CANADIAN APPLIED
MATHEMATICS QUARTERLY
Volume 6, Number 2, Summer 1998

UNIQUENESS OF LIMIT CYCLES FOR A PREDATOR-PREY SYSTEM OF HOLLING AND LESLIE TYPE

SZE-BI HSU AND TZY-WEI HWANG

ABSTRACT. In this paper we study the uniqueness of limit cycles of the Holling-Tanner model, an important predator-prey model in mathematical ecology. We first transform the system into a predator-prey system with Gause-type. The reduced system is different from the classical one since the prey isocline has two humps instead of one. We use the method of reflections to estimate the Floquet exponent of the limit cycle and derive a sufficient condition for the uniqueness of limit cycles.

1. Introduction. In this paper we study the uniqueness of limit cycles for the following Holling-Tanner model [**10, 11**],

$$\begin{aligned}
&\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \frac{mx}{A+x}y,\\
(1.1)\qquad &\frac{dy}{dt} = sy\left(1-\frac{hy}{x}\right),\\
&x(0)>0, \qquad y(0)>0,
\end{aligned}$$

where $r, m, s, h, A, K > 0$.

The predator-prey system (1.1) assumes the prey grows logistically with intrinsic growth rate r and carrying capacity K in the absence of predation. The predator consumes the prey according to Holling type II functional response $mx/(A+x)$ and grows logistically with intrinsic growth rate s and carrying capacity proportional to the population size of the prey. The Holling-Tanner model is an important and interesting

Received by the editors on November 13, 1996, and in revised form on April 17, 1997.

AMS *Mathematics Subject Classification*. 34C35, 92A17.

Key words and phrases. Uniqueness of limit cycle, Holling-Tanner model, predator-prey system, Liapunov function.

First author was supported by Grant NSC 85-2121-M-007-029 from the National Council of Science, Republic of China

model of a predator-prey system in both biological and mathematical sense. In [**13**] a study of several pairs of interacting species, ranging from house sparrows and European sparrow hawk to mule deer and mountain lion, shows that the theoretical predictions of (1.1) based on the estimated parameter values are broadly in line with practical reality. In [**14**] the authors use the model (1.1) to investigate the temperature-mediated stability of the predator-prey mite interaction between *M. occidentalis* and the McDaniel spider mite *Tetranychus mcdanieli McGregor* on apple trees. From the experimental data, they assume the parameters r, s, m in (1.1) are functions of temperature T, specifically

$$\begin{aligned} r &= r(T) = 0.048[exp(0.103T) - \exp(0.369T - 7.457)], \\ s &= s(T) = 0.089[\exp(0.055T) - \exp(0.483T - 11.648)], \\ m &= m(T) = 16\frac{(s(T))^2}{r(T)} \quad \text{and} \quad K = 300 \text{ mites/leaf}, \\ a &= \frac{1}{25} \text{ mite/leaf}, \qquad h = \frac{1}{0.15}. \end{aligned} \tag{1.2}$$

They examined the qualitative properties of (1.1) with data (1.2) by means of the numerical bifurcation code AUTO86. Their extensive numerical studies showed that the unique positive equilibrium of (1.1) is either globally stable or gives rise to a globally stable limit cycle, there can also exist a range wherein multiple stable states occur. These stable states consist of a focus and a limit cycle, separated from each other in the phase plane by an unstable limit cycle. For the detailed biological meaning, the reader may consult [**14, 15, 10, 11**].

Establishing the uniqueness of limit cycles for a predator-prey systems has been an interesting mathematical problem in the past decade. In 1981, K.S. Cheng [**1**] was the first one to prove the uniqueness of limit cycle for a specific predator-prey system with Holling type II functional response by using the symmetry of prey isocline. Subsequently, Liou and Cheng [**9**] further developed a method of reflection to extend the class of predator-prey model for which the results are valid. Kuang and Freedman [**8**] and Huang and Merrill [**6**] transform a class of predator-prey model with Gause-type to a generalized Lienard system where the results of uniqueness of limit cycles are available. S. Ding [**2**] studied a kind of predator-prey system with Holling-type III functional response and showed that the results hold.

The difference between the system (1.1) and the Gause-type models mentioned above is the shape of predator isocline. The predator isocline of (1.1) is a straight line through origin in the phase plane, while those of Gause-type models are vertical line, $x = x^*, x^* > 0$. To establish the uniqueness of the limit cycle for the system (1.1), our first step is to convert (1.1) into a Gause-type predator-prey model as we did in our previous paper [**5**]. It is interesting to note that the reduced system whose prey isocline has two humps, one local minimum and one local maximum, instead of one hump discussed in most predator-prey models [**12**].

We use the method of reflection developed in [**9**] to estimate the Floquet exponent of a limit cycle and derive a sufficient condition for the uniqueness of limit cycles.

The paper will be organized as follows. In Section 2 we introduce the nondimensional form of (1.1) and summarize the results of local stability of the positive equilibrium discussed in [**5**]. Then we transform the system (1.1) into a Gause-type predator-prey system whose prey isocline has two humps. In Section 3 we give a sufficient condition for the uniqueness of limit cycles of system (1.1). We employ the techniques in [**9**] as well as a Liapunov function to prove the results. Although the condition is not optimal, our numerical studies show, however, that there is a large region in the parameter space where our sufficient condition of the uniqueness of limit cycle holds.

2. Preliminary results. In this section we summarize some basic results in [**5**]. First we write the system (1.1) in nondimensional form. Let

$$\tilde{t} = rt, \qquad \tilde{x}(\tilde{t}) = \frac{x(t)}{K}, \qquad \tilde{y}(\tilde{t}) = \frac{my(t)}{rK},$$
$$\delta = \frac{s}{r}, \qquad \beta = \frac{sh}{m}, \qquad a = \frac{A}{K}.$$

Then (1.1) takes the form

$$
\begin{aligned}
&\frac{dx}{dt} = x(1-x) - \frac{x}{a+x}y \\
(2.1) \qquad &\frac{dy}{dt} = y\left(\delta - \beta\frac{y}{x}\right) \\
&x(0) > 0, \qquad y(0) > 0.
\end{aligned}
$$

Obviously, from (2.1), there exists a unique positive equilibrium $E^* = (x^*, y^*)$. Let

$$P(x) = 2x^2 + (a + \delta - 1)x + a\delta. \tag{2.2}$$

Lemma 1.2 [5]. *The equilibrium $E^* = (x^*, y^*)$ of (2.1) is locally asymptotically stable if $P(x^*) > 0$ and E^* is an unstable focus if $P(x^*) < 0$.*

We note from (2.2), $P(x) \geq 0$ for all $x > 0$ if

$$a + \delta \geq 1. \tag{2.3}$$

If

$$a + \delta < 1 \quad \text{and} \quad (1 - a - \delta)^2 - 8a\delta \leq 0, \tag{2.4}$$

then $P(x) \geq 0$ for all $x \neq 0$.

If

$$a + \delta < 1 \quad \text{and} \quad (1 - a - \delta)^2 - 8a\delta > 0, \tag{2.5}$$

then $P(x) = 2(x - \alpha_1)(x - \alpha_2)$ where

$$
\begin{aligned}
\alpha_1 &= \frac{1}{4}[1 - a - \delta - \sqrt{(1 - a - \delta)^2 - 8a\delta}], \\
\alpha_2 &= \frac{1}{4}[1 - a - \delta + \sqrt{(1 - a - \delta)^2 - 8a\delta}], \\
&0 < \alpha_1 < \alpha_2 < 1.
\end{aligned}
$$

For the case (2.5) the local asymptotic stability of E^* can be reformulated as

$$\alpha_2 < x^* < 1 \tag{2.6}$$

or

$$0 < x^* < \alpha_1 \tag{2.7}$$

and the instability condition for E^* is

$$\alpha_1 < x^* < \alpha_2. \tag{2.8}$$

We summarize the stability results from [**5**].

Theorem 2.2 [**5**].

(i) *Let* (2.3) *or* (2.4) *hold. Then the equilibrium* $E^* = (x^*, y^*)$ *is globally asymptotically stable in the interior of the first quadrant.*

(ii) *Let* (2.5) *and* (2.6) *hold. Then the conclusion of* (i) *holds.*

(iii) *Let* (2.5) *hold. For* $\beta > 0$ *sufficiently small,* $x^* = x^*(\beta)$ *is sufficiently close to zero and* (2.7) *holds. Furthermore, the conclusion of* (i) *holds for* $\beta > 0$ *sufficiently small.*

(iv) *Let* (2.8) *hold. Then there exists a limit cycle for* (2.1).

We note that, in Theorem 2.2 (iv), the existence of a limit cycle follows directly from the Poincaré-Bendixson theorem. The system (2.1) is persistent [**3**]. In fact, we can construct a compact positively invariant region [**10**]. So the Poincaré-Bendixson theorem is applicable.

