Nonlinear Singular Sturm-Liouville Problems and an Application to Transonic Flow Through a Nozzle

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Abstract

We consider a class of singular Sturm-Liouville problems with a nonlinear convection and a strongly coupling source. Our investigation is motivated by, and then applied to, the study of transonic gas flow through a nozzle. We are interested in such solution properties as the exact number of solutions, the location and shape of boundary and interior layers, and nonlinear stability and instability of solutions when regarded as stationary solutions of the corresponding convective reaction-diffusion equations. Novel elements in our theory include a priori estimate for qualitative behavior of general solutions, a new class of boundary layers for expansion waves, and a local uniqueness analysis for transonic solutions with interior and boundary layers.

1. Introduction

Consider the nonlinear Sturm-Liouville problem

$$\varepsilon u'' = f(x, u)' - c(x)h(u)$$

with u prescribed at $x = 0$ and $x = 1$. We study properties of the solutions: their number, their asymptotic shape for small ε, and their stability and instability, when viewed as stationary solutions of the corresponding time evolution equation. Our study is motivated by, and then applied to, the problem of transonic gas flow through a nozzle. Novel elements in our theory include a new class of boundary layers corresponding to expansion waves, a priori estimate for qualitative behavior of general solutions, and a local uniqueness analysis for transonic solutions with interior and boundary layers.

For simplicity of presentation, we shall carry out our analysis for the model problem

$$\varepsilon u'' = f(u)' - c(x)h(u), \quad u = u(x) \in \mathbf{R}^1, \qquad 0 \leqq x \leqq 1, \tag{1.1}$$

$$u(0) = u_l, \qquad u(1) = u_r. \tag{1.2}$$

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The flux function $f(u)$ is assumed to be convex. This is motivated by gas dynamics, where the sound speed depends monotonically on the density. By composition with a simple translation, we may assume that

$$f''(u) > 0 \quad \text{for all} \quad u \quad \text{under consideration}, \qquad f(0) = f'(0) = 0. \tag{1.3}$$

The function $h(u)$ represents the coupling of the source due to the geometry and the gas flow. The following strong nonlinear coupling assumption is dictated by physics:

$$h(u) \neq 0, \quad h'(u) \neq 0 \quad \text{for all} \quad u \quad \text{under consideration}. \tag{1.4}$$

The function $c(x)$ represents the strength of the source and may change sign. For simplicity and definiteness, we make the following assumption:

$$c(x)h(u) < 0 \quad \text{for} \quad 0 \leqq x \leqq 1 \quad \text{and all} \quad u \quad \text{under consideration}. \tag{1.5}$$

More general situations can also be treated with our analysis. There are two distinct cases:

$$c(x)h'(u) < 0 \qquad \text{(stable case; diverging duct)}, \tag{1.6}$$

$$c(x)h'(u) > 0 \qquad \text{(unstable case; converging duct)}. \tag{1.7}$$

Our equation (1.1) may be regarded as the stationary equation for the convective reaction-diffusion equation

$$u_t + f(u)_x = \varepsilon u_{xx} + c(x)h(u). \tag{1.8}$$

In the stable case, (1.1)–(1.2) has a unique solution which is time asymptotically stable with respect to (1.8). In the unstable case, there may exist three solutions, one of which has an interior layer and is nonlinearly unstable with respect to (1.8); the other two are stable.

We now relate our problem to nozzle flow. The quasi-one-dimensional model for isentropic viscous compressible flow through a nozzle is

$$(\rho A)_t + (\rho v A)_x = 0, \tag{1.9$_1$}$$

$$(\rho v A)_t + (\rho v^2 A)_x + A p_x = \mu (A v_x)_x, \tag{1.9$_2$}$$

where ρ, v, p, μ and $A = A(x)$ are the density, velocity, pressure, viscosity coefficient of the gas and the cross section of the nozzle. The pressure is a given function of the density ρ. We assume that

(1.10) $$p'(\rho) > 0, \qquad p''(\rho) > 0.$$

This is satisfied by the polytropic gases where $p(\rho) = a\rho^r$, $\frac{5}{3} \geq r > 1$. The stationary equations are

$$(\rho v A)_x = 0,$$

$$(\rho v^2 A)_x + A p_x = \mu (A v_x)_x.$$

The first equation can be integrated to yield a relation between v and ρ. This can be used to eliminate one of the dependent variables and so yield a scalar equation for one dependent variable. If ρ is eliminated, we get

(1.11) $$(g(v, x))_x = \mu v_{xx} + c(x) k(v),$$

$(1.12)_1$ $$g(v, x) = \frac{\rho_0 v_0 A_0}{A(x)} v + p\left(\frac{\rho_0 v_0 A_0}{v A(x)}\right) - \mu \frac{A'(x)}{A(x)} v,$$

$(1.12)_2$ $$c(x) = -\frac{A'(x)}{(A(x))^2} \rho_0 v_0 A_0 - \mu \left(\frac{A'(x)}{A(x)}\right)',$$

$(1.12)_3$ $$k(v) = v,$$

where $\rho_0 = \rho(0)$, $v_0 = v(0)$ and $A_0 = A(0)$. From (1.10) we have

$$g_v(v, x) = \frac{\rho_0 v_0 A_0}{A(x)} - \frac{\rho_0 v_0 A_0}{v^2 A(x)} p'\left(\frac{\rho_0 v_0 A_0}{v A(x)}\right) - \mu \frac{A'(x)}{A(x)},$$

$$g_{vv}(v, x) = \frac{2\rho_0 v_0 A_0}{v^3 A(x)} p'\left(\frac{\rho_0 v_0 A_0}{v A(x)}\right) + \left(\frac{\rho_0 v_0 A_0}{v^2 A(x)}\right)^2 p''\left(\frac{\rho_0 v_0 A_0}{v A}\right) > 0.$$

The first equation shows that, for small viscosity μ, $g_v > 0$ if the flow is supersonic (i.e., when $|v| > (p')^{1/2}$), while $g_v < 0$ if the flow is subsonic (i.e., when $|v| < (p')^{1/2}$). We are interested in transonic flow, i.e., when $|v|$ is close to the sonic speed $(p')^{1/2}$. Thus we assume that $|v| \neq 0$, and so $k(v) \neq 0$. We see that the nozzle equation (1.11) shares properties analogous to (1.3), (1.4) for the model (1.1). For this reason, we will call the positive (negative) states $u > 0$ ($u < 0$) for (1.1) supersonic (subsonic) and call the zero state sonic. It is clear from the above equations that for a converging duct, i.e., $vA'(x) < 0$, we have $c(x)k'(v) > 0$. This is the reason we called the unstable case (1.7) the converging duct and the stable case (1.6) the diverging duct.

