x_1 - x_2). \end{aligned} \tag{4}$$

We shall employ the following notation for the relevant rest points of system (2). We say that a rest point of (2) does not exist if any one of its components is negative. Since $\lim_{t \rightarrow \infty} \Sigma(t) = 0$, any rest point $\bar{E} = (\bar{S}, \bar{x}_1, \bar{x}_2)$ of (2) must satisfy

$$1 - \bar{S} - \bar{x}_1 - \bar{x}_2 = 0.$$

The washout rest point is denoted by $E_0^3 = (1, 0, 0)$. There is only one possible rest point involving plasmid-free organisms but no plasmid-bearing organisms, denoted by $E_2^3 = (\lambda_2, 0, 1 - \lambda_2)$ where λ_2 is defined as the unique value of S where

$$f_2(\lambda_2) = 1 \tag{5}$$

Case	Criteria	Rest points (stable one is bold)
I	$\lambda_{1q} > 1, \lambda_2 > 1$	$\{\mathbf{E}_0\}$
II	$\lambda_2 < 1 < \lambda_{1q}$	$\{E_0, \mathbf{E}_2\}$
III	$\lambda_2 < \lambda_{1q} < 1$	$\{E_0, \mathbf{E}_2\}$
IV	$\lambda_{1q} < 1 < \lambda_2$	$\{E_0, \mathbf{E}_c\}$
V	$\lambda_{1q} < \lambda_2 < 1$	$\{E_0, E_2, \mathbf{E}_c\}$

TABLE 1.

(if one exists). The mixed culture rest point is denoted by $E_c^3 = (\lambda_{1q}, x_1^*, x_2^*)$, where λ_{1q} is defined as the unique value that satisfies

$$f_1(\lambda_{1q}) = \frac{1}{1-q} \quad (6)$$

(if it exists). When E_c^3 exists (this will be discussed in section 3),

$$x_1^* = \frac{(1-\lambda_{1q})(1-f_2(\lambda_{1q}))}{f_1(\lambda_{1q})-f_2(\lambda_{1q})},$$

$$x_2^* = 1 - \lambda_{1q} - x_1^*.$$

It can easily be seen that no rest point can exist where there are plasmid-bearing organisms but no plasmid-free organisms.

The corresponding rest points of (4) are simply the projections on $(x_1 - x_2)$ -space and are denoted by:

$$E_0 = (0, 0), \quad E_2 = (0, 1 - \lambda_2), \quad E_c = (x_1^*, x_2^*).$$

We assume that these do not exist if the sum of the components exceeds 1.

3. Local Stability Analysis. The local stability analysis of system (4) breaks conveniently into five cases. These are summarized in **Table 1**.

The analysis for the local stability of E_0 and E_2 is standard, for details see Ellermeyer [4]. Note that Hayes' theorem (see Bellman and Cooke [1] pg.444) is the main tool for this analysis. However, Hayes' theorem cannot be applied directly to the analysis of the local stability of E_c . To analyze the local stability of this steady state, we develop an elementary but interesting method stated as follows.

We assume that $\lambda_{1q} < 1$ and $\lambda_{1q} < \lambda_2$, i.e., the steady state E_c exists. To simplify our statement, we fix τ and make q vary in the following argument. For each q , the stability of E_c with respect to (4) is determined by the stability of the trivial solution of the linearized system

$$\begin{aligned} z_1'(t) &= -z_1(t) + (1-q)f_1(\lambda_{1q})z_1(t-\tau) - (1-q)f_1'(\lambda_{1q})x_1^*z_1(t) \\ &\quad - (1-q)f_1'(\lambda_{1q})x_1^*z_2(t), \\ z_2'(t) &= -z_2(t) + qf_1(\lambda_{1q})z_1(t-\tau) - qf_1'(\lambda_{1q})x_1^*z_1(t) - qf_1'(\lambda_{1q})x_1^*z_2(t) \\ &\quad + f_2(\lambda_{1q})z_2(t-\tau) - f_2'(\lambda_{1q})x_2^*z_1(t) - f_2'(\lambda_{1q})x_2^*z_2(t). \end{aligned}$$

The above system has non-trivial solutions of the form $(z_1(t), z_2(t)) = (k_1 e^{\lambda t}, k_2 e^{\lambda t})$ if and only if λ is a solution of the characteristic equation

$$H(\lambda, \tau) = (\lambda + \alpha - e^{-\lambda\tau})(\lambda + \beta - e^{-\lambda\tau} f_2(\lambda_{1q})) + qf_1'(\lambda_{1q})x_1^* e^{-\lambda\tau} - (\alpha - 1)(\beta - 1) = 0,$$

where

$$\alpha = 1 + (1-q)f_1'(\lambda_{1q})x_1^*,$$

$$\beta = 1 + qf_1'(\lambda_{1q})x_1^* + f_2'(\lambda_{1q})x_2^*.$$

It is clear that Hayes' theorem is not applicable to this characteristic equation.

To show that the steady state E_c is stable, first we claim that there is no ω satisfying $H(i\omega, \tau) = 0$ for all $\tau > 0$. Assume that τ is fixed. From $ImH(i\omega, \tau) = 0$, we acquire that

$$\omega = \frac{f_2(\lambda_{1q}) \sin(2\omega\tau) - \gamma \sin(\omega\tau)}{\alpha + \beta - (1 + f_2(\lambda_{1q})) \cos(\omega\tau)}, \tag{7}$$

where $\gamma = 1 + f_2'(\lambda_{1q})x_2^* + \alpha f_2(\lambda_{1q})$. Substituting equation (7) into the ω in $ReH(i\omega, \tau) = 0$ and simplifying it with elementary trigonometric rules, we have

$$F(\cos(\omega\tau)) = -a_4 \cos^4(\omega\tau) + a_3 \cos^3(\omega\tau) - a_2 \cos^2(\omega\tau) + a_1 \cos(\omega\tau) - a_0 = 0, \tag{8}$$

where

$$\begin{aligned} a_4 &= 4f_2^2(\lambda_{1q}), \\ a_3 &= 2f_2(\lambda_{1q})[2\gamma + (\alpha + \beta)(1 + f_2(\lambda_{1q}))], \\ a_2 &= \gamma^2 + f_2(\lambda_{1q})(1 - f_2(\lambda_{1q}))^2 + (1 + f_2(\lambda_{1q}))^2(\alpha + \beta - 1) \\ &\quad + \gamma(1 + f_2(\lambda_{1q}))(\alpha + \beta) + 2f_2(\lambda_{1q})(\alpha + \beta)^2, \\ a_1 &= \gamma(1 - f_2(\lambda_{1q}))^2 + 2(1 + f_2(\lambda_{1q}))(\alpha + \beta)(\alpha + \beta - 1) + \gamma(\alpha + \beta)^2, \\ a_0 &= \gamma(\alpha + \beta)(1 + f_2(\lambda_{1q})) - \gamma^2 + (\alpha + \beta)^2(\alpha + \beta - 1 - f_2(\lambda_{1q})). \end{aligned} \tag{9}$$

Note that $a_i > 0$ for $i = 0, 1, 2, 3, 4$. Now we only need to prove that $F(X) = 0$ has no solution lies in $[-1, 1]$. It is not difficult to see $F(1) < 0$ by the fact that $\alpha + \beta + f_2(\lambda_{1q}) > 1 + \gamma$. Thus we only need to show that $F'(X) > 0$ for $X \in [-1, 1]$. This can be done if we can show that $G'(Y) > 0$ for $Y < 0$, where $Y = X - 1$ and $G(Y) = F(X)$. This is equivalent to show that

$$\begin{aligned} 0 &< a_4, \\ 0 &< -12a_4 + 3a_3, \\ 0 &< 12a_4 - 6a_3 + 2a_2, \\ 0 &< -4a_4 + 3a_3 - 2a_2 + a_1. \end{aligned} \tag{10}$$

From the facts $\alpha > 1$, $\beta > 1$, $\alpha + \beta > \gamma > 1 + f_2(\lambda_{1q})$, and $\alpha + \beta + f_2(\lambda_{1q}) > 1 + \gamma$, we are able to prove the above inequalities (for details, see **Appendix**). Therefore, this claim is proved.

To show that the steady state E_c is locally asymptotically stable, we claim that $H(\lambda, \tau) = 0$ has no roots with non-negative real parts for all $\tau \geq 0$. We know that the roots of $H(\lambda, 0) = 0$ are two roots with negative real parts by [12]. Then we note that, for a retarded equation, the supremum of the real parts of the roots of the transcendental equation varies continuously with τ (see Datko [3], Cooke and Grossman [2]). Now we define $\mathcal{S}(\tau)$ as a function of τ to denote the supremum of the real parts of the roots of $H(\lambda, \tau) = 0$. If there exists a $\tau_0 > 0$ and one root of $H(\lambda, \tau_0) = 0$ is with positive real part, i.e., $\mathcal{S}(\tau_0) > 0$, then we know that there exists a τ^* between 0 and τ_0 such that $\mathcal{S}(\tau^*) = 0$ by $\mathcal{S}(0) < 0$ and the above continuity argument. Now we show that there exists a ω^* satisfying $H(i\omega^*, \tau^*) = 0$. Since $H(\lambda, \tau^*)$ is analytic, it can have only a finite number of zeros in any compact set of the complex plane. Thus, assume that there exists a sequence $\{z_n\}$ of roots of $H(\lambda, \tau^*) = 0$ such that $Re z_n \rightarrow 0^-$, $Im z_n \rightarrow \infty$ as $n \rightarrow \infty$. However, note that

$$0 = \lim_{n \rightarrow \infty} \frac{H(z_n, \tau^*)}{z_n^2} = \lim_{n \rightarrow \infty} 1 + \frac{H(z_n, \tau^*) - z_n^2}{z_n^2} = 1,$$

where the last equality holds by the observation $\lim_{|\lambda| \rightarrow \infty} \frac{H(\lambda, \tau^*) - \lambda^2}{\lambda^2} = 0$. This leads to a contradiction. Thus, we know that there are only a finite number of roots for $H(\lambda, \tau^*) = 0$. Then, by $\mathcal{S}(\tau^*) = 0$, we know there exists a real number ω^* such

that $H(i\omega^*, \tau^*) = 0$. However, this contradicts to the conclusion that there is no ω satisfying $H(i\omega, \tau) = 0$ for all $\tau > 0$. Hence, such τ_0 does not exist. This proves the local stability of E_c . We extend the above argument in the form of a theorem:

Theorem 1. *Let $H(\lambda, \tau) = 0$ be a transcendental characteristic equation of an equilibrium E . Assume that there exists a positive integer n such that*

$$\lim_{|\lambda| \rightarrow \infty} \frac{H(\lambda, \tau) - \lambda^n}{\lambda^n} = 0$$

holds for all $\tau > 0$. If the equilibrium is locally stable when $\tau = 0$ and $H(\lambda, \tau) = 0$ has no solution for λ on the imaginary axis when $\tau > 0$, then the equilibrium E is locally stable for any delay $\tau > 0$.