Let

$$u = yl(x), \qquad l(x) = \left(\frac{1-x}{x}\right)^{\delta}. \tag{2.9}$$

Then we reduce (2.1) to the following system

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{x}{a+x}\frac{u}{l(x)} \\ \frac{du}{dt} &= \frac{u^2\beta}{xl(x)(1-x)(a+x)}\left(x + \frac{a}{x^*}\right)(x - x^*) \\ &\quad x(0) > 0, \qquad u(0) > 0. \end{aligned} \tag{2.10}$$

Consider the prey-isocline of (2.10),

$$u = h(x) = (1-x)(a+x)l(x). \tag{2.11}$$

From [5], if (2.5) holds then it follows that

$$h'(x) = -\frac{l(x)}{x}P(x) = -\frac{2l(x)}{x}(x-\alpha_1)(x-\alpha_2). \tag{2.12}$$

Thus, the prey-isocline $u = h(x)$ has two humps, namely, a local maximum at $x = \alpha_2$ and a local minimum at $x = \alpha_1$. Obviously, from (2.11), (2.9) and (2.12), we have $h(1) = 0$, $\lim_{x\to 0^+} h(x) = +\infty$ and $h'(x) > 0$ for $\alpha_1 < x < \alpha_2$ and $h'(x) < 0$ for $x \in (0, \alpha_1) \cup (\alpha_2, 1)$. In [5] we introduced the following Liapunov function for the system (2.10),

$$V(x,u) = \beta \int_{x^*}^{x} \frac{\eta - x^*}{q(\eta)}\, d\eta + \int_{u^*}^{u} \frac{\eta - u^*}{\eta^2}\, d\eta \tag{2.13}$$

where

$$q(x) = \frac{x^2(1-x)}{(x + a/x^*)}. \tag{2.14}$$

Then an easy computation yields

$$\dot{V} = \frac{\beta(x + a/x^*)}{x(1-x)(a+x)l(x)}(x - x^*)(h(x) - u^*).$$

We shall repeatedly use these properties of Liapunov function V in the next section. Finally, we note that the problem of uniqueness of a limit cycle of the Holling-Tanner model (2.1) is equivalent to that of the system (2.10).

3. Main results. In this section we restrict our attention to the reduced system (2.10). Rewrite (2.10) in the following form:

$$\begin{aligned} \frac{dx}{dt} &= \varphi(x)[h(x) - u] = F(x,u) \\ \frac{du}{dt} &= \psi(x)u^2 = G(x,u) \\ x(0) &> 0, \qquad u(0) > 0, \end{aligned} \tag{3.1}$$

where

$$\varphi(x) = \frac{x}{a+x}\frac{1}{l(x)},$$

$$\psi(x) = \frac{\beta}{xh(x)}(x - x^*)\left(x + \frac{a}{x^*}\right),$$

and $h(x), l(x)$ are defined in (2.11) and (2.9), respectively.

Let (2.8) hold, i.e., $\alpha_1 < x^* < \alpha_2$. Then there exists x_1^*, x_2^* satisfying (3.2)

$$0 < x_1^* < \alpha_1, \quad \alpha_2 < x_2^* < 1 \quad \text{and} \quad h(x_1^*) = h(x_2^*) = h(x^*) = u^*.$$

Introduce

$$R(x) = V(x, u^*) = \beta \int_{x^*}^{x} \frac{\eta - x^*}{q(\eta)}\, d\eta. \tag{3.3}$$

Now we state our main result.

Theorem 3.1. *Let $\alpha_1 < x^* < \alpha_2$. If*

$$R(x_1^*) \geq R(x_2^*), \tag{3.4}$$

then the system (2.1) *has a unique limit cycle.*

Remark 3.1. The condition (3.4) states that if x^* is near α_2, then we have uniqueness of limit cycles.

We shall prove Theorem 3.1 by the following lemmas.

Lemma 3.2. *Let $\alpha_1 < x^* < \alpha_2$. Then*

$$\frac{d}{dx}\left[\frac{\varphi(x)h'(x)}{\psi(x)h(x)}\right] < 0 \quad \textit{for } x \in [\alpha_1, x^*) \cup [\alpha_2, 1].$$

Proof. From (3.1) and (2.12), we have

$$\varphi(x)h'(x) = -\frac{2(x-\alpha_1)(x-\alpha_2)}{a+x},$$

$$\psi(x)h(x) = \frac{\beta}{x}(x - x^*)\left(x + \frac{a}{x^*}\right).$$

Let

$$Q(x) = \frac{x(x-\alpha_1)}{(x+a)(x+a/x^*)}.$$

Then

$$\frac{\varphi(x)h'(x)}{\psi(x)h(x)} = -\frac{2}{\beta}Q(x)\cdot\frac{x-\alpha_2}{x-x^*}$$

and

$$\frac{d}{dx}\left[\frac{\varphi(x)h'(x)}{\psi(x)h(x)}\right] = -\frac{2}{\beta}\left[Q'(x)\frac{x-\alpha_2}{x-x^*} + Q(x)\frac{\alpha_2-x^*}{(x-x^*)^2}\right].$$

Since $Q(x)$ is increasing on $(\alpha_1, 1)$, then from the above identity we complete the proof of Lemma 3.3. □

Before we prove our main results, we introduce the following notations. Let $\Omega_1 = [0,\alpha_1]\times\mathbf{R}^+$, $\Omega_2 = [\alpha_1,\alpha_2]\times\mathbf{R}^+$ and $\Omega_3 = [\alpha_2, 1]\times\mathbf{R}^1$. Define $h_1 : (0,\alpha_1) \to \mathbf{R}$, $h_2 : [\alpha_1,\alpha_2] \to \mathbf{R}$ and $h_3 : [\alpha_2, 1] \to \mathbf{R}$ by $h_i(x) = h(x)$, $i = 1,2,3$. Then $h_1'(x) < 0$ for $x \in (0,\alpha_1)$, $h_2'(x) > 0$ for $x \in (\alpha_1,\alpha_2)$ and $h_3'(x) < 0$ for $x \in (\alpha_2, 1]$. Next we define the following reflection maps

$$S_l : \Omega_2 \longrightarrow \Omega_1, \quad S_l(x,u) = (h_1^{-1}(h_2(x)), u) \tag{3.5}$$

and

$$S_r : \Omega_2 \longrightarrow \Omega_3, \quad S_r(x,u) = (h_3^{-1}(h_2(x)), u).$$

In the followings lemmas we denote Γ to be a periodic orbit of (3.1) with periodic T, $\Gamma = \{(x(t), u(t)) : 0 \le t < T\}$ and $\Omega(\Gamma)$ to be the region enclosed by Γ in the xu phase plane, $x_m = x_m(\Gamma) = \min\{x(t) : 0 \le t \le T\}$, $x_M = x_M(\Gamma) = \max\{x(t) : 0 \le t \le T\}$ and $I(\Gamma) = [x_m, x_M]$. Let

$$\begin{aligned} \Omega_+ &= \{(x,u) \in \Omega(\Gamma) : u - h(x) \ge 0\}, \\ \Omega_- &= \{(x,u) \in \Omega(\Gamma) : u - h(x) \le 0\}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &u_+ : I(\Gamma) \longrightarrow \mathbf{R}^+ \text{ such that graph } u_+ = \Gamma\cap\Omega_+, \\ &u_- : I(\Gamma) \longrightarrow \mathbf{R}^+ \text{ such that graph } u_- = \Gamma\cap\Omega_-. \end{aligned} \tag{3.7}$$

Then $u_\pm = u_\pm(x)$ satisfies

$$\frac{du_\pm}{dx} = \frac{\psi(x)u_\pm^2}{\phi(x)[h(x) - u_\pm]} \tag{3.8}$$

and

$$u_+(x^*) - u^* > 0, \qquad u_-(x^*) - u^* < 0.$$

Lemma 3.3. (i)

$$\iint_{\Omega(\Gamma)} \frac{h'(x)}{u^2}\, dx\, du = 0.$$

(ii) $I(\Gamma)$ *is not contained in* $[x_1^*, x_2^*]$ *where* x_1^*, x_2^* *are defined in* (3.2).

Proof. From (3.1), it follows that

$$\psi(x)h(x) = \frac{\psi(x)}{\varphi(x)}\frac{dx}{dt} + \frac{1}{u}\frac{du}{dt}. \tag{3.9}$$

Then we have

$$\begin{aligned} 0 &= \int_0^T \psi(x(t))h(x(t))\, dt = \int_0^T \frac{h(x(t))}{u^2}\frac{du}{dt}\, dt = \oint_\Gamma \frac{h(x)}{u^2}\, du \\ &= \iint_\Omega \frac{d}{dx}\left(\frac{h(x)}{u^2}\right) dx\, du = \iint_\Omega \frac{h'(x)}{u^2}\, dx\, du. \end{aligned}$$

Hence, we complete the proof of (i). Now we prove (ii) by contradiction. Suppose, on the contrary, $I(\Gamma) \subseteq [x_1^*, x_2^*]$. Rewrite the Liapunov function V in (2.13) as

$$V(x,u) = \int_{x^*}^{x} \frac{\psi(\eta)}{\varphi(\eta)}\, d\eta + \int_{u^*}^{u} \frac{\eta - u^*}{\eta^2}\, d\eta. \tag{3.10}$$

From the assumption $I(\Gamma) \subseteq [x_1^*, x_2^*]$, the derivative of V along the periodic orbit Γ satisfies

$$\frac{d}{dt}V(x(t), u(t)) = \psi(x(t))[h(x(t)) - h(x^*)] > 0.$$

However, this leads to the following contradiction:

$$\begin{aligned} 0 &< \int_0^T \frac{d}{dt} V(x(t), u(t))\, dt \\ &= V(x(T), u(T)) - V(x(0), u(0)) = 0. \end{aligned}$$

Hence $I(\Gamma)$ is not contained in $[x_1^*, x_2^*]$. □

Lemma 3.4. *If* $\operatorname{int} I(\Gamma) \supseteq [x_1^*, x_2^*]$, *then* Γ *is orbitally asymptotically stable.*