Our main interest is in the properties of solutions for small ε. The inviscid theory, $\varepsilon = 0_+$, has been worked out by Liu in [4]. This is reviewed in Section 2. We also remark that a new type of boundary layer with algebraic decay exists. They correspond to rarefaction waves connecting to the sonic state, Remark 2.1. In Section 3 we present an analysis which shows that any general solution of (1.1), (1.2) is close to one of the corresponding inviscid solutions. Our analysis differs markedly from the usual asymptotic analysis where solutions of viscous equations are constructed based on the inviscid solutions. In the stable case (1.6), it follows easily from the maximum principle that there exists at most one solution. The inviscid theory predicts that in the unstable case there are one or three solutions, depending on the data (1.2). With the a priori qualitative understanding obtained in Section 3, we show in Section 4 that (1.1), (1.2) with (1.7) has exactly one or three solutions by proving a local uniqueness theorem. For solutions with no interior layer, this is done by generalizing and refining the classical argument of Coddington and Levinson [2]. For solutions with interior layer, a new argument is introduced for the local uniqueness theorem.

In Section 5 we present a stability analysis for solutions of (1.1) viewed as stationary solutions of (1.8).

There have been studies on singular nonlinear Sturm-Liouville problems: see [3] and references therein. However, these studies do not consider models with the strong coupling property (1.4). As we show here, this coupling property has the regularizing effect that viscous solutions are close to the inviscid solution. In the absence of (1.4), the inviscid theory often offers too many solutions, most of which do not correspond to viscous solutions. With (1.4) there arise new analytical difficulties, some of which are resolved here. It would be interesting to study more general convective reaction-diffusion equations such as the full quasi-one-dimensional nozzle flow equations (cf. references in [4]).

2. Inviscid Theory

In this section we review the time-asymptotic states for

$$u_t + f(u)_x = c(x)h(u). \tag{2.1}$$

We then present a brief account of that theory. Except for boundary or interior layers, the solutions of (1.1) should satisfy

$$f(u)_x = c(x)h(u) \tag{2.2}$$

as $\varepsilon \to 0_+$. Interior layers would tend to stationary shock waves (u_-, u_+) of (2.1) satisfying the jump condition and entropy condition

$$f(u_+) = f(u_-), \qquad u_+ < u_-. \tag{2.3}$$

An inviscid boundary layer (u_l, u_0) at $x = 0$ would correspond to stationary waves of (1.1) at $x = 0$ which in turn satisfy approximately

$$f(u)_x = \varepsilon u_{xx}, \quad u(0) = u_l, \quad u(\infty) = u_0, \quad u_x(\infty) = 0,$$

since in the layer the source $c(x)h(u)$ has little effect. Integrate the above equation from $x = x$ to $x = \infty$ to get

$$(2.4)_0 \qquad \varepsilon u_x(x) = f(u(x)) - f(u_0), \quad u(0) = u_l, \quad u(\infty) = u_0.$$

In order for the above equation to have a solution, we need

$$(2.4) \quad (u - u_0)(f(u) - f(u_0)) < 0 \quad \text{for all } u \text{ between } u_l \text{ and } u_0.$$

Thus an inviscid boundary layer (u_l, u_0) of (2.1) at $x = 0$ would correspond to a limit of viscous boundary layers of (1.1) if and only if (2.4) holds. Similarly, a boundary layer (u_0, u_r) at $x = 1$ satisfies

$$(2.5)_0 \qquad \varepsilon u_x(x) = f(u(x)) - f(u_0), \quad u(1) = u_r, \quad u(-\infty) = u_0,$$

$$(2.5) \quad (u - u_0)(f(u) - f(u_0)) > 0 \quad \text{for all } u \text{ between } u_0 \text{ and } u_r.$$

Conditions (2.4) and (2.5) can be related directly to elementary waves for

$$(2.6) \qquad u_t + f(u)_x = 0.$$

There are two types of waves for (2.6). Two states u_- and u_+ can be connected by a shock wave (u_-, u_+) with speed

$$\sigma = \frac{f(u_+) - f(u_-)}{u_+ - u_-}$$

if $u_+ < u_-$. When $u_+ > u_-$, (u_-, u_+) is a rarefaction wave propagating with characteristic speed $f'(u)$. This is so under the convexity condition $f''(u) > 0$, (1.3). It can be seen easily that (2.4) holds if and only if (u_l, u_0) constitutes an elementary wave for (2.6) with negative speed. Similarly, (2.5) holds if and only if (u_0, u_r) has positive speed.

Remark 2.1. Formal asymptotic expansion for boundary layers at $x = 1$ can be performed to yield $(2.5)_0$ as follows: Write

$$u(x) = U(x) + v(\eta),$$

$$\eta = \frac{1 - x}{\varepsilon},$$

where $U(x)$ is valid outside the layer and therefore is close to being an inviscid solution satisfying (2.2), and $v(\eta)$ is valid in the layer. Plug the expression into (1.1) and compare the coefficients of ε^{-1} to obtain

$$v'' + f'(v)v' = 0,$$

$$v(0) = u_r, \qquad v(\infty) = v'(\infty) = 0.$$

Integrate above to yield $(2.5)_0$,

$$v' = -f(v), \quad v(0) = u_r, \quad v(\infty) = 0.$$

From this we notice that in the case of a *rarefaction* boundary layer, $u_r > 0$, $v(\eta)$ decays *algebraically* as a consequence of (1.3). When $f(u)$ has higher-order zeros at $u = 0$, the decay rate is lower. This is in contrast to the usual types of boundary layers, where the decay is exponential.

The above can be summarized as follows: Solutions of (1.1), (1.2) would tend to inviscid solutions which satisfy (2.2) except for possible discontinuities. The discontinuities at $0 < x < 1$, $x = 0$ and $x = 1$, would satisfy (2.3), (2.4) and (2.5), respectively. Such an inviscid wave pattern is called an asymptotic state because it represents the large-time state of solutions with given end states at $x = \pm\infty$ (see Liu [4]). We now describe all the possible types of asymptotic states with given end states u_l and u_r.