Thus, we complete the local stability analysis of **Table 1**.

4. Global stability analysis. In this section, we shall prove that the local stable equilibria in **Table 1** are global stable. We can prove that this is true for the cases I, II, and III by Barbalát's lemma and Fluctuation lemma. However, in cases IV and V, we can only prove it is true when $0 < q \ll 1$.

Now we claim, via the outline of Wolkowicz and Xia [19], that the equilibria E_0, E_2 , under desired conditions, are globally stable. First, we state the following Barbalát's lemma and Fluctuation lemma.

Lemma 1. *Let $a \in (-\infty, \infty)$ and $f : [a, \infty) \rightarrow R$ be a differentiable function. If $\lim_{t \rightarrow \infty} f(t)$ exists (finite) and the derivative function $f'(t)$ is uniformly continuous on (a, ∞) , then $\lim_{t \rightarrow \infty} f'(t) = 0$.*

Since we only consider the asymptotical behavior of system (2), we may assume the initial functions of it are bounded by equation (3). This assumption makes $\pi(\varphi; t)$ bounded. By the fact that f_1, f_2 are continuously differentiable and $\pi(\varphi; t)$ is bounded, we see that $\pi''(\varphi; t)$ is bounded. Thus, the uniform continuity condition in **Lemma 1** holds.

Lemma 2. *Let $f : R^+ \rightarrow R$ be a differentiable function. If $\liminf_{t \rightarrow \infty} f(t) < \limsup_{t \rightarrow \infty} f(t)$, then there are sequences $\{t_m\} \uparrow \infty$ and $\{s_m\} \uparrow \infty$ such that for all m*

$$\begin{aligned} f(t_m) &\rightarrow \limsup_{t \rightarrow \infty} f(t) \quad \text{as } m \rightarrow \infty, \quad f'(t_m) = 0 \quad \text{and} \\ f(s_m) &\rightarrow \liminf_{t \rightarrow \infty} f(t) \quad \text{as } m \rightarrow \infty, \quad f'(s_m) = 0. \end{aligned}$$

The proofs of **Lemma 1** and **Lemma 2** can be seen in Gopalsamy [6] and in Hirsch, Hanisch, and Gabriel [8], respectively.

Before proving the results, we introduce the following notation

$$\begin{aligned} \gamma_1 &= \limsup_{t \rightarrow \infty} x_1(t), & \delta_1 &= \liminf_{t \rightarrow \infty} x_1(t); \\ \gamma_2 &= \limsup_{t \rightarrow \infty} x_2(t), & \delta_2 &= \liminf_{t \rightarrow \infty} x_2(t). \end{aligned} \tag{11}$$

Now we prove the desired results in a series lemmas.

Lemma 3. *If $\lambda_{1q} < 1$ (or $\lambda_2 < 1$), then $S(t) < 1$ for all sufficiently large t .*

Proof. First note that if $S(\bar{t}) = 1$ for some $\bar{t} > 0$, then $S'(\bar{t}) < 0$ and so if $S(T) \leq 1$ for some $T \geq 0$ then $S(t) < 1$ for all $t > T$.

Suppose that $S(t) > 1$ for all large t . Then $S'(t) < 0$ and so $S(t) \downarrow S^* \geq 1$ for some S^* as $t \rightarrow \infty$. (Since $S(t)$ is bounded below and satisfies system (2), we must have $S^* = 1$.) Thus $S(t) > \lambda_{1q}$ (or $S(t) > \lambda_2$). Define

$$\begin{aligned} z_1(t) &= x_1(t) + \int_{t-\tau}^t (1-q)f_1(S(\theta))x_1(\theta)d\theta, \\ z_2(t) &= x_2(t) + \int_{t-\tau}^t (qf_1(S(\theta))x_1(\theta) + f_2(S(\theta))x_2(\theta))d\theta. \end{aligned} \tag{12}$$

Then

$$\begin{aligned} z'_1(t) &= -x_1(t) + (1-q)f_1(S(t))x_1(t) > 0, \\ (\text{or } z'_2(t) &= -x_2(t) + qf_1(S(t))x_1(t) + f_2(S(t))x_2(t) \\ &> -x_2(t) + f_2(S(t))x_2(t) > 0,) \end{aligned} \tag{13}$$

for all large t . Now set $j = 1$ when $\lambda_{1q} < 1$ (or set $j = 2$ when $\lambda_2 < 1$). Since $z_j(t)$ is clearly bounded above, equation (13) shows that $\lim_{t \rightarrow \infty} z_j(t) = z_j^* > 0$. By Barbălat's Lemma, $\lim_{t \rightarrow \infty} z'_j(t) = 0$. Since $\lim_{t \rightarrow \infty} S(t) = 1$, it follows that $x_1(t), x_2(t) \downarrow 0$ as $t \rightarrow \infty$. But from (12) $z_j(t) \rightarrow 0$ which contradicts $z_j(t) \uparrow z_j^* > 0$. \square

Lemma 4. *If $\lambda_{1q} \geq 1$, then $\lim_{t \rightarrow \infty} x_1(t) = 0$.*

Proof. It can be seen from the proof of **Lemma 3** that either $S(t) \downarrow 1$ as $t \rightarrow \infty$ or $S(t) < 1$ for all large t . Suppose that $S(t) \downarrow 1$ as $t \rightarrow \infty$. By Barbălat's Lemma, $\lim_{t \rightarrow \infty} S'(t) = 0$. Hence, $\lim_{t \rightarrow \infty} f_1(S(t))x_1(t) + f_2(S(t))x_2(t) = 0$. This leads to $x_1(t) \rightarrow 0$ as $t \rightarrow \infty$.

Now suppose that $S(t) < 1 \leq \lambda_{1q}$ for all large t . Then equation (13) gives $z'_1(t) = x_1(t)(-1 + (1-q)f_1(S(t))) \leq 0$. Thus $z_1(t) \downarrow z_1^*$ as $t \rightarrow \infty$. By Barbălat's Lemma again, we have $\lim_{t \rightarrow \infty} z'_1(t) = 0$. Thus, $\lim_{t \rightarrow \infty} x_1(t)(-1 + (1-q)f_1(S(t))) = 0$. If there exists a sequence $\{t_m\} \uparrow \infty$ such that $\lim_{m \rightarrow \infty} x_1(t_m) > 0$, then $\lim_{m \rightarrow \infty} f_1(S(t_m)) = 1/(1-q)$. Hence $\lim_{m \rightarrow \infty} S(t_m) = \lambda_{1q}$. When $\lambda_{1q} > 1$, this contradicts the fact that $S(t_m) < 1$ for all large t_m . When $\lambda_{1q} = 1$, equation (3) and the fact that $\lim_{m \rightarrow \infty} S(t_m) = \lambda_{1q} = 1$ leads to $\lim_{m \rightarrow \infty} x_1(t_m) = 0$. This also contradicts to the existence of $\{t_m\}$ and we complete this proof. \square

We prove the global stability of the steady state E_1 under the conditions of case I as follows,

Lemma 5. *If $\lambda_{1q} \geq 1$ and $\lambda_2 \geq 1$, then $\lim_{t \rightarrow \infty} (S(t), x_1(t), x_2(t)) = (1, 0, 0)$.*

Proof. By **Lemma 4** and the fact that $\lambda_{1q} \geq 1$, we have $\lim_{t \rightarrow \infty} x_1(t) = 0$. As the proof of **Lemma 4**, if $S(t) \downarrow 1$, we can see $x_2(t) \rightarrow 0$ as $t \rightarrow \infty$ from equation (3).

Now assume $S(t) < 1$ for all large t . To prove this lemma, we only need to claim that $\gamma_2 = 0$. If not, i.e., $\gamma_2 > 0$, we know, from Barbălat's lemma and Fluctuation lemma, that there exists a sequence $\{t_m\} \uparrow \infty$ such that $x_2(t_m) \rightarrow \gamma_2, x'_2(t_m) \rightarrow 0, S(t_m - \tau) \rightarrow S^\dagger \leq 1, x_2(t_m - \tau) \rightarrow x_2^\dagger \leq \gamma_2$, as $m \rightarrow \infty$. Substituting into the x_2 -equation in system (2), we have

$$0 = -\gamma_2 + f_2(S^\dagger)x_2^\dagger \leq -\gamma_2 + f_2(1)\gamma_2 \leq 0.$$

Then $f_2(1) = 1$, i.e., $\lambda_2 = 1$ and $\lim_{m \rightarrow \infty} S(t_m - \tau) = 1$, and $\lim_{m \rightarrow \infty} x_2(t_m - \tau) = \gamma_2$. However, this leads to $\lim_{m \rightarrow \infty} x_2(t_m - \tau) = 0$ by equation (3), which contradicts to $\gamma_2 > 0$. This ends the proof. \square

Next, we shall prove the global stability of E_2 under the conditions of cases II and III of **Table 1**. Again, we need the following lemmas.