Proof. From (3.1), we have

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial u} = \varphi(x)h'(x) + \varphi'(x)(h(x) - u) + 2\psi(x)u.$$

Since Γ is a periodic orbit of (3.1), then

$$\oint_\Gamma \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial u} \right) dt = \oint_\Gamma \varphi(x)h'(x)\, dt.$$

It suffices to show that [4]

$$\oint_\Gamma \varphi(x)h'(x)\, dt < 0. \tag{3.11}$$

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, see Figure 1, where

$$\begin{aligned} \Gamma_1 &= \{(x, u) \in \Gamma : x < x_1^*\} = \widehat{AB} \\ \Gamma_2 &= \{(x, u_-(x)) : x_1^* \le x \le x_2^*\} = \widehat{BC} \\ \Gamma_3 &= \{(x, u) \in \Gamma : x > x_2^*\} = \widehat{CD} \\ \Gamma_4 &= \{(x, u_+(x)) \in \Gamma : x_1^* \le x \le x_2^*\} \\ &= \widehat{DA}. \end{aligned}$$

Let $\widehat{AB} = \{(x(u), u) : u_B \le u \le u_A\}$. Since $\varphi(x)h'(x) < 0$ and $\psi(x) < 0$ for $(x, u) \in \Gamma_1$, from (3.1) it follows that

$$\int_{\Gamma_1} \varphi(x)h'(x)\, dt = -\int_{u_B}^{u_A} \frac{\varphi(x(u))h'(x(u))}{\psi(x(u))u^2}\, du < 0. \tag{3.12}$$

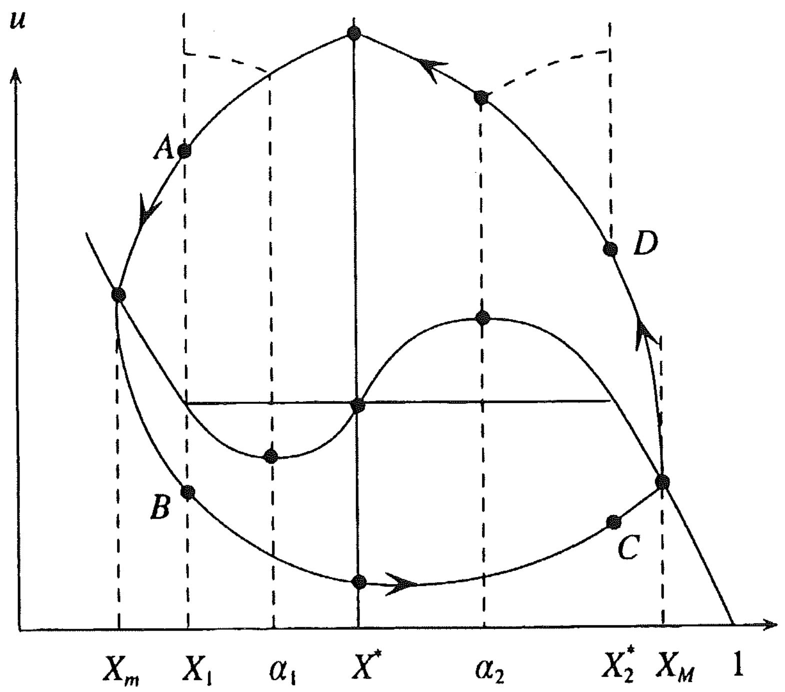

FIGURE 1.

From (3.1) we have

(3.13)

$$\int_{\Gamma_4} \varphi(x)h'(x)\,dt = \int_{x_2^*}^{x_1^*} \frac{h'(x)}{h(x)-u_+(x)}\,dx$$
$$= \int_{x_2^*}^{x^*} \frac{h'(x)}{h(x)-u_+(x)}\,dx + \int_{x^*}^{x_1^*} \frac{h'(x)}{h(x)-u_+(x)}\,dx.$$

The change of variable $y = h_2^{-1}(h_3(x))$ yields

$$I_1 = \int_{x_2^*}^{x^*} \frac{h'(x)}{h(x)-u_+(x)}\,dx$$
$$= -\int_{\alpha_2}^{x_2^*} \frac{h_3'(x)}{h_3(x)-u_+(x)}\,dx$$
$$+ \int_{\alpha_2}^{x^*} \frac{h_2'(y)}{h_2(y)-u_+(y)}\,dy$$
$$= -\int_{\alpha_2}^{x_2^*} \frac{h_3'(x)}{h_3(x)-u_+(x)}\,dx + \int_{\alpha_2}^{x_2^*} \frac{h_3'(x)}{h_3(x)-u_+(h_2^{-1}(h_3(x)))}\,dx$$
$$= \int_{\alpha_2}^{x_2^*} \frac{[u_+(h_2^{-1}(h_3(x))) - u_+(x)]h_3'(x)}{(h_3(x)-u_+(x))(h_3(x)-u_+(h_2^{-1}(h_3(x)))}\,dx.$$

Since $h_3'(x) < 0$, $u_+(h_2^{-1}(h_3(x))) > u_+(x)$, $h_3(x) - u_+(x) < 0$, $h_3(x) - u_+(h_2^{-1}(h_3(x))) < 0$ for all $x \in (\alpha_2, x_2^*)$, it follows that the integral $I_1 < 0$.

The above arguments and the change of variable $y = h_2^{-1}(h_1(x))$ yield

$$\begin{aligned}
I_2 &= \int_{x^*}^{x_1^*} \frac{h'(x)}{h(x) - u_+(x)}\, dx \\
&= -\int_{x_1^*}^{\alpha_1} \frac{h_1'(x)}{h_1(x) - u_+(x)}\, dx + \int_{x^*}^{\alpha_1} \frac{h_2'(y)}{h_2(y) - u_+(y)}\, dy \\
&= -\int_{x_1^*}^{\alpha_1} \frac{h_1'(x)}{h_1(x) - u_+(x)}\, dx + \int_{x_1^*}^{\alpha_1} \frac{h_1'(x)}{h_1(x) - u_+(h_2^{-1}(h_1(x)))}\, dx \\
&= \int_{x_1^*}^{\alpha_1} \frac{h_1'(x)[u_+(h_2^{-1}(h_1(x))) - u_+(x)]}{(h_1(x) - u_+(h_2^{-1}(h_1(x))))(h_1(x) - u_+(x))}\, dx \\
&< 0.
\end{aligned}$$

From (3.13), $\int_{\Gamma_4} \varphi(x)h'(x)\, dt < 0$. Similarly, we have $\int_{\Gamma_3} \varphi(x)h'(x)\, dt < 0$ and $\int_{\Gamma_2} \varphi(x)h'(x)\, dt < 0$. Hence, we establish (3.11) and complete the proof of Lemma 3.4. □

Lemma 3.5. *If $\alpha_1 \le x_m$, then Γ is orbitally asymptotically stable.*

Proof. From Lemma 3.3 (ii) and the assumption $\alpha_1 \le x_m$, we have $x_M > x_2^*$. Before we prove the lemma, we need to establish the following claims.

Claim 1. $(\Omega(\Gamma) \cap \Omega_3) \backslash S_r(\Omega(\Gamma) \cap \Omega_2) \neq \varnothing$ *where the reflection map S_r is defined in* (3.5). *From Lemma* 3.3 (i) *and* (3.5), *it follows that*

$$\begin{aligned}
0 &= \iint_{\Omega(\Gamma)} \frac{h'(x)}{u^2}\, dx\, du \\
&= \iint_{\Omega(\Gamma)\cap\Omega_2} \frac{h_2'(x)}{u^2}\, dx\, du + \iint_{\Omega(\Gamma)\cap\Omega_3} \frac{h_3'(x)}{u^2}\, dx\, du \\
&= -\iint_{S_r(\Omega(\Gamma)\cap\Omega_2)} \frac{h_3'(x)}{u^2}\, dx\, du + \iint_{\Omega(\Gamma)\cap\Omega_3} \frac{h_3'(x)}{u^2}\, dx\, du.
\end{aligned}$$

From the above identity, it is impossible that $S_r(\Omega(\Gamma)\cap\Omega_2) \supseteq \Omega(\Gamma)\cap\Omega_3$. Hence, we prove Claim 1.

From Claim 1 we have $S_r(\Gamma_l)\cap\Gamma_r \neq \phi$, where $\Gamma_l = \{(x,u)\in\Gamma : x < \alpha_2\}$ and $\Gamma_r = \{(x,u)\in\Gamma : x \geq \alpha_2\}$.

Claim 2. $\#(S_r(\Gamma_l)\cap\Gamma_r^{\pm}) \leq 1$, *i.e., $S_r(\Gamma_l)$ intersects $\Gamma_r^{\pm}$ in at most one point where $\Gamma_r^{\pm} = \Gamma_r\cap\Omega_{\pm}$, where $\Omega_{\pm}$ are defined in* (3.6).