Besides hypothesese (1.3) and (1.5) we further assume that

$$\lim_{|u|\to\infty} \frac{h(u)}{f(u)} = 0. \tag{2.7}$$

Since $c(x)h(u) < 0$, (1.5), a solution $u(x)$ of (2.2) moves toward the sonic state zero as x increases. Condition (2.7) ensures that given a state u there always exists a state $\tilde{u}$ such that $\tilde{u}$ and u are connected by a solution of (2.2) with values $\tilde{u}$ at $x = 0$ and u at $x = 1$. Since (2.2) is singular at $u = 0$, and $f'(0) = 0$ by (1.3), there are two states u^* and u_*, with $u^* > 0 > u_*$, each of which is connected to $u = 0$ by solutions of (2.2). Given a state $u \neq 0$, define $\bar{u}$, $\bar{u}u < 0$, satisfying

$$f(u) = f(\bar{u})$$

so that u and $\bar{u}$ form a standing shock wave, (2.3).

Consider first the stable case (1.6), $c(x)h'(u) < 0$. It is easily shown that in this case

$$u^* > \bar{u}_*. \tag{2.8}$$

Case A. $u_l \leqq \bar{u}_*$.

Case A1. When $u_r \geqq 0$, the asymptotic state consists of a backward wave (u_l, u_*) at $x = 0$, a subsonic stationary wave connecting u_* at $x = 0$ and $u = 0$ at $x = 1$, and a forward rarefaction wave $(0, u_r)$ at $x = 1$. The backward wave (u_l, u_*) is a shock (rarefaction) wave when $u_l > u_*$ $(u_l < u_*)$.

Case A2. When $u_r < 0$, the asymptotic state consists of a backward wave $(u_l, \bar{u}_r)$ and a stationary wave $(\bar{u}_r, u_r)$.

Case B. $\bar{u}_* < u_l < u^*$.

Case B1. When $u_r \geqq 0$, the asymptotic state consists of a supersonic stationary wave (u_l, u_-) for $0 \leqq x < x_0$, a stationary shock wave (u_-, u_+) at $x = x_0$, a subsonic stationary wave $(u_+, 0)$ for $x_0 < x < 1$ and a supersonic rarefaction wave $(0, u_r)$ at $x = 1$. The location $x = x_0$ and the states u_-, u_+ of the standing shock wave are determined uniquely by the left state u_l.

Case B2. When $u_r < 0$ and $\bar{u}_r < \bar{u}_l$, the asymptotic state consists of a backward shock wave $(u_l, \bar{u}_r)$ and a subsonic stationary wave $(\bar{u}_r, u_r)$.

Case B3. When $u_r < 0$ and $\bar{u}_r \geqq \bar{u}_l$, the asymptotic state consists of a supersonic stationary wave (u_l, u_-) for $0 \leqq x < x_0$, a standing shock wave (u_-, u_+) at $x = x_0$ and a subsonic stationary wave (u_+, u_r) for $x_0 < x \leqq 1$.

Case C. $u_l \geqq u^*$. Define u_1 with $\bar{u}_1 \equiv u_l$.

Case C1. When $u_r \geqq \bar{u}_1$, the asymptotic state consists of a stationary wave (u_l, u_1) and a forward wave (u_1, u_r) at $x = 1$.

Case C2. When $u_r < \bar{u}_1 < 0$ and $\bar{u}_r \geqq \bar{u}_l$, the asymptotic state consists of a stationary wave for $x \neq x_0$ and a stationary shock wave at $x = x_0$.

Case C3. When $u_r < \bar{u}_1 < 0$ and $\bar{u}_r < \bar{u}_l$, the asymptotic state consists of a backward shock wave $(u_l, \bar{u}_r)$ at $x = 0$ and a stationary wave $(\bar{u}_r, u_r)$.

Next we consider the unstable case (1.7), $c(x)h'(u) > 0$. In this case we have

$$\bar{u}_* > u^* > 0. \tag{2.9}$$

Case D. $u_l \leqq u^*$.

Case D1. When $u_r \geqq 0$, the asymptotic state consists of a backward wave (u_l, u_*) at $x = 0$, a subsonic stationary wave $(u_*, 0)$ for $0 < x < 1$, and a forward rarefaction wave $(0, u_r)$ at $x = 1$.

Case D2. When $u_r < 0$, the asymptotic state consists of a backward wave $(u_l, \bar{u}_r)$ at $x = 0$ and a subsonic stationary wave $(\bar{u}_r, u_r)$ for $0 < x \leqq 1$.

Case E. $u^* < u_l$. Define u_1 by $\bar{u}_1 \equiv u_l$.

Case E1. When $u_r \geqq 0$ and $u^* < u_l < \bar{u}_*$, there are three asymptotic states:

(i) a supersonic stationary wave (u_l, u_1) for $0 \leqq x < 1$ and a forward wave (u_1, u_r) at $x = 1$;
(ii) a supersonic stationary wave (u_l, u_-) for $0 \leqq x < x_0$, a stationary shock wave (u_-, u_+) at $x = x_0$, a subsonic stationary wave $(u_+, 0)$ for $x_0 < x < 1$ and a forward rarefaction wave $(0, u_r)$ at $x = 1$; x_0, u_-, u_+ are uniquely determined by u_l;
(iii) a backward shock wave (u_l, u_*) at $x = 0$, a subsonic stationary wave $(u_*, 0)$ for $0 < x < 1$, and a forward rarefaction wave $(0, u_r)$ at $x = 1$.

Case E2. When $u_r \geqq 0$ and $u_l \geqq \bar{u}_*$, the asymptotic state consists of a stationary wave (u_l, u_1) for $0 \leqq x < 1$ and a forward wave (u_1, u_r) at $x = 1$.

Case E3. When $u_r < 0$, there are three subcases: (i) if $\bar{u}_1 < u_r < 0$, then an asymptotic state consists of a stationary wave (u_l, u_1) and a forward shock wave; (ii) if $\bar{u}_l \geqq \bar{u}_r$, then it consists of a backward shock wave $(u_l, \bar{u}_r)$ and a stationary wave $(\bar{u}_r, u_r)$; (iii) if $\bar{u}_1 < u_r < 0$ and $\bar{u}_l > \bar{u}_r$, then there also exists an asymptotic state which consists of transonic stationary waves with a stationary shock wave at $x = x_0$, determined uniquely by u_l and u_r.

Thus, given end states u_l and u_r, the inviscid theory yields three solutions for Case E1 and also (iii) of Case E3. It can be shown easily that the above is a complete description of asymptotic states and that an asymptotic state depends smoothly on its end states.