Lemma 6. *If $\lambda_{1q} < 1$, then $\gamma_1 \leq 1 - \lambda_{1q}$.*

Proof. By Barbălat’s lemma and Fluctuation lemma, we know there exists a sequence $\{t_m\} \uparrow \infty$ such that $x_1(t_m) \rightarrow \gamma_1$, $x'_1(t_m) \rightarrow 0$, $x_2(t_m) \rightarrow x_2^\dagger \geq 0$, $x_1(t_m - \tau) \rightarrow x_1^\dagger \leq \gamma_1$, as $m \rightarrow \infty$. Substituting into the x_1 -equation in system (2) where $S(t - \tau)$ is replaced by $1 - x_1(t) - x_2(t) - \epsilon(t)$ (from equation (3)), we have

$$0 = -\gamma_1 + (1 - q)f_1(1 - \gamma_1 - x_2^\dagger)x_1^\dagger.$$

Therefore,

$$\gamma_1 \leq (1 - q)f_1(1 - \gamma_1)\gamma_1,$$

since $x_2^\dagger \geq 0$ and $x_1^\dagger \leq \gamma_1$. Hence, we conclude that either $\gamma_1 = 0$ or $\gamma_1 > 0$ and $1 \leq (1 - q)f_1(1 - \gamma_1)$. This inequality leads to $1 - \gamma_1 \geq \lambda_{1q}$ or, equivalently, $1 - \lambda_{1q} \geq \gamma_1$. Then this lemma holds in either case. \square

Lemma 7. *If $\lambda_2 < 1$ and $\lambda_2 < \lambda_{1q}$, then $\delta_2 > 0$.*

Proof. First, we find a constant $\varepsilon_0 > 0$ satisfies either $\varepsilon_0 < 1 - \lambda_2$, if $\lambda_2 < 1 \leq \lambda_{1q}$; or $\varepsilon_0 < \lambda_{1q} - \lambda_2$, if $\lambda_2 < \lambda_{1q} < 1$. When $\lambda_2 < 1 \leq \lambda_{1q}$, we know, by **Lemma 4**, that there exists $T > \tau$ such that $x_1(t) < \varepsilon_0/3$ and $|\epsilon(t)| < \varepsilon_0/3$, as $t \geq T$. While $\lambda_2 < \lambda_{1q} < 1$, we know, by Lemma 6, that there exists a $T > \tau$ such that $x_1(t) < 1 - \lambda_{1q} + \varepsilon_0/3$ and $|\epsilon(t)| < \varepsilon_0/3$, as $t \geq T$.

Suppose $\delta_2 = 0$. Then there exists $t_0 > t$ such that $x_2(t_0) < \varepsilon_0/3$. Define

$$\begin{aligned} \sigma &= \min_{t \in [t_0 - \tau, t_0]} x_2(t) > 0, \text{ and} \\ \bar{t} &= \sup\{t \geq t_0 - \tau : x_2(s) \geq \sigma \text{ for all } s \in [t_0 - \tau, t]\}. \end{aligned}$$

Then $t_0 \leq \bar{t} < \infty$ and

$$\begin{aligned} x_2(t) &\geq \sigma \text{ for all } t \in [t_0 - \tau, \bar{t}] \text{ and} \\ x_2(\bar{t}) &= \sigma, \quad x'_2(\bar{t}) \leq 0. \end{aligned} \tag{14}$$

Hence, when $\lambda_2 < 1 \leq \lambda_{1q}$, we know

$$S(\bar{t} - \tau) = 1 - x_1(\bar{t}) - x_2(\bar{t}) + \epsilon(\bar{t} - \tau) \geq 1 - \varepsilon_0/3 - \sigma - \varepsilon_0/3 \geq 1 - \varepsilon_0 > \lambda_2;$$

when $\lambda_2 < \lambda_{1q} < 1$, we know

$$S(\bar{t} - \tau) = 1 - x_1(\bar{t}) - x_2(\bar{t}) + \epsilon(\bar{t} - \tau) \geq 1 - (1 - \lambda_{1q} + \varepsilon_0/3) - \sigma - \varepsilon_0/3 \geq \lambda_{1q} - \varepsilon_0 > \lambda_2.$$

For both cases, we have

$$\begin{aligned} x'_2(\bar{t}) &\geq -x_2(\bar{t}) + f_2(1 - x_1(\bar{t}) - x_2(\bar{t}) + \epsilon(\bar{t} - \tau))x_2(\bar{t} - \tau) \\ &> -\sigma + \sigma = 0, \end{aligned}$$

contradicting (14). Therefore, $\delta_2 > 0$ and the proof is complete. \square

The next two lemmas are for proving $\lim_{t \rightarrow \infty} x_1(t) = 0$ as $\lambda_2 < \lambda_{1q} < 1$.

Lemma 8. *If $\lambda_2 < \lambda_{1q} < 1$, then $\delta_1 = \gamma_1$.*

Proof. If $\delta_1 < \gamma_1$. By Fluctuation lemma, there exists a sequence $\{t_m\} \uparrow \infty$ such that $x_1(t_m) \rightarrow \gamma_1, x_1'(t_m) = 0, x_2(t_m) \rightarrow x_2^\dagger, x_1(t_m - \tau) \rightarrow x_1^\dagger$, as $m \rightarrow \infty$, where x_1^\dagger and x_2^\dagger satisfy $x_1^\dagger \leq \gamma_1$ and $x_2^\dagger \geq \delta_2$. Since $x_1'(t_m) = 0$, we know, by the x_1 -equation of system (2), that

$$x_1(t_m) = (1 - q)f_1(1 - x_1(t_m) - x_2(t_m) + \epsilon(t_m - \tau))x_1(t_m - \tau).$$

Letting $m \rightarrow \infty$, we have

$$\gamma_1 \leq (1 - q)f_1(1 - \gamma_1 - \delta_2)\gamma_1.$$

Then we have either $\gamma_1 = 0$ (this leads to a contradiction to the assumption $\delta_1 < \gamma_1$, since $\gamma_1 = \delta_1 = 0$ in this case) or

$$1 - \gamma_1 - \delta_2 \geq \lambda_{1q}. \tag{15}$$

By Barbálat's lemma and Fluctuation lemma, we know there exists a sequence $\{s_n\} \uparrow \infty$ such that $x_2(s_n) \rightarrow \delta_2, x_2'(s_n) \rightarrow 0, x_2(s_n - \tau) \rightarrow x_2^\ddagger, x_1(s_n - \tau) \rightarrow x_1^\ddagger$, as $n \rightarrow \infty$, where $x_1^\ddagger, x_2^\ddagger$ satisfy $x_1^\ddagger \leq \gamma_1$ and $x_2^\ddagger \geq \delta_2$. Since $x_2'(s_n) \rightarrow 0$ as $n \rightarrow \infty$, we know, by the x_2 -equation of system (2), that

$$\begin{aligned} x_2(s_n) = & (1 - q)f_1(1 - x_1(s_n) - x_2(s_n) + \epsilon(s_n))x_1(s_n - \tau) \\ & + f_2(1 - x_1(s_n) - x_2(s_n) + \epsilon(s_n))x_2(s_n - \tau). \end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$\delta_2 \geq f_2(1 - \delta_2 - \gamma_1)\delta_2.$$

From **Lemma 7** $\delta_2 > 0$, the above inequality leads to

$$1 - \delta_2 - \gamma_1 \leq \lambda_2. \tag{16}$$

Hence, by (15) and (16), we know $\lambda_2 \geq \lambda_{1q}$ which contradicts to $\lambda_2 < \lambda_{1q} < 1$. \square

Lemma 9. *If $\lambda_2 < \lambda_{1q} < 1$, then $\delta_1 = \gamma_1 = 0$.*

Proof. Suppose that $\delta_1 > 0$. Note that $\lim_{t \rightarrow \infty} x_1(t) = \delta_1$. By Barbálat's lemma, $\lim_{t \rightarrow \infty} x_1'(t) = 0$. Then we have

$$\delta_1 = \lim_{t \rightarrow \infty} x_1(t) = \lim_{t \rightarrow \infty} (1 - q)f_1(1 - x_1(t) - x_2(t) + \epsilon(t - \tau))x_1(t - \tau).$$

So we know that $\lim_{t \rightarrow \infty} x_2(t) = x_2^\dagger$ exists and

$$\delta_1 = (1 - q)f_1(1 - \delta_1 - x_2^\dagger)\delta_1,$$

and so

$$1 - \delta_1 - x_2^\dagger = \lambda_{1q}. \tag{17}$$

Since $\lim_{t \rightarrow \infty} x_2(t) = x_2^\dagger$ exists, we know, by Barbálat's lemma and the argument in the proof of previous lemma, that

$$\delta_2 \geq f_2(1 - \delta_1 - x_2^\dagger)\delta_2.$$

Since $\delta_2 > 0$, the above inequality leads to

$$1 - \delta_1 - x_2^\dagger \leq \lambda_2.$$

Hence, $\lambda_2 \geq \lambda_{1q}$ which contradicts to $\lambda_2 < \lambda_{1q} < 1$. \square

So far, when $\lambda_2 < 1$ and $\lambda_2 < \lambda_{1q}$, we already know that $\lim_{t \rightarrow \infty} x_1(t) = 0$ from both **Lemma 4** and **Lemma 9**; that $\delta_2 > 0$ by **Lemma 7**. To achieve the desired result, we need the following lemma.