We shall prove $S_r(\Gamma_l)$ intersects Γ_r^+ in at most one point. Similar arguments show $\#(S_r(\Gamma_l)\cap\Gamma_r^-) \leq 1$. Let $u_2(x) = u_+(x)$ for $x \in [x_m, \alpha_2]$ and $u_3(x) = u_+(x)$ for $x\in[\alpha_2, x_M]$ where $u_+(x)$ is defined in (3.7). Set $\tilde{u}(x) = u_2(h_2^{-1}(h_3(x)))$ for $x \in [\alpha_2, h_3^{-1}(h_2(x_m))]$. Clearly, $\tilde{u}(x) > u_3(x)$ for all $x\in(\alpha_2, x_2^*)$. Suppose $S_r(\Gamma_l)\cap\Gamma_r^+ \neq \varnothing$. Then there exists $\theta\in(x_2^*, 1)$ such that $\tilde{u}(x) > u_3(x)$ for all $x\in(x_2^*,\theta)$ and

$$\tilde{u}(\theta) = u_3(\theta), \qquad \tilde{u}'(\theta) < u_3'(\theta). \tag{3.14}$$

To show $\#(S_r(\Gamma_l)\cap\Gamma_r^+)\leq 1$, it suffices to show that, for $x>\theta$, as long as $\tilde{u}(x), u_3(x) > h_3(x)$, we have $\tilde{u}(x) < u_3(x)$.

Let $y = h_2^{-1}(h_3(x))$. From (3.8), we have

$$\begin{aligned}\tilde{u}'(x) &= u_2'(h_2^{-1}(h_3(x)))\frac{d}{dx}h_2^{-1}(h_3(x))\\ &= \frac{\psi(y)}{\varphi(y)}\frac{u_2^2(y)}{[h_2(y)-u_2(y)]}\frac{h_3'(x)}{h_2'(y)}\\ &= \frac{\psi(y)}{\varphi(y)}\frac{\tilde{u}^2(x)}{[h_3(x)-\tilde{u}(x)]}\frac{h_3'(x)}{h_2'(y)}.\end{aligned} \tag{3.15}$$

Set $\hat{\theta} = h_2^{-1}[h_3(\theta)]$. Then (3.14), (3.15) and (3.8) yield

$$\begin{aligned}\tilde{u}'(\theta) - u_3'(\theta) &= \frac{\psi(\hat\theta)}{\varphi(\hat\theta)}\frac{\tilde{u}^2(\theta)}{(h_3(\theta)-\tilde{u}(\theta))}\frac{h_3'(\theta)}{h_2'(\hat\theta)}\\ &\quad -\frac{\psi(\theta)u_3^2(\theta)}{\varphi(\theta)[h_3(\theta)-u_3(\theta)]}\\ &= \frac{\tilde{u}^2(\theta)}{h_3(\theta)-\tilde{u}(\theta)}\left[\frac{\psi(\hat\theta)}{\varphi(\hat\theta)h_2'(\hat\theta)}h_3'(\theta) - \frac{\psi(\theta)}{\varphi(\theta)}\right] < 0.\end{aligned} \tag{3.16}$$

Obviously, $h_3(\theta) - \tilde{u}(\theta) < 0$ and $\theta > x_2^*$. Then, from (3.16), it follows that

$$\frac{\psi(\hat{\theta})}{\varphi(\hat{\theta})h_2'(\hat{\theta})} < \frac{\psi(\theta)}{\varphi(\theta)h_3'(\theta)}. \tag{3.17}$$

Since $h_3(\theta) = h_2(\hat{\theta}) > 0$, (3.17) can be rewritten as

$$\frac{\varphi(\hat{\theta})h_2'(\hat{\theta})}{\psi(\hat{\theta})h_2(\hat{\theta})} > \frac{\varphi(\theta)h_3'(\theta)}{\psi(\theta)h_3(\theta)}. \tag{3.18}$$

The inequality $\alpha_1 \le x_m < \hat{\theta} < x^* < \alpha_2 < x_2^* < \theta < x_M$, (3.18) and Lemma 3.2 yield that, for any $x \in (\theta, 1)$, $y \in [\alpha_1, \hat{\theta}]$,

$$\frac{\varphi(y)h_2'(y)}{\psi(y)h_2(y)} > \frac{\varphi(\hat{\theta})h_2'(\hat{\theta})}{\psi(\hat{\theta})h_2(\hat{\theta})} > \frac{\varphi(\theta)h_3'(\theta)}{\psi(\theta)h_3(\theta)} > \frac{\varphi(x)h_3'(x)}{\psi(x)h_3(x)}. \tag{3.19}$$

In particular, if we set $y = h_2^{-1}(h_3(x))$, $x \in [\theta, x_M]$ in (3.19), then $h_2(y) = h_3(x) > 0$. Since $h_3'(x) < 0$, $\psi(x) > 0$, $h_2'(y) > 0$, $\psi(y) < 0$, from (3.19) it follows that

$$\frac{\psi(y)}{\varphi(y)}\frac{h_3'(x)}{h_2'(y)} > \frac{\psi(x)}{\varphi(x)}. \tag{3.20}$$

For $x > \theta$ as long as $u_3(x), \tilde{u}(x) > h_3(x)$, from (3.15) and (3.8) we have

$$\begin{aligned}(\tilde{u} - u_3)'(x) &= \frac{\psi(y)}{\varphi(y)}\frac{h_3'(x)}{h_2'(y)}\frac{\tilde{u}^2(x)}{[h_3(x) - \tilde{u}(x)]} \\ &\quad - \frac{\psi(x)}{\varphi(x)}\frac{u_3^2(x)}{[h_3(x) - u_3(x)]} \\ &= \frac{u_3^2(x)}{[h_3(x) - u_3(x)]}\left[\frac{\psi(y)}{\varphi(y)}\frac{h_3'(x)}{h_2'(y)} - \frac{\psi(x)}{\varphi(x)}\right] \\ &\quad + \frac{\psi(y)h_3'(x)}{\varphi(y)h_2'(y)} \\ &\quad \cdot \frac{h_3(x)(\tilde{u}(x) + u_3(x)) - \tilde{u}(x)\tilde{u}_3(x)}{(h_3(x) - \tilde{u}(x))(h_3(x) - u_3(x))} \\ &\quad \cdot (\tilde{u}(x) - u_3(x)).\end{aligned} \tag{3.21}$$

We claim that the curves $u = \tilde{u}(x)$ and $u = u_3(x)$ will not intersect for $x > \theta$. If they do, then there exist $\hat{x}$, $\theta < \hat{x} < x_m$, such that $\tilde{u}(\hat{x}) = u_3(\hat{x})$ and $\tilde{u}'(\hat{x}) \geq u_3'(\hat{x})$. Setting $x = \hat{x}$ and $y = \hat{y} = h_2^{-1}(h_3(\hat{x}))$ in (3.20) and (3.21), yields

$$\tilde{u}'(\hat{x}) - u_3'(\hat{x}) = \frac{u_3^2(\hat{x})}{[h_3(\hat{x}) - u_3(\hat{x})]} \cdot \left[\frac{\psi(\hat{y})}{\varphi(\hat{y})}\frac{h_3'(\hat{x})}{h_2'(\hat{y})} - \frac{\psi(\hat{x})}{\varphi(\hat{x})}\right] < 0.$$

This leads to a contradiction. Thus, we complete the proof of Claim 2.

Now we are in a position to prove Lemma 3.5. From Claim 2, $S_r(\Gamma_l)$ intersects Γ_r^+, Γ_r^- at exactly one point C^+ and C^-. Let $C^+ = (x_+, v_+)$, $C^- = (x_-, v_-)$ and $D^+ = (y_+, v_+)$, $D^- = (y_-, v_-)$ satisfying $S_r(D^+) = C_+$, $S_r(D_-) = C_-$. The points P, Q are the leftmost and rightmost points of Γ, respectively. Assume $\Gamma \cap \{x = \alpha_2\} = \{A, F\}$ with $A = (\alpha_2, u_A)$, $F = (\alpha_2, u_F)$, $u_A > u_F$ and $\Gamma \cap \{x = x_2^*\} = \{B, E\}$ with $B = (x_2^*, u_B)$, $E = (x_2^*, U_E)$, $u_B > u_E$, see Figure 2. As in Lemma 3.4, it suffices to show that (3.11) holds. Write

$$\oint_\Gamma \varphi(x)h'(x)\,dt = \int_{C+\widehat{BA}D_+} + \int_{D_+\widehat{P}D_-} + \int_{D_-\widehat{FE}C_-} + \int_{C-\widehat{Q}C+} \varphi h'\,dt.$$

Let $y = h_2^{-1}(h_3(x))$. Then

$$\begin{aligned}
\int_{C_+\widehat{BA}D_+} \varphi(x)h'(x)\,dt &= \int_{x_+}^{y_+} \frac{h'(x)}{(h(x) - u_+(x))}\,dx \\
&= \int_{x_+}^{\alpha_2} \frac{h_3'(x)}{h_3(x) - u_3(x)}\,dx \\
&\quad + \int_{\alpha_2}^{y_+} \frac{h_2'(y)}{h_2(y) - u_2(y)}\,dy \\
&= \int_{x_+}^{\alpha_2} \frac{h_3'(x)}{h_3(x) - u_3(x)}\,dx \\
&\quad + \int_{\alpha_2}^{x_+} \frac{h_3'(x)}{h_3(x) - \tilde{u}(x)}\,dx \\
&= \int_{\alpha_2}^{x_+} \frac{h_3'(x)(\tilde{u}(x) - u_3(x))}{(h_3(x) - \tilde{u}(x))(h_3(x) - u_3(x))}\,dx \\
&< 0.
\end{aligned}$$