3. A Priori Properties

As in Section 2, hereafter besides (1.3)–(1.5) we also assume that

$$\lim_{|u| \to \infty} \frac{h(u)}{f(u)} = 0. \tag{3.1}$$

LEMMA 3.1. *Any solution* $u(x)$ *of* (1.1) *belongs to one of the following three types*:

Type I: $u(x)$ *is strictly increasing*;
Type II: $u(x)$ *is strictly decreasing*;
Type III: $u(x)$ *has a unique critical point which is an absolute minimum.*

Proof: This is an immediate consequence of the hypothesis (1.5), $c(x)h(u) < 0$.

LEMMA 3.2. *There exists a positive constant M depending on u_l and u_r and not on ε such that any solution of $u(x) = u(x;\varepsilon)$ of (1.1), (1.2) satisfies*

$$|u(x)| < M, \qquad 0 \leqq x \leqq 1.$$

Proof: The lemma holds trivially for monotone solutions. Let $u(x)$ be a Type III solution of Lemma 3.1 with minimum $u(x_0)$ at $x = x_0$. Integrate (1.1) from $x = x_0$ to $x = 1$, and use (1.5), $ch < 0$, to obtain

$$\begin{aligned}\varepsilon u'(1) + f(u(x_0)) &= f(u_r) - \int_{x_0}^{1} c(x)h(u(x))\,dx \\ &= f(u_r) + \int_{x_0}^{1} |c(x)h(u(x))|\,dx.\end{aligned} \tag{3.2}$$

Since $h'(u) \neq 0$ and $u(x)$ is strictly increasing for $x_0 < x < 1$, $h(u(x))$ is monotone for $x_0 < x < 1$. When

$$\max_{x_0 \leq x \leq 1} h(u(x)) = h(u(1)) = h(u_r),$$

we have from (3.2) and $u'(1) > 0$ that

$$f(u(x_0)) < f(u_r) + h(u_r)\int_{x_0}^{1} c(x)\,dx$$

which is bounded independent of ε and so the lemma is proved. When

$$\max_{x_0 \leq x \leq 1} h(u(x)) = h(u(x_0)),$$

we have again from (3.2) that

$$f(u(x_0)) < f(u_r) + h(u(x_0))\int_{x_0}^{1} c(x)\,dx.$$

This estimate and hypothesis (3.1) yield an upper bound independent of ε for $|u(x_0)|$ and therefore for $|u(x)|$, $0 \leq x \leq 1$.

LEMMA 3.3. *Suppose that $u(x)$, $a < x < b$, is a strictly increasing (decreasing) solution of (1.1), (1.2) and that $u(x) > -C\varepsilon^{1/4}$ ($u(x) < C\varepsilon^{1/4}$), $a < x < b$, for some $C > 0$. Then $b - a = O(1)\varepsilon^{1/2}$ as $\varepsilon \to 0_+$.*

Proof: We consider only the case when $u(x)$ is increasing; the other case is similar. Integrate (1.1) from a to x to obtain

$$\varepsilon u'(x) - \varepsilon u'(a) = f(u(x)) - f(u(a)) - \int_a^x c(y)h(u(y))\,dy.$$

Since u is increasing, $u(x) > u(a)$ for $x > a$, and $u(a) > -C\varepsilon^{1/4}$, we have from (1.3) that $f(u(x)) - f(u(a)) > -\frac{1}{2}C^2\varepsilon^{1/2}$. Thus the above yields

$$\varepsilon u'(x) \geqq -\frac{1}{2}C^2\varepsilon^{1/2} - \int_a^x c(y)h(u(y))\,dy \quad \text{for} \quad a < x < b.$$

Integrate this from a to b to yield

$$\varepsilon u(b) - \varepsilon u(a) \geqq -\frac{1}{2}C^2(b-a)\varepsilon^{1/2} - \int_a^b\int_a^x c(y)h(u(y))\,dy\,dx.$$

By assumption, $c(x)h(u) < 0$, and from Lemma 3.2, $|u| < M$. Thus, the above estimate yields

$$2\varepsilon M + \frac{1}{2}C^2\varepsilon^{1/2}(b-a) \geqq D(b-a)^2,$$

for some positive constant D. The lemma follows immediately from this inequality.

In the following two lemmas we study the solution of (1.1), (1.2) outside the layers. In all cases we assume that $x_1 - x_0$ is of order one, i.e., $x_1 - x_0 > C\varepsilon^{3/8}$ for sufficiently large C, and $0 \leqq x_0 < x_1 \leqq 1$.

LEMMA 3.4. *Let $u(x)$ be a solution of* (1.1), (1.2). *Set*

$$C_1 \equiv 3\Big(\max_{|u|\leq M} f''(u)\Big)\Big(\max_{\substack{0\leq x\leq 1\\ |u|\leq M}} |c(x)h(u)|\Big)^2,$$

$$\varphi(x) \equiv f'(u(x))u'(x) - c(x)h(u(x)).$$

(i) *Suppose that $u'(x) > 0$, $f'(u(x)) < -\varepsilon^{1/4}$ for $x_0 \leqq x \leqq x_1$ and $|\varphi(x_0)| < C_1\varepsilon^{1/4}$. Then $|\varphi(x)| \leqq C_1\varepsilon^{1/4}$, $x_0 \leqq x \leqq x_1$, for ε sufficiently small.*

(ii) *Suppose that $u'(x) < 0$, $f'(u(x)) > \varepsilon^{1/4}$ for $x_0 \leqq x \leqq x_1$ and $|\varphi(x_1)| < C_1\varepsilon^{1/4}$. Then $|\varphi(x)| \leqq C_1\varepsilon^{1/4}$, $x_0 \leqq x \leqq x_1$, for ε sufficiently small.*

Proof: We shall prove (i); (ii) follows by similar arguments. Suppose that the conclusion fails. Then there exists $\bar{x}$, $x_0 \leqq \bar{x} < x_1$, such that

$$|\varphi(x)| \leqq C_1\varepsilon^{1/4} \quad \text{for} \quad x_0 < x < \bar{x},$$

and either

$$\text{(I)} \qquad \varphi(\bar{x}) = C_1\varepsilon^{1/4}, \qquad \varphi'(\bar{x}) \geqq 0,$$

or

$$\text{(II)} \qquad \varphi(\bar{x}) = -C_1\varepsilon^{1/4}, \qquad \varphi'(\bar{x}) \leqq 0.$$

We treat (I); (II) is similar. Relation (1.1) is the same as $\varepsilon u''(x) = \varphi(x)$. With (I) we have

$$u''(\bar{x}) = C_1\varepsilon^{-3/4},$$

$$f'(u(\bar{x}))u''(\bar{x}) + f''(u(\bar{x}))(u'(\bar{x}))^2 - c'(\bar{x})h(u(\bar{x})) - c(\bar{x})h'(u(\bar{x}))u'(\bar{x}) \geqq 0.$$