Lemma 10. *If $\lambda_2 < 1$ and $\lambda_2 < \lambda_{1q}$, then $\delta_2 = \gamma_2 = 1 - \lambda_2$.*

Proof. Suppose that $\delta_2 < \gamma_2$, then we know, by Fluctuation lemma, that there exists a sequence $\{t_m\} \uparrow \infty$ such that $x_2(t_m) \rightarrow \gamma_2$, $x'_2(t_m) = 0$, $x_2(t_m - \tau) \rightarrow x_2^\ddagger$, $x_1(t_m) \rightarrow x_1^\ddagger$, as $m \rightarrow \infty$, where $x_1^\ddagger, x_2^\ddagger$ satisfy $x_1^\ddagger \geq 0$ and $x_2^\ddagger \leq \gamma_2$. Since $x'_2(t_m) = 0$, we know, by the x_2 -equation of system (2), that

$$x_2(t_m) = (1 - q)f_1(1 - x_1(t_m) - x_2(t_m) + \epsilon(t_m - \tau))x_1(t_m - \tau) + f_2(1 - x_1(t_m) - x_2(t_m) + \epsilon(t_m - \tau))x_2(t_m - \tau).$$

Letting $m \rightarrow \infty$, we have, by the fact that $\lim_{t \rightarrow \infty} x_1(t) = 0$, that

$$\gamma_2 \leq f_2(1 - \gamma_2)\gamma_2.$$

Since $\delta_2 > 0$ (this implies $\gamma_2 > 0$), the above inequality leads to $1 - \gamma_2 \geq \lambda_2$. Hence $1 - \lambda_2 \geq \gamma_2$.

By Fluctuation Lemma again, we know there exists a sequence $\{s_n\} \uparrow \infty$ such that $x_2(s_n) \rightarrow \delta_2$, $x'_2(s_n) = 0$, $x_2(s_n - \tau) \rightarrow x_2^\ddagger$, $x_1(s_n) \rightarrow 0$, as $n \rightarrow \infty$, where x_2^\ddagger satisfy $x_2^\ddagger \geq \delta_2$. We know, by the x_2 -equation of system (2), that

$$x_2(s_n) = (1 - q)f_1(1 - x_1(s_n) - x_2(s_n) + \epsilon(s_n - \tau))x_1(s_n - \tau) + f_2(1 - x_1(s_n) - x_2(s_n) + \epsilon(s_n - \tau))x_2(s_n - \tau).$$

Note that $\lim_{t \rightarrow \infty} x_1(t) = 0$. Letting $n \rightarrow \infty$, we have

$$\delta_2 \geq f_2(1 - \delta_2)\delta_2.$$

Since $\delta_2 > 0$, the above inequality leads to

$$1 - \delta_2 \leq \lambda_2.$$

Hence, $1 - \lambda_2 \leq \delta_2$. Combining with the previous result $1 - \lambda_2 \geq \gamma_2$, we know this leads to a contradiction of the assumption $\delta_2 < \gamma_2$.

Thus, we have $\delta_2 = \gamma_2 > 0$. By Barbălat's lemma, we have $\lim_{t \rightarrow \infty} x'_2(t) = 0$. Note that $\lim_{t \rightarrow \infty} x_1(t) = 0$. By the x_2 -equation of system (2) again, we have $0 = -\gamma_2 + f_2(1 - \gamma_2)\gamma_2$. Therefore, $\gamma_2 = 1 - \lambda_2 = \delta_2$. \square

Combining the above lemmas, we conclude the following theorem.

Theorem 2. *If $\lambda_2 < 1$ and $\lambda_2 < \lambda_{1q}$, then $\lim_{t \rightarrow \infty} (S(t), x_1(t), x_2(t)) = (\lambda_2, 0, 1 - \lambda_2)$.*

Next, we shall prove the global stability of E_c under conditions $\lambda_{1q} < 1$ and $\lambda_{1q} < \lambda_2$ when q is perturbed from 0. The proof follows the outline of a similar result of Hsu and Waltman [10].

Theorem 3. *For $q > 0$ sufficiently small, then the equilibrium E_c^3 of system (2) is globally stable.*

Since our model is of infinite dimensional nature, the proof of Theorem 3 is slightly different from that of **Theorem 4.2.** in Hsu and Waltman [10]. We introduce the notation in Smith and Waltman [16]:

Let $T : U \times [0, \infty) \times \Lambda \rightarrow U$ be continuous and define a family of semi-dynamical systems on U parameterized by $\lambda \in \Lambda$. Note that $U \subset X$, X is a Banach space, and Λ is a metric space. Often $T(x, t, \lambda)$ is written as $T_\lambda^t x$. More precisely, the continuous map T defines a family of semi-dynamical systems on U parameterized by $\lambda \in \Lambda$ provided that for each $\lambda \in \Lambda$: (i) $T_\lambda^0 = id_X$; (ii) $T_\lambda^t \circ T_\lambda^s = T_\lambda^{t+s}$ for $t, s \geq 0$. The following theorem is cited from Smith and Waltman [16]:

Theorem 4. *Let T be a family of semi-dynamical systems parameterized by Λ . Let $(x_0, \lambda_0) \in U \times \Lambda$, $B_X(x_0, \delta) \subset U$ for some $\delta > 0$ and assume that $D_x T(x, t, \lambda)$ exists for $(x, t, \lambda) \in B_X(x_0, \delta) \times [0, \infty) \times \Lambda$ and for each fixed $t \geq 0$, $D_x T(x, t, \lambda)$ is continuous on $B_X(x_0, \delta) \times \Lambda$. Suppose that $T_{\lambda_0}^t x_0 = x_0$ for all $t \geq 0$, $U(t) \equiv D_x T_{\lambda_0}^t(x_0)$ defines a strongly continuous semigroup with negative growth bound ($r(U(t)) = \exp(-\omega t)$ with $\omega > 0$), and $T_{\lambda_0}^t(x) \rightarrow x_0$ for each $x \in U$. In addition, suppose that:*

(H1) *For each $\lambda \in \Lambda$, there is a subset B_λ of U such that for each $x \in U$, $T_\lambda^t(x) \in B_\lambda$ for all large t .*

(H2) *$C = \bigcup_{\lambda \in \Lambda} T_\lambda^s(B_\lambda)$ is compact in U for some $s > 0$.*

Then there exist $\epsilon_0 > 0$ and a continuous map $\hat{x} : B_\Lambda(\lambda_0, \epsilon_0) \rightarrow U$ such that $\hat{x}(\lambda_0) = x_0$, $T_\lambda^t \hat{x}(\lambda) = \hat{x}(\lambda)$ for $t \geq 0$, and

$$T_\lambda^t x \rightarrow \hat{x}(\lambda), \quad x \in U, \lambda \in B_\Lambda(\lambda_0, \epsilon_0).$$

Note that, in practical situation, we employ the following

(H3) *there exists a compact set $D \subset U$ such that for each $\lambda \in \Lambda$ and each $x \in U$, $T_\lambda^t(x) \in D$ for all large t .*

to replace **(H1)** and **(H2)**, since **(H1)** and **(H2)** hold if **(H3)** holds. Note that, in system (2), Λ is $[0, 1)$, λ is q . We already know, from [19], that, when $q = 0$, the corresponding equilibrium, $E_c^3 (= E_1^3)$, is both locally and globally stable. The set U in the previous theorem is $\{(S, x_1, x_2) : S \geq 0, x_1 \geq 0, x_2 \geq 0\}$ in our case. Though E_2^3 is not an interior point of U as the theorem required, this theorem is still applicable by the fact that $B(E_1^3, \delta) \cap U$ is convex and the following remark in [16]:

"Remark 2.1 The assumption that x_0 is an interior point of U is unnecessarily restrictive. An examination of the proof indicates that it is sufficient that T can be extended to $B_X(x_0, \delta) \times \Lambda$ for some $\delta > 0$ and has a continuous derivative on the set and that $B_X(x_0, \delta) \cap U$ is convex. Alternatively, one-sided derivatives with respect to some cone or wedge in X may also be used."

We have already noted that all trajectories with initial conditions in the non-negative octant eventually lie in the compact set $\bar{Q} = U \cap \{(S, x_1, x_2) : 1 - S - x_1 - x_2 \leq 0\}$. Constructing the compact set required in the previous theorem will be a question of uniform persistence uniformly in the parameter q so that the other equilibria different from E_c^3 are away from the compact set $D = \bar{Q} \cap \{(S, x_1, x_2) : x_2 \leq \epsilon\}$ for some small $\epsilon > 0$.

The differential inequalities theorem for delay systems is required. We extend the result of differential inequalities for ordinary differential equations (see Hale [7] p.31) as follows

Lemma 11. *Let $x(t)$ be a scalar, differentiable function satisfying*

$$\begin{aligned} x'(t) &\geq f(t, x(t), x(t - \tau)), \quad 0 \leq t \leq b, \\ x(t) &= \psi(t), \quad \psi(t) \geq \phi(t), \quad -\tau \leq t \leq 0, \end{aligned}$$

where $\phi(t) \in C_1^+ = \{\psi : \psi(\theta) \geq 0 \text{ and continuous for all } \theta \in [-\tau, 0]\}$ and $f(t, x, z)$ is non-decreasing in z . Let φ be the solution of

$$\begin{aligned} x'(t) &= f(t, x(t), x(t - \tau)), \quad 0 \leq t \leq b, \\ x(t) &= \phi(t), \quad -\tau \leq t \leq 0, \end{aligned}$$

Then $x(t) \geq \varphi(t)$, for $-\tau \leq t \leq b$.

Proof. To show this lemma, we need the continuous dependence theorem for delay systems, which is stated in Kuang [13] p.13.