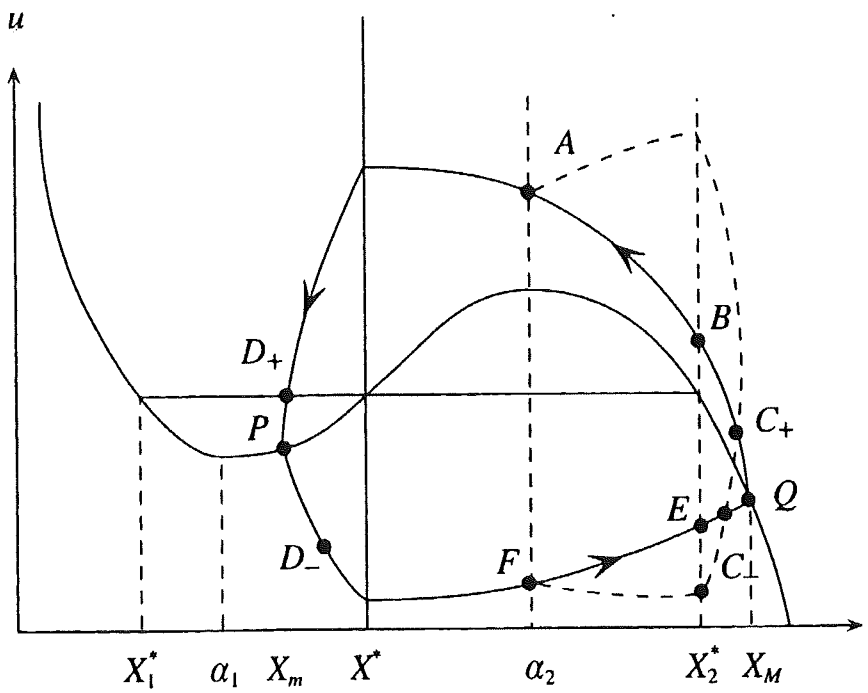

FIGURE 2.

Similarly,

$$\int_{D_-\widehat{FEC}_-} \varphi(x)h'(x)\,dt < 0.$$

Let $D_+\widehat{P}D_- = \{(y(u),u) : u \in [v_-,v_+]\}$ and $C_-\widehat{Q}C_+ = \{(x(u),u) : u \in [v_-,v_+]\}$. Rewrite (3.18) as

$$\frac{\varphi(y_\pm)h_2'(y_\pm)}{\psi(y_\pm)h_2(y_\pm)} > \frac{\varphi(x_\pm)h_3'(x_\pm)}{\psi(x_\pm)h_3(x_\pm)}. \tag{3.22}$$

If $y_+ \geq y_-$, then $x_+ \leq x_-$. Obviously $y(u) < y_+$, $x(u) \geq x_+$ for all $v_- \leq u \leq v_+$. Then (3.22) and Lemma 3.2 yield

$$\left.\frac{\varphi h_2'}{\psi h_2}\right|_{x=y(u)} \geq \frac{\varphi(y_+)h_2'(y_+)}{\psi(y_+)h_2(y_+)} > \frac{\varphi(x_+)h_3'(x_+)}{\psi(x_+)h_3(x_+)} \geq \left.\frac{\varphi h_3'}{\psi h_3}\right|_{x=x(u)}.$$

On the other hand, if $y_+ < y_-$, then $x_+ > x_-$ and

$$\left.\frac{\varphi h_2'}{\psi h_2}\right|_{x=y(u)} \geq \frac{\varphi(y_-)h_2'(y_-)}{\psi(y_-)h_2(y_-)} > \frac{\varphi(x_-)h_3'(x_-)}{\psi(x_-)h_3(x_-)} \geq \left.\frac{\varphi h_3'}{\psi h_3}\right|_{x=x(u)}.$$

Hence, in either case we have

$$(3.23) \qquad \left.\frac{\varphi h_2'}{\psi h_2}\right|_{x=y(u)} > \left.\frac{\varphi h_3'}{\psi h_3}\right|_{x=x(u)} \qquad \text{for all } u \in [v_-, v_+].$$

Consider

(3.24)

$$\begin{aligned}
\int_{D_+\widehat{P}D_-} \varphi h'\,dt &= \int_{D_+\widehat{P}D_-} \frac{\varphi h'}{\psi h}\psi h\,dt \\
&= \int_{D_+\widehat{P}D_-} \frac{\varphi h'}{\psi h}\left(\frac{\psi}{\varphi}x' + \frac{u'}{u}\right)dt, \\
&\quad \text{(from (3.9))} \\
&= \int_{D_+\widehat{P}D_-} \frac{h'}{h}\,dx + \frac{\varphi h'}{\psi h}(y(u))\frac{1}{u}\,du, \\
&= \int_{y_+}^{y_-} \frac{h'}{h}\,dx + \int_{v_+}^{v_-} \frac{\varphi h_2'}{\psi h_2}(y(u))\frac{1}{u}\,du, \\
&= \ln h_2(y_-) - \ln h_2(y_+) - \int_{v_-}^{v_+} \frac{\varphi h_2'}{\psi h_2}(y(u))\frac{1}{u}\,du, \\
&= \ln h_3(x_-) - \ln h_3(x_+) - \int_{v_-}^{v_+} \frac{\varphi h_2'}{\psi h_2}(y(u))\frac{1}{u}\,du.
\end{aligned}$$

Similarly, we have

$$(3.25) \qquad \begin{aligned}
\int_{C_-\widehat{Q}C_+} \varphi h'\,dt &= \int_{x_-}^{x_+} \frac{h'}{h}\,dx + \int_{v_-}^{v_+} \frac{\varphi h_3'}{\psi h_3}(x(u))\frac{1}{u}\,du, \\
&= \ln h_3(x_+) - \ln h_3(x_-) \\
&\quad + \int_{v_-}^{v_+} \frac{\varphi h_3'}{\psi h_3}(x(u))\frac{1}{u}\,du.
\end{aligned}$$

From (3.23), (3.24) and (3.25), it follows that

$$\int_{D_+\widehat{P}D_-} + \int_{C_-\widehat{Q}C_+} \varphi h'\,dt < 0.$$

Hence, we establish (3.11) and complete the proof of Lemma 3.5. □

Lemma 3.6. *If $x_1^* \le x_m < \alpha_1$, then Γ is orbitally asymptotically stable.*

Proof. From Lemma 3.3 (ii) and the assumption $x_1^* \le x_m < \alpha_1$, we have $x_M > x_2^*$. Let $\tilde{z} = h_2^{-1}(h_1(x_m))$. Then $\alpha_2 < \tilde{z} < x^*$. The following shows

$$
\begin{aligned}
(3.26) \quad & \iint_{\Omega(\Gamma)\cap((0,\tilde{z}]\times\mathbf{R}^+)} \frac{h'(x)}{u^2}\,dx\,du \\
&= \iint_{\Omega(\Gamma)\cap\Omega_1} \frac{h_1'(x)}{u^2}\,dx\,du \\
&\quad + \iint_{\Omega(\Gamma)\cap([\alpha_1,\tilde{z}]\times\mathbf{R}^+)} \frac{h_2'(x)}{u^2}\,dx\,du \\
&= \iint_{S_l(\Omega(\Gamma)\cap([\alpha_1,\tilde{z}]\times\mathbf{R}^+))\backslash(\Omega(\Gamma)\cap\Omega_1)} \left(-\frac{h_1'(x)}{u^2}\right)dx\,du \\
&> 0
\end{aligned}
$$

where the reflection map S_l is defined in (3.5). Since the map

$$\eta(\theta) \overset{\text{def}}{=} \iint_{\Omega(\Gamma)\cap((0,\theta]\times\mathbf{R}^+)} \frac{h'(x)}{u^2}\,dx\,du$$

is continuous and strictly increasing on $[\alpha_1, \tilde{z}]$, then from (3.26), i.e., $\eta(\tilde{z}) > 0$ and $\eta(\alpha_1) < 0$, there exists a unique $z_1 \in (\alpha_1, \tilde{z})$ such that $\eta(z_1) = 0$ or

$$(3.27) \qquad \iint_{\Omega(\Gamma)\cap((0,z_1]\times\mathbf{R}^+)} \frac{h'(x)}{u^2}\,dx\,du = 0.$$