From the hypothesis $f'(u) < -\varepsilon^{1/4}$ the above yields

$$C_1\varepsilon^{-1/2} \leqq f''(u(\bar{x}))(u'(\bar{x}))^2 - c'(\bar{x})h(u(\bar{x})) - c(\bar{x})h'(u(\bar{x}))u'(\bar{x}). \tag{3.3}$$

On the other hand, from (I) we have

$$f'(u(\bar{x}))u'(\bar{x}) - c(\bar{x})h(u(\bar{x})) = C_1\varepsilon^{1/4},$$

and by Lemma 3.2, $|u| < M$. Since, by hypothesis, $f'(u) < -\varepsilon^{1/4}$, it follows that

$$|u'(\bar{x})| \leqq \left(\max_{\substack{0\le x\le 1\\ |u|\le M}} |c(x)h(u)| + C_1\varepsilon^{1/4}\right)\varepsilon^{-1/4}.$$

This and (3.3) yield

$$\begin{aligned} C_1\varepsilon^{-1/2} \leqq 2\max_{|u|\le M} f''(u)&\left(\left(\max_{\substack{0\le x\le 1\\ |u|\le M}} |c(x)h(u)|\right)^2\varepsilon^{-1/2} + C_1^2\right) \\ &+ \max_{\substack{0\le x\le 1\\ |u|\le M}} |c'(x)h(u)| \\ &+ \max_{\substack{0\le x\le 1\\ |u|\le M}} |c(x)h'(u)|\left(\max_{\substack{0\le x\le 1\\ |u|\le M}} |c(x)h(u)|\varepsilon^{-1/4} + C_1\right). \end{aligned}$$

This contradicts the definition of C_1 when ε is sufficiently small.

PROPOSITION 3.5. *Suppose that $u(x)$ is a solution of* (1.1), (1.2) *and is strictly monotone over a subinterval (x_0, x_1) of $(0,1)$. Then there exist $C > 0$ and a subinterval (x_2, x_3) of (x_0, x_1) such that, for sufficiently small ε, $|\varphi(x)| \equiv |f'(u(x))u'(x) - c(x)h(u(x))| \leqq C\varepsilon^{1/4}$ over (x_2, x_3) and*

$$|x_2 - x_0| + |x_3 - x_1| \leqq D\varepsilon^{3/8}$$

for some positive constant D independent of ε.

Proof: In view of Lemmas 3.3 and 3.4, we only need to show that, for strictly increasing $u(x)$, there exists x_2 with $|\varphi(x_2)| \leqq C\varepsilon^{1/4}$ and $|x_2 - x_0| = O(1)\varepsilon^{3/8}$ and, for strictly decreasing $u(x)$, there exists x_3 with $|\varphi(x_3)| \leqq C\varepsilon^{3/8}$ and $|x_3 - x_1| = O(1)\varepsilon^{3/8}$ for any given $C > 0$. Consider the case $u(x)$ is increasing. Suppose that $|\varphi(x)| \geqq C\varepsilon^{1/4}$, $x_0 < x < \bar{x}$, for some $\bar{x} > x_0$ and $C > 0$. There are two cases:

$$\text{(I)} \qquad \varphi(x) > C\varepsilon^{1/4} \quad \text{for} \quad x_0 < x < \bar{x},$$

or

$$\text{(II)} \qquad \varphi(x) < -C\varepsilon^{1/4} \quad \text{for} \quad x_0 < x < \bar{x}.$$

In Case (I) we have from (1.1), $\varepsilon u''(x) = \varphi(x)$, that $u''(x) > C\varepsilon^{-3/4}$ for $x_0 < x < \bar{x}$ and so by integration

$$u'(x) \geqq C\varepsilon^{-3/4}(x - x_0) + u'(x_0) \geqq C\varepsilon^{-3/4}(x - x_0), \quad x_0 \leqq x \leqq \bar{x}.$$

Integrate again to obtain

$$u(\bar{x}) - u(x_0) \geqq \frac{C}{2}\varepsilon^{-3/4}(\bar{x} - x_0)^2.$$

Since u is increasing, $2M \geqq u(\bar{x}) - u(x_0)$, and also the above yields

$$\bar{x} - x_0 < \left(\frac{4M}{C}\right)^{1/2} \varepsilon^{3/8} \equiv E.$$

Thus we have shown that $|\varphi(x_2)| \leqq C\varepsilon^{1/4}$ for some x_2, $x_0 < x_2 < x_0 + E$. This proves the lemma for Case (I). Other cases are treated similarly.

We next study boundary and interior layers.

LEMMA 3.6. *Use the same notations as in Proposition* 3.5.

(i) *When $u'(x) > 0$, $x_0 < x < x_1$, then $u(x) < 0$, $x_0 < x < x_3$, and x_2 can be chosen so that either $\varepsilon u''(x) \geqq C_1\varepsilon^{1/4}$ and $|u(x) - u(x_0)| = O(1)\varepsilon^{3/8}$ for*

$x_0 \leqq x \leqq x_2$ or $\varepsilon u''(x) \leqq -C_1\varepsilon^{1/4}$ for $x_0 \leqq x \leqq x_2$; x_3 can be chosen so that either $x_3 = x_1$ and $f'(u(x_3)) \leqq -C_1\varepsilon^{1/4}$ or $x_3 < x_1$ and $f'(u(x_3)) = -C_1\varepsilon^{1/4}$.

(ii) *When $u'(x) < 0$, $x_0 < x < x_1$, then $u(x) > 0$, $x_0 < x < x_3$, and x_2 and x_3 can be chosen so that $x_2 = x_0$ and either $x_3 = x_1$ or $x_3 < x_1$. In the case $x_3 < x_1$ we have either $|u(x) - u(x_1)| \leqq D\varepsilon^{3/8}$ for $x_3 \leqq x \leqq x_1$ or $\varphi(x_3) = -C_1\varepsilon^{1/4}$ and $f(u(x_3)) - f(u(x_1)) \geqq -D\varepsilon^{3/8}$ for some $D > 0$ independent of ε.*

Proof: (i) The second part is a direct consequence of (i) of Lemma 3.4. It follows from Lemma 3.3 that $u(x_3) < 0$. Since $u'(x) > 0$ and $u(x)$ is close to the inviscid wave, by Proposition 3.5 we have $u(x) < -C$, $x_0 < x < x_2$, for some positive C independent of ε. When $x_2 = x_0$, (i) holds trivially. If $x_2 > x_0$, then by (i) of Lemma 3.4 we may assume that $|\varphi(x_2)| = C_1\varepsilon^{1/4}$ and $|\varphi(x)| \geqq C_1\varepsilon^{1/4}$ for $x_0 < x < x_2$. There are two possibilities:

(I) $\varphi(x) \geqq C_1\varepsilon^{1/4}$ for $x_0 < x < x_2$ and $\varphi(x_2) = C_1\varepsilon^{1/4}$,

(II) $\varphi(x) \leqq -C_1\varepsilon^{1/4}$ for $x_0 < x < x_2$ and $\varphi(x_2) = -C_1\varepsilon^{1/4}$.