Now we follow the outline of Hale [7] to complete the proof. Consider the following systems

$$\begin{aligned}x'_n(t) &= f(t, x_n(t), x_n(t - \tau)) - \frac{1}{n}, \quad 0 \leq t \leq b, \\x_n(t) &= \phi(t) - \frac{1}{n}, \quad -\tau \leq t \leq 0.\end{aligned}$$

By the continuous dependence theorem, we know that $x_n(t) \rightarrow \varphi(t)$ uniformly on $[-\tau, b]$.

It suffices to show that, for n sufficiently large, $x(t) \geq x_n(t)$, on $t \in [0, b]$. If not, then there exists n large and $0 < t_1 < t_2 < b$ such that $x(t) \geq x_n(t)$ for $t \leq t_1$ and $x(t) < x_n(t)$ for $t \in (t_1, t_2)$. Note that $x(t_1) = x_n(t_1)$. Then, for $t > t_1$ and t near t_1 , we have

$$\frac{x(t) - x(t_1)}{t - t_1} < \frac{x_n(t) - x_n(t_1)}{t - t_1}.$$

Let $t \rightarrow t_1$, then we have

$$\begin{aligned}x'(t_1) &\leq x'_n(t_1) \\&= f(t_1, x_n(t_1), x_n(t_1 - \tau)) - \frac{1}{n} \\&< f(t_1, x(t_1), x(t_1 - \tau)).\end{aligned}$$

The last inequality holds when $f(t, x, z)$ is non-decreasing in z and it also leads to a contradiction. \square

Note that the inequality holds for the other direction. Note also that we can extend the result of this lemma to more general delay systems easily.

To proceed the following discussion, we need to consider the behavior of the scalar delay system

$$\begin{aligned}y' &= -y(t) + f_2(1 - y(t))y(t - \tau), \\y(t) &= \psi(t), \quad -\tau \leq t \leq 0, \quad \psi \in C_1^+.\end{aligned}\tag{18}$$

Note that the previous lemma is applicable to this system since $f_2(1 - y(t)) > 0$.

Now we employ the following lemma to analyze these systems.

Lemma 12. *Let y be the solution of system (18). If $\lambda_2 > 1$, then the only equilibrium 0 is globally stable. If $\lambda_2 < 1$, then the equilibrium $1 - \lambda_2$ is globally stable.*

Proof. When $\lambda_2 \geq 1$, it is easy to see that there is a unique rest point $y = 0$. Let γ be $\limsup_{t \rightarrow \infty} y(t)$. By Barbălat's lemma and Fluctuation lemma, we know that there exist $\{t_m\} \uparrow \infty$ such that $y'(t_m) \rightarrow 0$, $y(t_m) \rightarrow \gamma$, $y(t_m - \tau) \rightarrow y^b$, as $m \rightarrow \infty$, where y^b satisfies $y^b \leq \gamma$. Thus, we know from system (18)

$$\begin{aligned}0 &= -\gamma + f_2(1 - \gamma)y^b \\&\leq -\gamma + f_2(1 - \gamma)\gamma.\end{aligned}$$

Since $\lambda_2 > 1$, we know $-1 + f_2(1 - \gamma) < 0$. Hence, $\gamma = 0$. Thus, we know that $y = 0$ is globally stable.

On the other hand, when $\lambda_2 < 1$, it is easy to see that there are two rest points $y = 0$ and $y = 1 - \lambda_2$. Again, let γ be $\limsup_{t \rightarrow \infty} y(t)$ and δ be $\liminf_{t \rightarrow \infty} y(t)$.

By Barbālat’s lemma and Fluctuation lemma, we know that there exists $\{t_m\} \uparrow \infty$ so that $y'(t_m) \rightarrow 0$, $y(t_m) \rightarrow \delta$, $y(t_m - \tau) \rightarrow y^\dagger$ as $m \rightarrow \infty$, where y^\dagger satisfies $y^\dagger \geq \delta$. Then we have

$$0 = -\delta + f_2(1 - \delta)y^\dagger \geq -\delta + f_2(1 - \delta)\delta.$$

Thus, we know $\delta \leq 0$ or $\delta \geq 1 - \lambda_2$.

By Barbālat’s lemma and Fluctuation lemma again, we know that there exists $\{s_n\} \uparrow \infty$ so that $y'(s_n) \rightarrow 0$, $y(s_n) \rightarrow \gamma$, $y(s_n - \tau) \rightarrow y^\ddagger$ as $n \rightarrow \infty$, where y^\ddagger satisfies $y^\ddagger \leq \gamma$. Then we have

$$0 = -\gamma + f_2(1 - \gamma)y^\ddagger \leq -\gamma + f_2(1 - \gamma)\gamma.$$

Thus, we know $0 \leq \gamma \leq 1 - \lambda_2$.

Due to the well-posed property of this system, we know $\delta \geq 0$. Now we claim that $\delta > 0$. If $\delta = 0$, then there exists an $\varepsilon^* < 1 - \lambda_2$ and $t_0 > 0$ such that $y(t_0) < \varepsilon^*$. Let $\sigma = \min_{t \in [t_0 - \tau, t_0]} y(t)$, then $\sigma \leq \varepsilon^*$. Let $\bar{t} = \sup\{t \geq t_0 \mid y(s) \geq \sigma, \text{ for all } s \in [t - \tau, t]\}$. Then $y(\bar{t}) = \sigma$, $y'(\bar{t}) < 0$, and

$$\begin{aligned} y'(\bar{t}) &= -y(\bar{t}) + f_2(1 - y(\bar{t}))y(\bar{t} - \tau) \\ &= -\sigma + f_2(1 - \sigma)y(\bar{t} - \tau) \\ &\geq -\sigma + f_2(1 - (1 - \lambda_2))\sigma = 0. \end{aligned}$$

This contradicts to $y'(\bar{t}) < 0$.

Summarizing the above results, we conclude that $1 - \lambda_2$ is globally stable in system (18). □

Next, we want to see some systems which are perturbed from system (18) have similar dynamical behavior. Here we separate the results in the following lemmas.

Before proceeding our claims, we note that to acquire the equilibria of a system

$$y'(t) = F(y(t), y(t - \tau))$$

is equivalent to solve the equation $F(y, y) = 0$. Now we consider the equation

$$-y + f_2(1 + \epsilon - y)y + \epsilon_0 = 0,$$

where ϵ, ϵ_0 are small positive numbers. When $\lambda_2 < 1$ and $\epsilon = \epsilon_0 = 0$, it is easy to see 0 and $1 - \lambda_2$ are two solutions. Hence, by the monotonicity of f_2 , we know that there are also two solutions $-\xi_+ (< 0)$ and $\eta_+ (> 1 - \lambda_2)$ when ϵ, ϵ_0 are small (see FIGURE 1 for an example). Since ϵ, ϵ_0 are small, we may assume that $\eta_+ < 1$ by the continuity of f_2 .

Lemma 13. *Suppose that $\lambda_2 < 1$. Let y be the solution of system*

$$y'(t) = -y(t) + f_2(1 + \epsilon - y(t))y(t - \tau) + \epsilon_0,$$

where ϵ and ϵ_0 are small positive numbers. Then $y(t) \rightarrow \eta_+$ as $t \rightarrow \infty$, where η_+ is the equilibrium of the system which is close to $1 - \lambda_2$.

Note that all values of the initial function cannot be less than $-\xi_+$ for all $-\tau \leq t \leq 0$.

Proof. Let $\gamma_\epsilon = \limsup_{t \rightarrow \infty} y(t)$ and $\delta_\epsilon = \liminf_{t \rightarrow \infty} y(t)$.

By Barbālat’s lemma and Fluctuation lemma, we know that there exists $\{t_m\} \uparrow \infty$ so that $y'(t_m) \rightarrow 0$, $y(t_m) \rightarrow \delta_\epsilon$, $y(t_m - \tau) \rightarrow y^\dagger$ as $m \rightarrow \infty$, where y^\dagger satisfies $y^\dagger \geq \delta_\epsilon$. Then we have

$$0 = -\delta_\epsilon + f_2(1 + \epsilon - \delta_\epsilon)y^\dagger + \epsilon_0 \geq -\delta_\epsilon + f_2(1 + \epsilon - \delta_\epsilon)\delta_\epsilon + \epsilon_0.$$

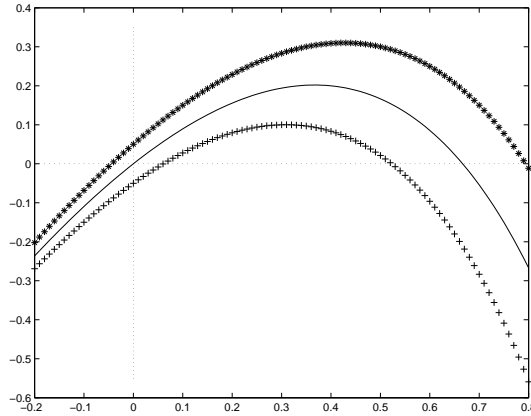


FIGURE 1. The solid curve, '*'-curve, and '+'-curve denotes the function $g_1(y) = -y + \frac{4y(1-y)}{1+(1-y)}$, $g_2(y) = -y + \frac{4y(1+0.1-y)}{1+(1+0.1-y)} + 0.05$, $g_3(y) = -y + \frac{4y(1-0.1-y)}{1+(1-0.1-y)} - 0.05$, respectively.

Thus, by FIGURE 1, we know $\delta_\epsilon \leq -\xi_+$ or $\delta_\epsilon \geq \eta_+$.