Set $z_0 = h_1^{-1}(h_2(z_1))$ and $z_2 = h_3^{-1}(h_2(z_1))$. Then $x_m < z_0 < \alpha_1$ and $z_2 > x_2^*$. Let

$$
\begin{aligned}
\tilde{\Gamma} &= \{(x,u) \in \Gamma : x < z_1\}, \\
\gamma_+ &= \{(x,u) \in \Gamma \cap \Omega_+ : z_1 \le x \le \alpha_2\}, \\
\gamma_- &= \{(x,u) \in \Gamma \cap \Omega_- : z_1 \le x \le \alpha_2\}, \\
\Gamma_r^+ &= \{(x,u) \in \Gamma \cap \Omega_+ : x \ge \alpha_2\}, \\
\Gamma_r^- &= \{(x,u) \in \Gamma \cap \Omega_- : x \ge \alpha_2\},
\end{aligned}
$$

where Ω_+, Ω_- are defined in (3.6), see Figure 3(a). First we claim that

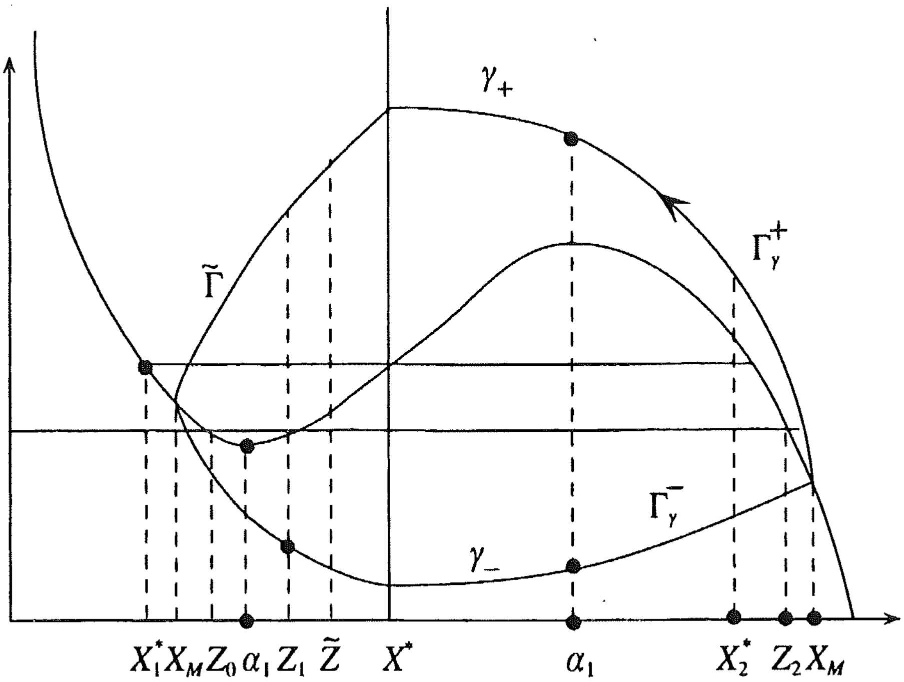

FIGURE 3(a).

$$\int_{\tilde{\Gamma}} \varphi(x)h'(x)\,dt < 0. \tag{3.28}$$

Let $\tilde{\Gamma} = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where

$$\begin{aligned}
\gamma_1 &= \{(x,u) \in \tilde{\Gamma} : x_m \leq x \leq z_0\},\\
\gamma_2 &= \{(x,u) \in \tilde{\Gamma} \cap \Omega_+ : z_0 \leq x \leq z_1\},\\
\gamma_3 &= \{(x,u) \in \tilde{\Gamma} \cap \Omega_- : z_0 \leq x \leq z_1\}.
\end{aligned}$$

Obviously, $\int_{\gamma_1} \varphi(x)h'(x)\,dt < 0$ and

$$\int_{\gamma_2 \cup \gamma_3} \varphi(x)h'(x)\,dt = \int_{z_1}^{z_0} \frac{h'(x)}{h(x) - u_+(x)}\,dx + \int_{z_0}^{z_1} \frac{h'(x)}{h(x) - u_-(x)}\,dx$$

where $u_+(x), u_-(x)$ are defined in (3.7). Using the same arguments as in the proof of Lemma 3.4, we have $\int_{\gamma_i} \varphi(x)h'(x)\,dt < 0$ for $i = 2, 3$. Hence

$$\int_{\tilde{\Gamma}} \varphi(x)h'(x)\,dt = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \varphi(x)h'(x)\,dt < 0.$$

Using the same arguments of Claim 2 in Lemma 3.5 yields

$$\#(S_r(\gamma_+) \cap \Gamma_r^+) \leq 1,$$

and

$$\#(S_r(\gamma_-) \cap \Gamma_r^-) \leq 1.$$

Next we claim that

$$z_2 < x_M. \tag{3.29}$$

Suppose, on the contrary, $z_2 \geq x_M$. Let $u_2(x) = u_+(x)$, $u_4(x) = u_-(x)$ for $x \in [z_1, \alpha_2]$ and $u_3(x) = u_+(x)$, $u_5(x) = u_-(x)$ for $x \in [\alpha_2, x_M]$ and $\tilde{u}(x) = u_2(h_2^{-1}(h_3(x)))$, $\underset{\sim}{u}(x) = u_4(h_2^{-1}(h_3(x)))$ for $x \in [\alpha_2, z_2]$. Then $\tilde{u}(x_2^*) > u_3(x_2^*)$, $\tilde{u}(x_M) > u_3(x_M)$, see Figure 3(a). Obviously, $\tilde{u}(x) > u_3(x)$ for $x \in (\alpha_2, x_M)$. Similarly, $\underset{\sim}{u}(x) < u_5(x)$ for $x \in (\alpha_2, x_M)$. Hence, we have

$$\Omega(\Gamma) \cap \Omega_3 \subseteq S_r(\Omega(\Gamma) \cap ([z_1, \alpha_2] \times \mathbf{R}^+)). \tag{3.30}$$

From Lemma 3.3(i), (3.27) and (3.30), it follows that

$$\begin{aligned}
0 &= \iint_{\Omega(\Gamma)} \frac{h'(x)}{u^2}\,dx\,du - \iint_{\Omega(\Gamma)\cap((0,z_1]\times\mathbf{R}^+)} \frac{h'(x)}{u^2}\,dx\,du \\
&= \iint_{\Omega(\Gamma)\cap([z_1,\alpha_2]\times\mathbf{R}^+)} \frac{h_2'(x)}{u^2}\,dx\,du + \iint_{\Omega(\Gamma)\cap\Omega_3} \frac{h_3'(x)}{u^2}\,dx\,du \\
&= \iint_{S_r(\Omega(\Gamma)\cap([z_1,\alpha_2]\times\mathbf{R}^+)\setminus(\Omega(\Gamma)\cap\Omega_3)} \frac{-h_3'(x)}{u^2}\,dx\,du \\
&> 0.
\end{aligned}$$

The above is a contradiction. Hence, we complete the proof of the claim (3.29).

Now we are in a position to prove Lemma 3.6. Since

$$\int_\Gamma \varphi(x)h'(x)\,dt = \int_{\tilde{\Gamma}} + \int_{\gamma_-} + \int_{\Gamma_r^-} + \int_{\Gamma_r^+} + \int_{\gamma_+} \varphi h'\,dt,$$

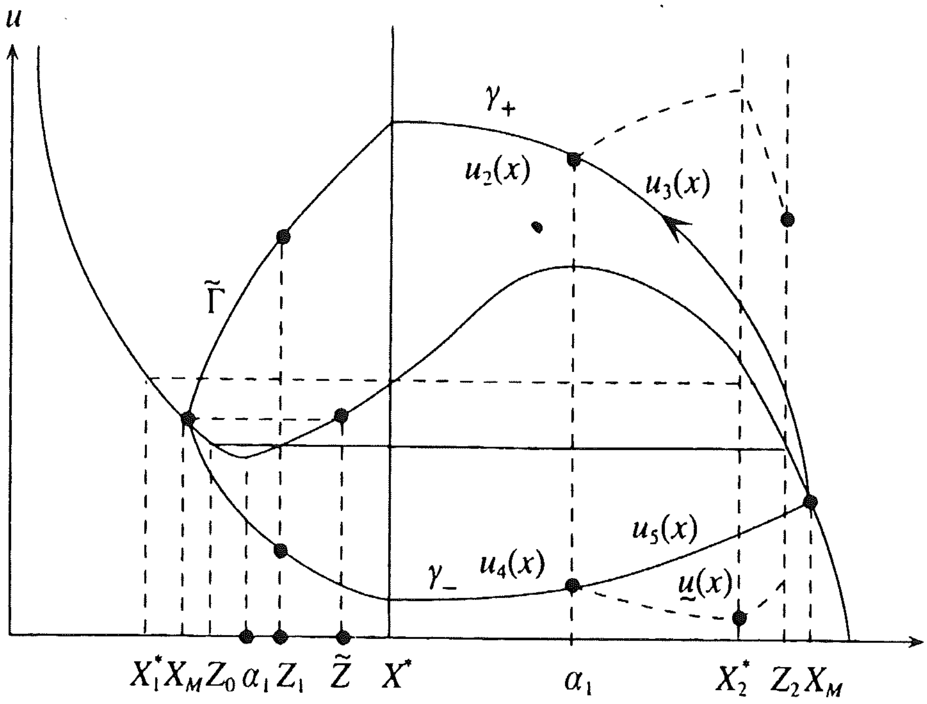

FIGURE 3(b).

from (3.28) it suffices to show that

$$\int_{\gamma_-} + \int_{\Gamma_r^-} + \int_{\Gamma_r^+} + \int_{\gamma_+} \varphi h' \, dt < 0. \tag{3.31}$$

There are three cases as follows.

Case 1. $\#(S_r(\gamma_+) \cap \Gamma_r^+) = \#(S_r(\gamma_-) \cap \Gamma_r^-) = 0$, see Figure 3(b). Using similar arguments as in the proof of Lemma 3.5 yields $\int_{\gamma_+ \cup \Gamma_\gamma^+} \varphi(x)h'(x)\, dt < 0$, $\int_{\gamma_- \cup \Gamma_\gamma^-} \varphi(x)h'(x)\, dt < 0$.

Hence, (3.31) holds.

Case 2. $\#(S_r(\gamma_+) \cap \Gamma_r^+) = \#(S_r(\gamma_-) \cap \Gamma_r^-) = 1$, see Figure 3(c). Let $S_r(\gamma_+) \cap \Gamma_r^+ = \{C_+\}$ and $S_r(\gamma_-) \cap \Gamma_r^- = \{C_-\}$. Denote $C_+ = (x_+, v_+)$, $C_- = (x_-, v_-)$ and $D_+ = (y_+, v_+)$, $D_- = (y_-, v_-)$ satisfying $S_r(D_+) = C_+$, $S_r(D_-) = C_-$.