In (I), $u''(x) > 0$. Since $\varphi(x_2) = C_1\varepsilon^{1/4}$ and $u(x_2) < -C$,

$$u'(x_2) = \frac{c(x_2)h(u(x_2))}{f'(u(x_2))} + \frac{C_1\varepsilon^{1/4}}{f'(u(x_2))} < C',$$

for some C' independent of ε. Thus $u'(x) < C'$ for $x_0 < x < x_2$. From Proposition 3.5, $|x_2 - x_0| = O(1)\varepsilon^{3/8}$ and so we have

$$u(x_2) - u(x_0) = \int_{x_0}^{x_2} u'(x)\,dx = O(1)\varepsilon^{3/8}.$$

Case (II) corresponds to the case $\varepsilon u''(x) \leqq -C_1\varepsilon^{1/4}$ for $x_0 < x < x_2$ in (i). This proves (i).

(ii) That $x_2 = x_0$ follows from (ii) of Lemma 3.4. If $x_3 < x_1$, then by (ii) of Lemma 3.4 we may assume that $|\varphi(x_3)| = C_1\varepsilon^{1/4}$ and $|\varphi(x)| \geqq C_1\varepsilon^{1/4}$ for $x_3 < x < x_1$ so long as $f'(u(x)) > \varepsilon^{1/4}$. When $\varphi(x_3) = C_1\varepsilon^{1/4}$ we can show, just as in Case I above, that $|u(x) - u(x_2)| \leqq D\varepsilon^{3/8}$ for $x_3 \leqq x \leqq x_1$. Consider next the case $\varphi(x_3) = -C_1\varepsilon^{1/4}$ and $\varphi(x) \leqq -C_1\varepsilon^{1/4}$ for $x_3 \leqq x \leqq x_2$ and $f'(u(x)) \geqq \varepsilon^{1/4}$. In the case $f'(u(x)) \geqq \varepsilon^{1/4}$ for $x_3 \leqq x \leqq x_1$, we see by integrating (1.1) that

$$\varepsilon u'(x_1) - \varepsilon u'(x_3) = f(u(x_1)) - f(u(x_3)) + \int_{x_3}^{x_1} c(x)h(u(x))\,dx.$$

From $\varphi(x_3) = -C_1\varepsilon^{1/4}$ and $f'(u(x_3)) \geqq \varepsilon^{1/4}$ we have

$$u'(x_3) = \frac{-C_1\varepsilon^{1/4} + c(x_3)h(u(x_3))}{f'(u(x_3))} = O(1)\varepsilon^{-1/4}.$$

Thus in view of the above estimates and Proposition 3.5 we conclude that

$$\begin{aligned} f(u(x_3)) - f(u(x_1)) &= \int_{x_3}^{x_1} c(x)h(u(x))\,dx - \varepsilon u'(x_1) + \varepsilon u'(x_3) \\ &> \int_{x_3}^{x_1} c(x)h(u(x))\,dx + \varepsilon u'(x_3) \\ &= O(1)|x_1 - x_3| + O(1)\varepsilon^{3/4} \\ &= O(1)\varepsilon^{3/8}. \end{aligned}$$

This completes the proof of Lemma 3.6.

LEMMA 3.7. *Suppose that $u(x)$ is a solution of* (1.1), (1.2) *and has a minimum at $x = x_0$ not near the boundary $x = 0$ or $x = 1$, i.e., $C\varepsilon^{3/8} < x_0 < 1 - C\varepsilon^{3/8}$ for sufficiently large positive constant C. Then there exists $x_1, x_2, x_1 < x_0 < x_2$, such that $\varphi(x_1) = -C_1\varepsilon^{1/4}$, $\varphi(x_2) = C_1\varepsilon^{1/4}$, $|x_2 - x_1| \leqq D\varepsilon^{3/8}$, $u(x_1) > 0 > u(x_2)$ and $|f(u(x_2)) - f(u(x_1))| \leqq D\varepsilon^{3/8}$ for some positive constant D.*

Proof: Since $u'(x) > 0$ for $x_0 < x \leqq 1$, and $\varphi(x_0) = -c(x_0)h(u(x_0)) > 0$ and $u''(x_0) = \varepsilon^{-1}\varphi(x_0)$, it follows from (i) of Lemma 3.6 that there exists x_2, $x_2 > x_0$, with $\varphi(x_2) = C_1\varepsilon^{1/4}$ and $|u(x_0) - u(x_2)| + |x_2 - x_0| = O(1)\varepsilon^{3/8}$. Moreover, $u(x_2) < -C$ for some positive C independent of ε. Similarly, from (ii) of Lemma 3.6, there exists x_1 with $x_1 < x_0$ and $\varphi(x_1) = -C_1\varepsilon^{1/4}$, $u(x_1) > C$ and $|x_1 - x_0| = O(1)\varepsilon^{3/8}$. It remains to verify the last estimate in the lemma. Integrate (1.1) from x_1 to x_0 to obtain

$$\varepsilon u'(x_2) - \varepsilon u'(x_1) = f(u(x_2)) - f(u(x_1)) - \int_{x_1}^{x_2} c(x)h(u(x))\,dx.$$

Since $ch = O(1)$, $|x_1 - x_2| = O(1)\varepsilon^{3/8}$ and

$$u'(x) = \frac{\varphi(x) + c(x)h(u(x))}{f'(u(x))},$$

which is bounded at $x = x_1, x_2$ because $|u(x)| > C$ there, we have

$$f(u(x_2)) - f(u(x_1)) = O(1)\varepsilon + O(1)(x_2 - x_1) = O(1)\varepsilon^{3/8}.$$

This proves the lemma.