By Barbálat's lemma and Fluctuation lemma again, we know that there exists $\{s_n\} \uparrow \infty$ so that $y'(s_n) \rightarrow 0$, $y(s_n) \rightarrow \gamma_\epsilon$, $y(s_n - \tau) \rightarrow y^\ddagger$ as $n \rightarrow \infty$, where y^\ddagger satisfies $y^\ddagger \leq \gamma_\epsilon$. Then we have

$$0 = -\gamma_\epsilon + f_2(1 + \epsilon - \gamma_\epsilon)y^\ddagger + \epsilon_0 \leq -\gamma_\epsilon + f_2(1 + \epsilon - \gamma_\epsilon)\gamma_\epsilon + \epsilon_0.$$

Therefore, we know $-\xi_+ \leq \gamma_\epsilon \leq \eta_+$.

Due to the well-posed property of this system, we know $\delta_\epsilon \geq -\xi_+$. Now we claim that $\delta_\epsilon > -\xi_+$. If $\delta_\epsilon = -\xi_+$, then there exists an $\epsilon^* > -\xi_+$ and $t_0 > 0$ such that $y(t_0) = \epsilon^*$. Let $\sigma_\epsilon = \min_{t \in [t_0 - \tau, t_0]} y(t)$, then $-\xi_+ \leq \sigma_\epsilon \leq \epsilon^*$. Let $\bar{t} = \sup\{t \geq t_0 \mid y(s) \geq \sigma_\epsilon, \text{ for all } s \in [t_0 - \tau, t_0]\}$. Then $y(\bar{t}) = \sigma_\epsilon$, $y'(\bar{t}) < 0$, and

$$\begin{aligned} y'(\bar{t}) &= -y(\bar{t}) + f_2(1 + \epsilon - y(\bar{t}))y(\bar{t} - \tau) + \epsilon_0 \\ &= -\sigma_\epsilon + f_2(1 + \epsilon - \sigma_\epsilon)y(\bar{t} - \tau) + \epsilon_0 \\ &\geq -\sigma_\epsilon + f_2(1 + \epsilon - \sigma_\epsilon)\sigma_\epsilon + \epsilon_0 \geq 0. \end{aligned}$$

The last inequality holds for $-\xi_+ \leq \sigma_\epsilon \leq \epsilon^*$. This contradicts to $y'(\bar{t}) < 0$.

Summarizing the above results, we conclude that η_+ is globally stable in this system and note that η_+ is close to $1 - \lambda_2$. □

Now we consider the equation

$$-y + f_2(1 - \epsilon - y)y - \epsilon_0 = 0.$$

By an argument similar to the one before **Lemma 13**, we know that there are two solutions $\xi_- (> 0$ and close to 0) and $\eta_- (< 1 - \lambda_2$ and close to $1 - \lambda_2$) when ϵ, ϵ_0 are small positive numbers. The proof is very similar to that of **Lemma 13**, we do not repeat the argument.

Lemma 14. *Suppose that $\lambda_2 < 1$. Let y be the solution of system*

$$y'(t) = -y(t) + f_2(1 - \epsilon - y(t))y(t - \tau) - \epsilon_0,$$

where ϵ and ϵ_0 are small positive numbers. Then $y(t) \rightarrow \eta_-$ as $t \rightarrow \infty$.

Now consider the case $\lambda_2 > 1$ and the equation

$$-y + f_2(1 + \epsilon - y)y + \epsilon_0 = 0.$$

When $\epsilon = \epsilon_0 = 0$, it is easy to see 0 is the only solution. Hence, by the monotonicity of f_2 , we know that there is also one solution ξ_+ (> 0 and close to 0) when ϵ, ϵ_0 are small (see FIGURE 2 for an example).

Lemma 15. *Suppose that $\lambda_2 > 1$. Let y be the solution of system*

$$y'(t) = -y(t) + f_2(1 + \epsilon - y(t))y(t - \tau) + \epsilon_0,$$

where ϵ and ϵ_0 are small positive numbers. Then $y(t) \rightarrow \xi_+$ as $t \rightarrow \infty$.

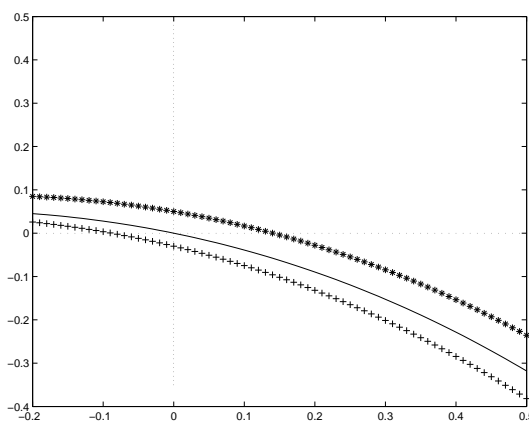


FIGURE 2. The solid curve, '*'-curve, and '+'-curve denotes the function $h_1(y) = -y + \frac{4y(1-y)}{5+(1-y)}$, $h_2(y) = -y + \frac{4y(1+0.1-y)}{5+(1+0.1-y)} + 0.05$, $h_3(y) = -y + \frac{4y(1-0.1-y)}{5+(1-0.1-y)} - 0.03$, respectively.

Here we note that the values of the initial function have to be no less than ξ_+ for all $-\tau \leq t \leq 0$.

Proof. Again, let $\gamma_\epsilon = \limsup_{t \rightarrow \infty} y(t)$. By Barbălat's lemma and Fluctuation lemma, we know that there exist $\{t_m\} \uparrow \infty$ such that $y'(t_m) \rightarrow 0$, $y(t_m) \rightarrow \gamma_\epsilon$, $y(t_m - \tau) \rightarrow y^b$, as $m \rightarrow \infty$, where y^b satisfies $y^b \leq \gamma_\epsilon$. Thus, we know from this system

$$\begin{aligned} 0 &= -\gamma_\epsilon + f_2(1 + \epsilon - \gamma_\epsilon)y^b + \epsilon_0 \\ &\leq -\gamma_\epsilon + f_2(1 + \epsilon - \gamma_\epsilon)\gamma_\epsilon + \epsilon_0. \end{aligned}$$

It is easy to see that $\gamma_\epsilon \leq \xi_+$. Thus, by the well-posed property of this system, we know $\gamma_\epsilon = \xi_+$ and that $y = \xi_+$ is globally stable. \square

Now we consider the equation

$$-y + f_2(1 - \epsilon - y)y - \epsilon_0 = 0,$$

where ϵ and ϵ_0 are small positive numbers. It is easy to see that there is only one solution $-\xi_- < 0$ and close to 0 (see FIGURE 2 as an example). Since the argument of the proof is very similar to that of **Lemma 15**, we state the lemma without proof.

Lemma 16. *Suppose that $\lambda_2 < 1$. Let y be the solution of system*

$$y'(t) = -y(t) + f_2(1 - \epsilon - y(t))y(t - \tau) - \epsilon_0,$$

where ϵ and ϵ_0 are small positive numbers. Then $y(t) \rightarrow -\xi_-$ as $t \rightarrow \infty$.

With the above comments and lemmas, we shall prove **Theorem 3**.

Proof of Theorem 3. For convenience, we use $\lambda_1(q)$ to represent λ_{1q} . As indicated before, we know that the equilibrium $E_1^3 = (\lambda_1(0), 1 - \lambda_1(0), 0)$ is global stable from Wolkowicz and Xia [19] if $\lambda_1(0) < 1$ and $\lambda_1(0) < \lambda_2$. We also know that $E_c^3 \rightarrow E_1^3$ as $q \rightarrow 0$. Hence, we only need to show that system (2) is uniformly persistent, uniformly in q near 0, or, equivalently, that there exist $\eta > 0$ and $q_0 > 0$ such that if $0 < q < q_0$, then $\liminf_{t \rightarrow \infty} \| (x_1)_t \| > \eta$. Proposition 1.2. in Thieme [18] shows that one can prove $\liminf_{t \rightarrow \infty} \| (x_1)_t \| > \eta$ provided that one can prove $\limsup_{t \rightarrow \infty} \| (x_1)_t \| > \epsilon$ for some $\epsilon > 0$, uniformly in q . Suppose, on the contrary, there exists $q_n \rightarrow 0$, $q_n > 0$ such that the corresponding solutions, $((S_n)_t, (x_{1n})_t, (x_{2n})_t)$, of system (2) with $q = q_n$ satisfy $\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \| (x_{1n})_t \| = 0$.

From system (2), it follows that

$$\begin{aligned} S'_n(t) &= 1 - S_n(t) - x_{2n}(t)f_2(S_n(t)) - \eta_n(t) \\ x'_{2n}(t) &= -x_{2n}(t) + x_{2n}(t - \tau)f_2(S_n(t - \tau)) + \xi_n(t - \tau) \end{aligned} \tag{19}$$

where

$$\eta_n(t) = f_1(S_n(t))x_{1n}(t), \text{ and } \xi_n(t - \tau) = qf_1(S_n(t - \tau))x_{1n}(t - \tau).$$

For $\epsilon > 0$ and n sufficiently large,

$$0 < \eta_n(t) < \epsilon, \text{ and } 0 < \xi_n(t - \tau) < \epsilon. \tag{20}$$

From (19) and (20), we have that for $n \geq N_0$, N_0 large,

$$\begin{aligned} (S_n(t) + x_{2n}(t + \tau))' &= 1 - (S_n(t) + x_{2n}(t + \tau)) - \eta_n(t) + \xi_n(t) \\ &\leq 1 - (S_n(t) + x_{2n}(t + \tau)) + \epsilon \end{aligned} \tag{21}$$

and that

$$1 - (S_n(t) + x_{2n}(t + \tau)) - \epsilon \leq (S_n(t) + x_{2n}(t + \tau))'. \tag{22}$$

By the differential inequalities (21) and (22), for $\epsilon, \epsilon_1 > 0$ and N_0 as above, there exist $T = T(\epsilon, \epsilon_1, N_0)$ such that, when $t \geq T$,

$$1 - \epsilon - \epsilon_1 \leq S_n(t) + x_{2n}(t + \tau) \leq 1 + \epsilon + \epsilon_1. \tag{23}$$