The points P, Q are the leftmost and rightmost points of Γ, respectively. Assume $\Gamma \cap \{x = z_1\} = \{P_1, P_2\}$ with $P_1 = (z_1, u_{p_1})$, $P_2 = (z_1, u_{p_2})$, $u_{p_1} > u_{p_2}$; $\Gamma \cap \{x = x^*\} = \{A, F\}$ with $A = (x^*, u_A)$, $F = (x^*, u_F)$, $u_A > u_F$; $\Gamma \cap \{x = x_2^*\} = \{B, E\}$ with $B = (x_2^*, u_B)$,

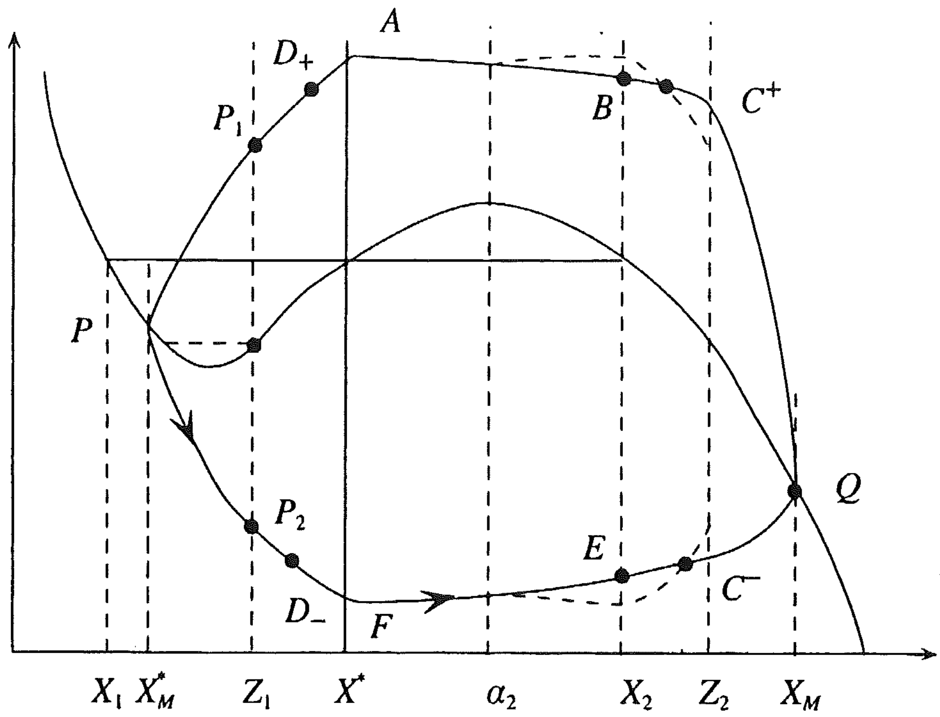

FIGURE 3(c).

$E = (x_2^*, u_E)$, $u_B > u_E$. Obviously, $x_2^* < x_+$, $x_- < z_2$ and $z_1 < y_+$, $y_- < x^*$. Write

$$\int_{\gamma_-} + \int_{\Gamma_r^-} + \int_{\Gamma_r^+} + \int_{\gamma_+} \varphi h'\,dt$$
$$= \int_{P_2\widehat{D}_-} + \int_{D_-\widehat{F}EC_-} + \int_{C_-\widehat{Q}C_+} + \int_{C_+\widehat{B}AD_+} + \int_{D_+\widehat{P}_1} \varphi h'\,dt.$$

Let $y = h_2^{-1}(h_3(x))$, $\underset{\sim}{u}(x) = u_-(y)$ for $\alpha_2 \leq x \leq z_2$. Then

$$\begin{aligned}
\int_{D_-\widehat{F}EC_-} \varphi h'\,dt &= \int_{y_-}^{x_-} \frac{h'(x)}{h(x) - u_-(x)}\,dx, \\
&= \int_{y_-}^{\alpha_2} \frac{h_2'(y)}{h_2(y) - u_-(y)}\,dy \\
&\quad + \int_{\alpha_2}^{x_-} \frac{h_3'(x)}{h_3(x) - u_-(x)}\,dx \\
&= -\int_{\alpha_2}^{x_-} \frac{h_3'(x)}{h_3(x) - \underset{\sim}{u}(x)}\,dx \\
&\quad + \int_{\alpha_2}^{x_-} \frac{h_3'(x)}{h_3(x) - u_-(x)}\,dx
\end{aligned}$$

$$= \int_{\alpha_2}^{x_-} \frac{h_3'(x)(u_-(x) - \underset{\sim}{u}(x))}{(h_3(x) - \underset{\sim}{u}(x))(h_3(x) - u_-(x))} \, dx$$
$$< 0.$$

Similarly we have

$$\int_{C_+\widehat{BAD}_+} \varphi h' \, dt < 0.$$

Let

$$\begin{aligned} D_+\widehat{P}_1 &= \{(y_+(u), u) : u \in [u_{p_1}, v_+]\}, \\ P_2\widehat{D}_- &= \{(y_-(u), u) : u \in [v_-, u_{p_2}]\}, \\ C_-\widehat{Q}C_+ &= \{(x(u), u) : u \in [v_-, v_+]\}. \end{aligned}$$

As in (3.24) and (3.25), it follows that

$$\int_{D_+\widehat{P}_1} \varphi h' \, dt = \ln h_2(z_1) - \ln h_2(y_+) - \int_{u_{p_1}}^{v_+} \frac{\varphi h_2'}{\psi h_2}(y_+(u)) \frac{1}{u} \, du,$$
$$\int_{P_2\widehat{D}_-} \varphi h' \, dt = \ln h_2(y_-) - \ln h_2(z_1) - \int_{v_-}^{u_{p_2}} \frac{\varphi h_2'}{\psi h_2}(y_-(u)) \frac{1}{u} \, du,$$
$$\int_{C_-\widehat{Q}C_+} \varphi h' \, dt = \ln h_3(x_+) - \ln h_3(x_-) + \int_{v_-}^{v_+} \frac{\varphi h_3'}{\psi h_3}(x(u)) \frac{1}{u} \, du.$$

Since $h_2(y_-) = h_3(x_-)$, $h_2(y_+) = h_3(x_+)$, then from the above it follows that

$$(3.32) \quad \int_{D_+\widehat{P}_1} + \int_{P_2\widehat{D}_-} + \int_{C_-\widehat{Q}C_+} \varphi h' \, dt$$
$$= \int_{v_-}^{u_{p_2}} + \int_{u_{p_2}}^{u_{p_1}} + \int_{u_{p_1}}^{v_+} \frac{\varphi h_3'}{\psi h_3}(x(u)) \frac{1}{u} \, du$$
$$- \int_{v_-}^{u_{p_2}} \frac{\varphi h_2'}{\psi h_2}(y_-(u)) \frac{1}{u} \, du$$
$$- \int_{u_{p_1}}^{v_+} \frac{\varphi h_2'}{\psi h_2}(y_+(u)) \frac{1}{u} \, du.$$

From the inequalities

$$\begin{aligned} y_-(u) < y_-, &\quad v_- < u < u_{p_2} \\ y_+(u) < y_+, &\quad u_{p_1} < u < u_+ \\ x(u) > x_-, x_+, &\quad v_- < u < v_+ \end{aligned}$$

and the fact that $(d/dx)[\varphi h'/(\psi h)] < 0$ for $x \in [\alpha_1, x^*] \cup [\alpha_2, 1]$ (Lemma 3.2), (3.32) yields

$$\begin{aligned}
I &= \int_{\widehat{D_+P_1}} + \int_{\widehat{P_2D_-}} + \int_{\widehat{C_-QC_+}} \varphi h'\,dt \\
&< \int_{u_{p_2}}^{u_{p_1}} \frac{\varphi h_3'}{\psi h_3}(x(u))\frac{1}{u}\,du + \int_{v_-}^{u_{p_2}} \frac{\varphi h_3'}{\psi h_3}(x_-)\frac{1}{u}\,du \\
&\quad + \int_{u_{p_1}}^{v_+} \frac{\varphi h_3'}{\psi h_3}(x_+)\frac{1}{u}\,du - \int_{v_-}^{u_{p_2}} \frac{\varphi h_2'}{\psi h_2}(y_-)\frac{1}{u}\,du \\
&\quad + \int_{u_{p_1}}^{v_+} \frac{\varphi h_2'}{\psi h_2}(y_+)\frac{1}{u}\,du \\
&= \int_{u_{p_2}}^{u_{p_1}} \frac{\varphi h_3'}{\psi h_3}(x(u))\frac{1}{u}\,du \\
&\quad + \int_{v_-}^{u_{p_2}} \left[\frac{\varphi h_3'}{\psi h_3}(x_-) - \frac{\varphi h_2'}{\psi h_2}(y_-)\right]\frac{1}{u}\,du \\
&\quad + \int_{u_{p_1}}^{v_+} \left[\frac{\varphi h_3'}{\psi h_3}(x_+) - \frac{\varphi h_2'}{\psi h_2}(y_+)\right]\frac{1}{u}\,du.
\end{aligned}$$

Obviously the first term in the above is negative. Then, from (3.22), we have $I < 0$ and hence (3.31) holds.