THEOREM 3.8. *Let $u(x) = u(x, \varepsilon)$ be a solution of* (1.1), (1.2). *Then for ε sufficiently small, $u(x, \varepsilon)$ is close to an inviscid time-asymptotic solution of* (2.1) *with the same boundary data u_l and u_r. More precisely, there exists an inviscid solution $u(x,0)$ with boundary data u_l and u_r such that*

(i) *if $u(x,0)$ has a boundary layer (u_l, u_1) at $x = 0$, then there exists $0 < x_1 = O(1)\varepsilon^{3/8}$ such that $u(x_1) - u_1 = O(1)\varepsilon^{3/8}$ and $u(x)$ is monotone over $(0,x_1)$;*

(ii) *if $u(x,0)$ has a standing shock wave (u_2, u_3) at $x = x_0$, then there exist $x_2, x_3, 0 < x_2 < x_3 < 1$, such that $|u(x_2) - u_2| + |u(x_3) - u_3| = O(1)\varepsilon^{3/8}$, $|x_3 - x_2| = O(1)\varepsilon^{3/8}$ and $u'(x) < 0$ for $x_2 < x < x_3$;*

(iii) *if $u(x,0)$ has a boundary layer (u_4, u_r) at $x = 1$, then there exists $x_4, 1 - O(1)\varepsilon^{3/8} < x_4 < 1$, such that $u(x_4) - u_4 = O(1)\varepsilon^{3/8}$ and $u(x)$ is monotone over $(x_4, 1)$,*

(iv) *outside all possible layers of the types* (i)–(iii),

$$|f(u(x))' - c(x)h(u(x))| = O(1)\varepsilon^{1/4} \quad \text{and} \quad |u(x) - u(x,0)| = O(1)\varepsilon^{3/8}.$$

Proof: From Proposition 3.5 we see that a monotone solution $(u(x)$ of (1.1), (1.2) is close to being inviscid except for possible boundary layers. Lemma 3.6 says that the boundary layers almost satisfy the inviscid boundary conditions (2.4) and (2.5). Consequently, we can find $\bar{u}_l, \bar{u}_1, \bar{u}_4$ and $\bar{u}_r$ with the property that $(\bar{u}_l, \bar{u}_1)$ is an admissible inviscid boundary layer at $x = 0$, $(\bar{u}_1, \bar{u}_4)$ an inviscid stationary wave, $(\bar{u}_4, \bar{u}_r)$ an admissible inviscid boundary layer at $x = 1$, and $|u_l - \bar{u}_l| + |\bar{u}_1 - u(x_1)| + |\bar{u}_4 - u(x_4)| + |u_r - \bar{u}_r| = O(1)\varepsilon^{3/8}$, for some x_1, x_4 with the prescribed properties in (i), (iii), (iv) of the theorem. We denote by $v(x)$ the inviscid time-asymptotic state with boundary values $\bar{u}_l$ and $\bar{u}_r$. From the inviscid theory of Section 2, there are at most three solutions to the inviscid problem with given end states. Moreover, the inviscid stationary solutions and the layers depend smoothly on its boundary values. Since $|u_l - \bar{u}_l| + |u_r - \bar{u}_r| = O(1)\varepsilon^{3/8}$, we conclude that there exists an inviscid solution $(u(x,0)$ with boundary values u_l and u_r such that $u(x,0)$ is close to $u(x)$ in the sense of (i)–(iv).

For a nonmonotone solution $u(x)$, it follows from Lemma 3.1 that $u(x)$ has a unique minimum at $x = x_0$. There are two cases: (I) x_0 is close to $x = 0$ or $x = 1$; (II) $|x_0| > C\varepsilon^{3/8}$ and $|x_0 - 1| > C\varepsilon^{3/8}$ for some large positive constant C. In the case where x_0 is close to $x = 0$, we have $u'(x) < 0$ for $0 \leqq x \leqq x_0$, $u'(x_0) = 0$ and so, by integrating (1.1) from 0 to x_3,

$$\varepsilon u'(x_0) - \varepsilon u'(0) = f(u(x_0)) - f(u(0)) + \int_0^{x_0} c(x)h(u(x))\,dx,$$

$$-f(u(x_0)) + f(u(0)) < \int_0^{x_0} c(x)h(u(x))\,dx = O(1)x_0 = O(1)\varepsilon^{3/8}.$$

Thus $u(x)$ has a boundary layer near $x = 0$ which satisfies the inviscid boundary condition (2.4) except for a small error $O(1)\varepsilon^{3/8}$. Similarly, if x_0 is close to 1, then $u(x)$ has an admissible boundary layer at $x = 1$. Thus the above arguments for monotone solutions can be applied to $u(x)$ outside the layers and Case (I) is treated accordingly. For Case (II) we apply Lemma 3.7 to locate x_2 and x_3 close to x_0, $x_2 < x_0 < x_3$, so that $u(x)$ has an interior layer in (x_2, x_3). The solution $u(x)$ is monotone over $(0, x_2)$ and over $(x_3, 1)$ and so the above arguments for monotone solutions apply again. This completes the proof of the theorem.

Remark 3.9. The above theorem does not yield an optimal thickness of the layers and the distance between viscous and inviscid solutions outside the layers. Suppose, for instance, that $u(x)$ has a boundary layer at $x = 0$ and that $u_l > 0$. From Theorem 3.8, it is a shock layer, $u'(x) < 0$ for $0 \leqq x < x_0$, $u'(x_0) = 0$, $u(x_0) < 0$ for some $x_0 > 0$. Moreover, for ε sufficiently small, $f(u_l) < f(u(x_0))$. From (1.1) we have

$$u''(x) = \frac{1}{\varepsilon}\big(f'(u(x))u'(x) - c(x)h(u(x))\big),$$

$$\varepsilon u'(x) = f(u(x)) - f(u(x_0)) + \int_x^{x_0} c(y)h(u(y))\,dy, \quad 0 \leqq x \leqq x_0.$$