To simplify our notations, we may rewrite $\epsilon + \epsilon_1$ as ϵ . Using the second equation in (19), and (23) for $t \geq T$, one has

$$x'_{2n}(t) \leq -x_{2n}(t) + f_2(1 + \epsilon - x_{2n}(t))x_{2n}(t - \tau) + \epsilon_0 \tag{24}$$

$$x'_{2n}(t) \geq -x_{2n}(t) + f_2(1 - \epsilon - x_{2n}(t))x_{2n}(t - \tau) - \epsilon_0 \tag{25}$$

By the differential inequalities (24), (25) and **Lemma 11**, we have, for t sufficiently large,

$$\widehat{\varphi}_n(t) \leq x_{2n}(t) \leq \widetilde{\varphi}_n(t), \tag{26}$$

and, by **Lemma 13 - 16**,

$$\lim_{t \rightarrow \infty} \widehat{\varphi}_n(t) = \hat{x}_{2n}, \quad \lim_{t \rightarrow \infty} \widetilde{\varphi}_n(t) = \tilde{x}_{2n}.$$

Note that $\widehat{\varphi}_n(t)$ and $\widetilde{\varphi}_n(t)$ are solutions of the corresponding equalities. By **Lemma 12 - 16**, for n sufficiently large, the limits \hat{x}_{2n} and \tilde{x}_{2n} are either close to 0 when $\lambda_2 > 1$, or close to $1 - \lambda_2$ when $\lambda_2 < 1$. From equation (23) and equation (26),

$S_n(t)$ is either close to 1 when $\lambda_2 > 1$, or close to λ_2 when $\lambda_2 < 1$. However, the x_1 -equation of system (2) implies that, in both cases, $x_{1n}(t) \rightarrow \infty$ as $t \rightarrow \infty$, contradicting that $x_{1n}(t)$ is bounded. This completes the proof of this theorem. \square

5. Discussion. We discuss a chemostat model including discrete time delay which concerns the time period which lapses between uptake of nutrient and the assimilation of nutrient into viable biomass. This extends the model in [12] to a scenario which is closer to the realistic world. Though Ellermeyer [4], and Wolkowicz and Xia [19] had made complete analysis for the chemostat models with delayed response in growth, the factor of plasmid lost in our model, i.e., $q \neq 0$, makes the analysis in their paper unapplicable to our model. For example, we only need Hayes' theorem to accomplish the local analysis of the models when $q = 0$. However, we need to develop a new method to analyze the local stability of the interior equilibrium for the case $q \neq 0$. While analyzing the global behavior of the equilibria, Ellermeyer, etc, use the differential inequalities (or, equivalently, the Barbălat's lemma and Fluctuation lemma) due to these equilibria are boundary equilibria. The global analysis of the interior equilibrium in our model is difficult, since their methods are not applicable. This difficulty forces us to use perturbation theory which is similar to that in Hsu and Waltman [10] to prove the global stability of the interior equilibrium under the scenario that $0 < q \ll 1$ instead of proving that for $0 \leq q \leq 1$. Another difficulty is that the model that Hsu and Waltman concerned is an ODE system, which is finite dimensional. Unlike their case, our model is infinite dimensional. Though our models are infinitely dimensional, the theorems in [16] and [18] are still applicable. Note that a differential inequality theorem for discrete time delay equations is developed. The proof of this theorem is easy but this theorem is useful for application. Even if we cannot prove the global stability of the interior equilibrium, but we cannot exclude the possibility of the existence of periodic solutions and chaotic solutions, which are often seen in the models with discrete time delays. Nevertheless, the numerical simulations suggests that the interior equilibrium is globally stable.

A model of competition for two complementary nutrients between plasmid-bearing and plasmid-free organisms in a chemostat was established and its global behavior was discussed by Hsu and Tzeng (see [9]). We modify their ODE system into a discrete time delay system as follows

$$\begin{aligned}
 S'(t) &= (S^{(0)} - S(t))D - \frac{x_1(t)}{y_{S1}} f_1(S, R) - \frac{x_2(t)}{y_{S2}} f_2(S, R), \\
 R'(t) &= (R^{(0)} - R(t))D - \frac{x_1(t)}{y_{R1}} f_1(S, R) - \frac{x_2(t)}{y_{R2}} f_2(S, R), \\
 x_1'(t) &= -Dx_1(t) + (1 - q)x_1(t - \tau)f_1(S(t - \tau), R(t - \tau)), \\
 x_2'(t) &= -Dx_2(t) + qx_1(t - \tau)f_1(S(t - \tau), R(t - \tau)) \\
 &\quad + x_2(t - \tau)f_2(S(t - \tau), R(t - \tau)), \\
 S(t) &= \psi_S(t), \quad R(t) = \psi_R(t), \quad x_1(t) = \psi_{x_1}(t), \quad x_2(t) = \psi_{x_2}(t), \\
 &\quad -\tau \leq t \leq 0, \quad (\psi_S, \psi_R, \psi_{x_1}, \psi_{x_2}) \in C_4^+,
 \end{aligned}
 \tag{27}$$

where

$$f_1(S, R) = \min \left(\frac{m_{S1}S}{K_{S1} + S}, \frac{m_{S2}S}{K_{S2} + S} \right),$$

$$f_2(S, R) = \min \left(\frac{m_{R1}R}{K_{R1} + R}, \frac{m_{R2}R}{K_{R2} + R} \right),$$
(28)

and C_4^+ is an extension of C_3^+ . Here S and R denotes the concentrations of two complementary nutrients; the parameters m_{Si} , m_{Ri} , K_{Si} , K_{Ri} , $i = 1, 2$, are the maximal growth of the i th competitor and the Michaelis-Menten (or half-saturation) constants with respect to the nutrients S and R alone. Again, y_{Ri} , $i = 1, 2$, are the yield constants with respect to nutrient R . Biologically we may assume that $y_{S1} = y_{S2} = y_S$ and $y_{R1} = y_{R2} = y_R$ as noted in Hsu and Tzeng [9].

Though system (27) appears more complicated than system (1), we observe a special partial order relation by using an assumption on the equality of the yield constants for both plasmid-bearing organisms and plasmid-free organisms and this relation makes the analysis for system (1) applicable to system (27). Due to the similarity, we skip the details of the analysis.

Unlike discrete delay models, Li, Wolkowicz, and Kuang had made a complete asymptotic analysis for a distributed delay chemostat model with two complementary resources in [14]. However, their approach is different and we cannot convert their model into a discrete delay model.

Finally, remark that we cannot prove the global stability of the interior equilibrium when q is not close to 0. Lyapunov functional may be a good method to achieve this, though it may be difficult to find such a functional.

Appendix. Here we use $\alpha > 1$, $\beta > 1$, $\alpha + \beta > \gamma > 1 + f_2(\lambda_{1q})$, and $\alpha + \beta + f_2(\lambda_{1q}) > 1 + \gamma$ to show that the inequalities in (10) hold. It is clear that $a_4 > 0$.

Note that we abbreviate λ_{1q} by λ_1 .

$$\begin{aligned} a_3 - 4a_4 &= 2f_2(\lambda_1)[2\gamma + (\alpha + \beta)(1 + f_2(\lambda_1))] - 16f_2^2(\lambda_1) \\ &= 2f_2(\lambda_1)[2\gamma + (\alpha + \beta)(1 + f_2(\lambda_1)) - 8f_2(\lambda_1)] > 0. \end{aligned}$$

The last inequality holds for $\gamma > 2f_2(\lambda_1)$, $\alpha + \beta \geq 2$, $1 + f_2(\lambda_1) \geq 2f_2(\lambda_1)$. Thus, we prove that $0 < -12a_4 + 3a_3$.

Now we claim that $-12a_4 + 6a_3 - 2a_2 < 0$ by

$$\begin{aligned} &6a_4 + a_2 - 3a_3 \\ &= 24f_2^2(\lambda_1) + \gamma^2 + f_2(\lambda_1)(1 - f_2(\lambda_1))^2 + (1 + f_2(\lambda_1))^2(\alpha + \beta - 1) \\ &\quad + \gamma(1 + f_2(\lambda_1))(\alpha + \beta) + 2f_2(\lambda_1)(\alpha + \beta)^2 - 12f_2(\lambda_1)\gamma \\ &\quad - 6f_2(\lambda_1)(\alpha + \beta)(1 + f_2(\lambda_1)) \\ &= -12f_2(\lambda_1)(\gamma - 2f_2(\lambda_1)) + (\gamma + 1 + f_2(\lambda_1))(\gamma - 1 - f_2(\lambda_1)) \\ &\quad + (1 + f_2(\lambda_1))(\alpha + \beta)(1 - f_2(\lambda_1)) + (\alpha + \beta)(1 + f_2(\lambda_1))(\gamma - 2f_2(\lambda_1)) \\ &\quad + 2f_2(\lambda_1)(\alpha + \beta)(\alpha + \beta - 1 - f_2(\lambda_1)) + f_2(\lambda_1)(1 - f_2(\lambda_1))^2 \\ &\geq (\gamma - 2f_2(\lambda_1))[(\alpha + \beta)(1 + f_2(\lambda_1)) - 4f_2(\lambda_1)] \\ &\quad + 2f_2(\lambda_1)(\alpha + \beta)(\alpha + \beta - 1 - f_2(\lambda_1) - \gamma + 2f_2(\lambda_1)) \\ &\quad + (\gamma - 1 - f_2(\lambda_1))(\gamma + 1 + f_2(\lambda_1) - 4f_2(\lambda_1)) \\ &\quad + (1 - f_2(\lambda_1))[(1 + f_2(\lambda_1))(\alpha + \beta) - 4f_2(\lambda_1)] + f_2(\lambda_1)(1 - f_2(\lambda_1))^2 \\ &> 0. \end{aligned}$$