Case 3. $\#(S_r(\gamma_+) \cap \Gamma_r^+) = 1$ and $\#(S_r(\gamma_-) \cap \Gamma_r^-) = 0$, see Figure 3(d), or $\#(S_r(\gamma_+) \cap \Gamma_r^+) = 0$ and $\#(S_r(\gamma_-) \cap \Gamma_r^-) = 1$.

We only consider the first subcase, since the same arguments apply to the second subcase. The proof is similar to those in Case 2 and Case 1. Let $\Gamma_r^- \cap \{x = z_2\} = \{G\}$, $G = (z_2, u_G)$, see Figure 3(d). As in Case 2, we have

$$\int_{\widehat{C_+BAD_+}} \varphi h'\,dt < 0.$$

Using the arguments in establishing Case 1 yields

$$\int_{\widehat{P_2FEG}} \varphi h'\,dt < 0.$$

Let

$$\widehat{D_+P_1} = \{(y(u), u) : u \in [u_{p_1}, v_+]\}$$

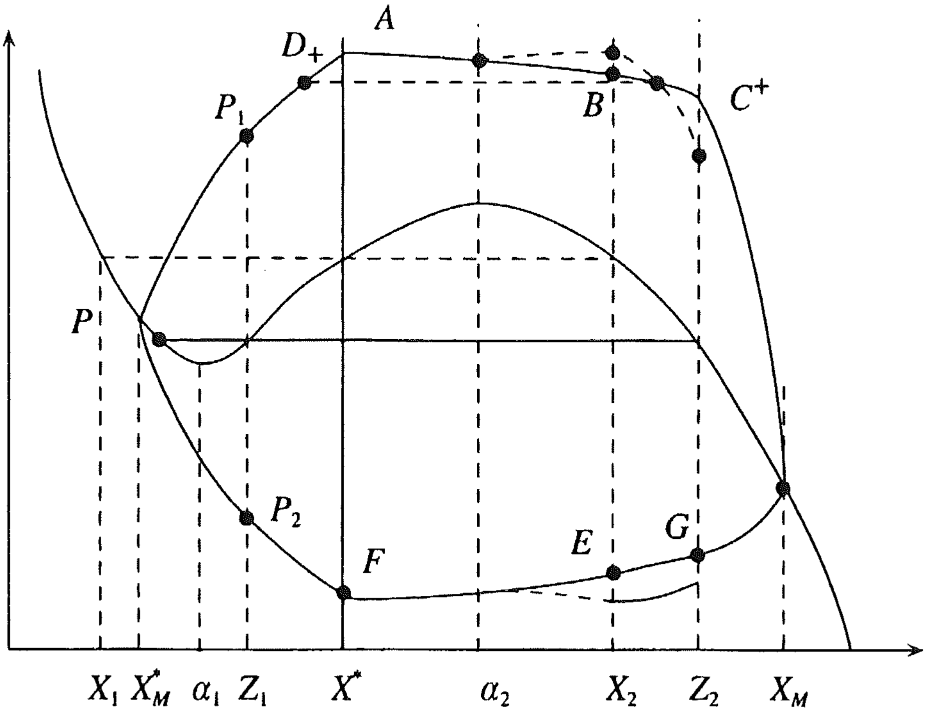

FIGURE 3(d).

and

$$G\widehat{Q}C_+ = \{(x(u), u) : u \in [u_G, v_+]\}.$$

Similar to the arguments in (3.24) and (3.25), we establish

$$\int_{D_+\widehat{P}_1} \varphi h' \, dt = \ln h_2(z_1) - \ln h_2(y_+) - \int_{u_{P_1}}^{v_+} \frac{\varphi h_2'}{\psi h_2}(y(u)) \frac{1}{u} \, du$$

and

$$\begin{aligned}
\int_{G\widehat{Q}C_+} \varphi h' \, dt &= \ln h_3(x_+) - \ln h_3(z_2) + \int_{u_G}^{v_+} \frac{\varphi h_3'}{\psi h_3}(x(u)) \frac{1}{u} \, du \\
&= \ln h_2(y_+) - \ln h_2(z_1) + \int_{u_{P_1}}^{v_+} \frac{\varphi h_3'}{\psi h_3}(x(u)) \frac{1}{u} \, du \\
&\quad + \int_{u_G}^{u_{P_1}} \frac{\varphi h_3'}{\psi h_3}(x(u)) \frac{1}{u} \, du \\
&< \ln h_2(y_+) - \ln h_2(z_1) + \int_{u_{P_1}}^{v_+} \frac{\varphi h_3'}{\psi h_3}(x(u)) \frac{1}{u} \, du.
\end{aligned}$$

Then

$$\int_{\widehat{D_+P_1}} + \int_{\widehat{GQC_+}} \varphi h' \, dt < \int_{u_{P_1}}^{v_+} \frac{\varphi h_3'}{\psi h_3}(x(u)) \frac{1}{u} \, du$$
$$- \int_{u_{P_1}}^{v_+} \frac{\varphi h_2'}{\psi h_2}(y(u)) \frac{1}{u} \, du$$
$$< \int_{u_{P_1}}^{v_+} \left[\frac{\varphi h_3'}{\psi h_3}(x_+) - \frac{\varphi h_2'}{\psi h_3}(y_+) \right] \frac{1}{u} \, du$$
$$< 0.$$

Hence, (3.31) holds. □

Proof of Theorem 3.1. First we show that any periodic orbit Γ of (3.1) satisfies $\Gamma \cap \{(x,u) : x \geq x_2^*\} \neq \varnothing$, i.e., $x_M > x_2^*$. Consider the Liapunov function V defined in (2.14) or (3.10). From the assumption (3.4),

$$R(x_1^*) = V(x_1^*, u^*) \geq R(x_2^*) = V(x_2^*, u^*),$$

the level curve $L_c = \{(x,u) : V(x,u) = c\}$ where $c = \min\{R(x_1^*), R(x_2^*)\} = R(x_2^*)$ is a closed Jordan curve passing through (x_2^*, u^*). The region Ω_c enclosed by L_c is indeed a region of repulsion due to the fact that $\dot{V} > 0$ on Ω_c. Then the periodic orbit must pass through the line $x = x_2^*$ and hence $x_M > x_2^*$. Since we have shown that when $x_M > x_2^*$, for any possible locations of Γ ($x_m < x_1^*$ in Lemma 3.4, $\alpha_1 \leq x_m$ in Lemma 3.5 and $x_1^* \leq x_m < \alpha_1$ in Lemma 3.6), Γ is orbitally asymptotically stable. Then system (3.1) or equivalently system (2.1) has a unique limit cycle. □

Remark 3.7. We note that the above techniques and arguments can be applied to a class of predator-prey systems whose prey isocline has two humps. For example, we can obtain similar results about the uniqueness of limit cycles for the system with Holling's type III functional response:

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{x}{(a+x)(b+x)} y, \\ \frac{dy}{dt} &= y\left(\delta - \beta \frac{y}{x}\right), \end{aligned} \tag{3.33}$$

as we did in [5] for the global stability of the equilibrium of (3.33).

REFERENCES

1. K.S. Cheng, *Uniqueness of a limit cycle for a predator-prey system*, SIAM J. Math. Anal. **12** (1981), 541–548.

2. S.H. Ding, *On a kind of predator-prey system*, SIAM J. Math. Anal. **20** (1989), 1426–1435.

3. H.I. Freedman and R.M. Mathsen, *Persistence in predator-prey systems with ratio-dependence predation influence*, Bull. Math. Biol. **55** (1993), 817–827.

4. J. Hale, *Ordinary differential equations*, Wiley-Interscience, New York, 1969.

5. S.B. Hsu and T.W. Huang, *Global stability for a class of predator-prey systems*, SIAM J. Appl. Math. **55** (1995), 763–783.

6. X.C. Huang and S. Merrill, *Condition for uniqueness of limit cycles in general predator-prey systems*, Math. Biosci. **96** (1989), 47–60.

7. Y. Kuang, *Nonuniqueness of limit cycles of Gause-type predator-prey systems*, Appl. Anal. **29** (1988), 269–287.

8. Y. Kuang and H.I. Freedman, *Uniqueness of limit cycles in Gause-type models of predator-prey systems*, Math. Biosci. **88** (1988), 67–84.

9. L.P. Liou and K.S. Cheng, *On the uniqueness of a limit cycle for a predator-prey system*, SIAM J. Math. Anal. **19** (1988), 867–878.

10. J.D. Murray, *Mathematical biology*, Springer Verlag, Berlin, 1989.

11. R.M. May, *Stability and complexity in model ecosystems*, Princeton University Press, Princeton, NJ, 1974.

12. M.L. Rosenzweig, *Why the prey curve has a hump*, Amer. Natur. **103** (1969), 81–87.

13. J.T. Tanner, *The stability and the intrinsic growth rates of prey and predator populations*, Ecology **56** (1975), 855–867.

14. D. Wollkind, J. Collings and J. Logan, *Metastability in a temperature-dependent model system for predator-prey mite outbreak interactions on fruit trees*, Bull. Math. Biol. **50** (1988), 379–490.

15. D. Wollkind and J. Logan, *Temperature-dependent predator-prey mite ecosystem on apple tree foliage*, J. Math. Biol. **6** (1978), 265–283.

Department of Mathematics, National Tsing Hua University, Hsinchu 30043, Taiwan, Republic of China
E-mail address: sbhsu.am.nthu.edu.tw

Department of Mathematics, Kaohsiung Normal University, Kaohsiung, Taiwan, Republic of China