Now we claim that $-4a_4 + 3a_3 - 2a_2 + a_1 > 0$ by

$$\begin{aligned}
& 3a_3 + a_1 - 4a_4 - 2a_2 \\
= & 12f_2(\lambda_1)\gamma + 6f_2(\lambda_1)(\alpha + \beta)(1 + f_2(\lambda_1)) + \gamma(1 - f_2(\lambda_1))^2 \\
& + 2(1 + f_2(\lambda_1))(\alpha + \beta)(\alpha + \beta - 1) + \gamma(\alpha + \beta)^2 - 16f_2^2(\lambda_1) \\
& - 2\gamma^2 - 2f_2(\lambda_1)(1 - f_2(\lambda_1))^2 - 2(1 + f_2(\lambda_1))^2(\alpha + \beta - 1) \\
& - 2\gamma(1 + f_2(\lambda_1))(\alpha + \beta) - 4f_2(\lambda_1)(\alpha + \beta)^2 \\
= & 8f_2(\lambda_1)\gamma + [\gamma(1 + f_2(\lambda_1))^2 + \gamma(\alpha + \beta)^2 - 2(1 + f_2(\lambda_1))(\alpha + \beta)] \\
& + 6f_2(\lambda_1)(\alpha + \beta)(1 + f_2(\lambda_1)) \\
& + [2(1 + f_2(\lambda_1))(\alpha + \beta)^2 - 2(1 + f_2(\lambda_1))(\alpha + \beta)] \\
& - 16f_2^2(\lambda_1) - 2\gamma^2 - [2f_2(\lambda_1)(1 + f_2(\lambda_1))^2 - 8f_2^2(\lambda_1)] \\
& - [2(1 + f_2(\lambda_1))^2(\alpha + \beta) - 2(1 + f_2(\lambda_1))^2] - 4f_2(\lambda_1)(\alpha + \beta)^2 \\
= & 8f_2(\lambda_1)\gamma + \gamma(\alpha + \beta - 1 - f_2(\lambda_1))^2 + 6f_2(\lambda_1)(\alpha + \beta)(1 + f_2(\lambda_1)) \\
& + 2(1 + f_2(\lambda_1))(\alpha + \beta)^2 + 8f_2^2(\lambda_1) + 2(1 + f_2(\lambda_1))^2 \\
& - 2(1 + f_2(\lambda_1))(\alpha + \beta) - 16f_2^2(\lambda_1) - 2\gamma^2 - 2f_2(\lambda_1)(1 + f_2(\lambda_1))^2 \\
& - 2(1 + f_2(\lambda_1))^2(\alpha + \beta) - 4f_2(\lambda_1)(\alpha + \beta)^2 \\
= & 8f_2(\lambda_1)(\gamma - 2f_2(\lambda_1)) + \gamma(\alpha + \beta - 1 - f_2(\lambda_1))^2 + 6f_2^2(\lambda_1)(\alpha + \beta) \\
& + 6f_2(\lambda_1)(\alpha + \beta) + 2(1 + f_2(\lambda_1))(\alpha + \beta)^2 + 8f_2^2(\lambda_1) + 2(1 + f_2(\lambda_1))^2 \\
& - 2(1 + f_2(\lambda_1))(\alpha + \beta) - 2\gamma^2 - 2f_2(\lambda_1)(1 + f_2(\lambda_1))^2 - 2(\alpha + \beta) \\
& - 4f_2(\lambda_1)(\alpha + \beta) - 2f_2^2(\lambda_1)(\alpha + \beta) - 4f_2(\lambda_1)(\alpha + \beta)^2 \\
= & 8f_2(\lambda_1)(\gamma - 2f_2(\lambda_1)) + \gamma(\alpha + \beta - 1 - f_2(\lambda_1))^2 + 2f_2(\lambda_1)(\alpha + \beta) \\
& + 4f_2^2(\lambda_1)(\alpha + \beta) + 2(1 + f_2(\lambda_1))(\alpha + \beta)^2 + 8f_2^2(\lambda_1) + 2(1 + f_2(\lambda_1))^2 \\
& - 4(\alpha + \beta) - 2f_2(\lambda_1)(\alpha + \beta) - 2\gamma^2 - 2f_2(\lambda_1)(1 + f_2(\lambda_1))^2 \\
& - 4f_2(\lambda_1)(\alpha + \beta)^2 \\
= & 8f_2(\lambda_1)(\gamma - 2f_2(\lambda_1)) + \gamma(\alpha + \beta - 1 - f_2(\lambda_1))^2 + 4f_2^2(\lambda_1)(\alpha + \beta) \\
& + 2(1 - f_2(\lambda_1))(\alpha + \beta)^2 + 8f_2^2(\lambda_1) + 2(1 + f_2(\lambda_1))^2 - 4(\alpha + \beta) \\
& - 2\gamma^2 - 2f_2(\lambda_1)(1 + f_2(\lambda_1))^2 \\
= & 8f_2(\lambda_1)(\gamma - 2f_2(\lambda_1)) + \gamma(\alpha + \beta - 1 - f_2(\lambda_1))^2 + 2(1 - f_2(\lambda_1))(\alpha + \beta)^2 \\
& + 8f_2^2(\lambda_1) + 2(1 + f_2(\lambda_1))^2(1 - f_2(\lambda_1)) - 4(\alpha + \beta)(1 - f_2^2(\lambda_1)) - 2\gamma^2 \\
= & 8f_2(\lambda_1)(\gamma - 2f_2(\lambda_1)) + \gamma(\alpha + \beta - 1 - f_2(\lambda_1))^2 \\
& + 2(1 - f_2(\lambda_1))(\alpha + \beta - 1 - f_2(\lambda_1))^2 - 2(\gamma + 2f_2(\lambda_1))(\gamma - 2f_2(\lambda_1)) \\
= & 2(1 + \frac{\gamma}{2} - f_2(\lambda_1))(\alpha + \beta - 1 - f_2(\lambda_1))^2 - 2(\gamma - 2f_2(\lambda_1))^2 \\
> & 0.
\end{aligned}$$

The last inequality holds for $1 + \frac{\gamma}{2} - f_2(\lambda_1) > 1$ and $\alpha + \beta - 1 - f_2(\lambda_1) > \gamma - 2f_2(\lambda_1)$.

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REFERENCES

- [1] R. Bellman and K.L. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963.

- [2] K. L. Cooke and Z. Grossman, *Discrete delay, distributed delay and stability switches*, J. Math. Analysis and Application, **86** (1982), 592–627.
- [3] R. Datko, *A procedure for determination of exponential stability of certain differential-difference equation*, Quart. Appl. Math., **36** (1978), 279–292.
- [4] S.F. Ellermeyer, *Competition in the chemostat: Global asymptotic behavior of a model with delayed response in growth*, SIAM J. Appl. Math., **54** (1994), 456–465.
- [5] H.I. Freedman, J.W.-H. So and P. Waltman, *Chemostat competition with time delays*, in Biomedical Modelling and Simulation, J. Eisenfeld and D. S. Levine, eds., Scientific Publishing Co., 1989, pp. 171–173.
- [6] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population SDynamics*, Kluwer Academic Publishers, Dordrecht, 1992.
- [7] J. K. Hale, *Ordinary Differential Equations*, Wiley, 1969.
- [8] W.M. Hirsch, H. Hanisch and J.P. Gabriel, *Differential equation models of some parasitic infections: methods for the study of asymptotic behavior*, Comm. Pure Appl. Math., **38** (1985), 733–753.
- [9] S.B. Hsu and Y.H. Tzeng, *Plasmid-bearing, plasmid-free organisms competing for two complementary nutrients in a chemostat*, Math. Biosci., **179** (2002), 183–206.
- [10] S.B. Hsu and P. Waltman, *A model of effect of anti-competitor toxins on plasmid-bearing, plasmid-free competition*, Taiwanese J. Math., **6** (2002), no. 1, 135–155.
- [11] S.B. Hsu, P. Waltman and S.F. Ellermeyer, *A remark on the global asymptotic stability of a dynamical system modeling two species competition*, Hiroshima Math. J., **24** (1994), 435–445.
- [12] S.B. Hsu, P. Waltman and G.S.K. Wolkowicz, *Global analysis of a model of plasmid-bearing, plasmid-free competition in a chemostat*, J. Math. Biol., **32** (1994), 731–742.
- [13] Y. Kuang, *Delay Differential Equations With Applications in Population Dynamics*, Academic Press, Boston, 1993.
- [14] B. Li, G.S.K. Wolkowicz and Y. Kuang, *Global asymptotic behavior of a chemostat model with two perfectly complementary resources and distributed delay*, SIAM J. Appl. Math., **60** (2000), 2058–2086.
- [15] H. Smith and P. Waltman, *Theory of Chemostat*, Cambridge University, Cambridge, 1995.
- [16] H. Smith and P. Waltman, *Perturbation of a globally stable steady state*, Proc. A.M.S., **127** (2) (1999), 447–453.
- [17] G. Stephanopoulos and G. Lapidus, *Chemostat dynamics of plasmid-bearing plasmid-free mixed recombinant cultures*, Chem. Eng. Sci., **43** (1988), 49–57.
- [18] H.R. Thieme, *Persistence under relaxed point-dissipativity (with application to an epidemic model)*, SIAM J. Math. Anal., **24** (1993), 407–435.
- [19] G.S.K. Wolkowicz and H. Xia, *Global asymptotic behavior of a cheostat model with discrete delays*, SIAM J. Appl. Math., **57** (1997), 1281–1310.

Received February 2004; revised November 2004.

E-mail address: hsu@math.tamu.edu; sbhsu@math.nthu.edu.tw

E-mail address: cli@mail.sjsmit.edu.tw