From the first identity and (1.5), $ch < 0$, we see that $u''(x) > 0$ for $\bar{x} \leqq x \leqq x_0$, $u(\bar{x}) = 0$. Choose x_1, $\bar{x} < x_1 < x_0$, with

$$f(u(x_0)) - f(u(x_1)) = \int_{x_1}^{x_0} c(y)h(u(y))\,dy,$$

or

$$|u(x_0) - u(x_1)| = O(1)|x_0 - x_1|,$$

for some positive bounded $O(1)$. Since $u'(x_0) = 0$, we have

$$|f(u(x)) - f(u(x_0))| < \int_x^{x_0} c(y)h(u(y))\,dy \quad \text{for} \quad x_1 < x < x_0,$$

and so

$$\varepsilon u'(x) = O(1)(x - x_0),$$

for some bounded and positive $O(1)$ and $x_1 < x < x_0$. Integrate to yield

$$|u(x_0) - u(x_1)| = \frac{O(1)}{\varepsilon}|x_0 - x_1|^2.$$

Thus we have

$$\frac{O(1)}{\varepsilon}|x_0 - x_1|^2 = O(1)|x_0 - x_1|,$$

or

$$|x_0 - x_1| = O(1)\varepsilon, \qquad |u(x_0) - u(x_1)| = O(1)\varepsilon.$$

Between $\bar{x}$ and x_1, we have

$$|f(u(x)) - f(u(x_0))| > \int_x^{x_0} c(y)h(u(y))\,dy,$$

so

$$\varepsilon u'(x) = O(1)(f(u(x)) - f(u(x_0)), \qquad \bar{x} < x < x_1.$$

Integrate it from $\bar{x}$ to x_1:

$$\varepsilon\int_{u(x_1)}^{0} \frac{du}{f(u(x_0)) - f(u)} = O(1)(x_1 - \bar{x}),$$

or

$$O(1)\varepsilon\log\varepsilon = O(1)(x_1 - \bar{x}).$$

Thus we conclude that $x_0 - \bar{x} = O(1)\varepsilon\log\varepsilon$, a similar estimate holding for $\bar{x} = 0$. Thus the thickness of the boundary layer is of the order $\varepsilon\log\varepsilon$. The same holds for other layers as well. Details are omitted.

4. Local Uniqueness and Bifurcation

We want to establish a local uniqueness theorem which, when combined with the a priori estimates of the last section, determines the exact number of solutions of (1.1), (1.2). For this we employ the shooting method. Let $u(x) = u(x, \varepsilon, \beta)$ be a solution of (1.1), with initial slope β:

(4.1) $$u(0) = u_l, \qquad u'(0) = \beta.$$

The following crucial lemmas establish the dependence of $u(x)$, $x > 0$, on the initial slope β. Set

(4.2) $$w(x) \equiv \frac{\partial u(x)}{\partial \beta}.$$

It follows from differentiating (1.1) and using (4.1) that

$$\varepsilon w'' - f'(u)w' + mw = 0, \quad m \equiv m(x) \equiv c(x)h'(u(x)) - f''(u(x))u'(x), \tag{4.3}$$

$$w(0) = 0, \qquad w'(0) = 1. \tag{4.4}$$

It is often convenient to rewrite (4.3) as

$$\varepsilon w'' - (f'(u)w)' + c(x)h'(u)w = 0. \tag{4.5}$$

The first lemma deals with the easy stability case, (1.6).

LEMMA 4.1. *Suppose that* $c(x)h'(u) < 0$. *Then* $w(x) > 0$ *for all* $0 < x < 1$.

Proof: Integrate (4.5) repeatedly to yield, for $1 \geq x > x' \geq 0$,

$$\varepsilon w'(x) = \varepsilon w'(x') + f'(u(x))w(x) - f'(u(x'))w(x') - \int_{x'}^{x} ch'w(\tau)\,d\tau,$$

$$\begin{aligned} w(x) = {} & w(x')\exp\left\{\int_{x'}^{x}\frac{f'(u)(\xi)}{\varepsilon}\,d\xi\right\} \\ & + \left[w'(x') - \frac{f'(u(x'))}{\varepsilon}w(x')\right]\int_{x'}^{x}\exp\left\{-\int_{x}^{y}\frac{f'(y(\xi))}{\varepsilon}\,d\xi\right\}dy \\ & - \frac{1}{\varepsilon}\int_{x'}^{x}dy\int_{x'}^{y}c(\tau)h'(u(\tau))w(\tau)\exp\left\{-\int_{x}^{y}\frac{f'(\xi)}{\varepsilon}\,d\xi\right\}d\tau. \end{aligned}$$

Setting $x' = 0$, it follows since, by hypothesis, $ch' < 0$ that $w(x) > 0$ so long as $w(y) > 0$ for $0 < y < x$. By (4.4) it is clear that $w(y) > 0$ for y close to 0. This proves the lemma.

THEOREM 4.2. *Suppose that* $c(x)h'(u) < 0$. *Then* (1.1), (1.2) *have a unique solution which tends to the corresponding inviscid solution as* $\varepsilon \to 0_+$.

Proof: Uniqueness follows from Lemma 4.1 or by the maximum principle. The existence of solutions is established by the shooting method. We defer this until later when we deal with the issue for the unstable case in Theorem 4.7.

For the remainder of this section we treat the more interesting and much harder instability case $c(x)h'(u) > 0$. For this we need to look at the boundary layers and interior layers separately. The first lemma on a single boundary layer refines the classical result of Coddington and Levinson [2]. Throughout, we assume that ε is small.

LEMMA 4.3. *Suppose that $u(x)$ is a solution of (1.1), (1.2) and has only a boundary layer at $x = 0$. That is, there exists $x_0 = O(1)\varepsilon$ such that (2.4) holds with $u_0 = u(x_0)$ and $u'(x) = O(1)$ independent of ε for $x > x_0$. Then the solution $w(x)$ of (4.3), (4.4) satisfies $w(x) > 0$ for $1 \geqq x > 0$ and $w'(x) > -\lambda\varepsilon^{-1}w(x)$ for $x > D\varepsilon^{1/2}$ and a constant $\lambda = C\varepsilon^{1/2}$, where C and D are positive constants independent of ε.*

Proof: We first show that, for some $x_1 = O(1)\varepsilon^{1/2}$,

$$(4.6)\qquad \begin{aligned} &w(x_1) > C\varepsilon, \quad w'(x_1) = O(1)\varepsilon^{1/2}, \\ &u'(x) = O(1), \qquad x_1 \leqq x \leqq 1. \end{aligned}$$

The latter estimate follows from the results of the last section, Remark 3.9, since x_1 lies outside the layer. Our proof for (4.6) is carried out along the same line as similar arguments in Coddington-Levinson [2], except for some refinement due to the boundary layer. We note that in [2] the authors dealt with only the subsonic case and we need to treat the transonic case. Integrate (4.5) from 0 to x to yield

$$(4.7)_1 \qquad \varepsilon w'(x) - f'(u)w(x) = \varepsilon - \int_0^x c(\tau)h'(u(\tau))w(\tau)\,d\tau.$$

Another integration results in an expression for $w(x)$:

$$(4.7)_2 \quad w(x) = \int_0^x E(x,s)\,ds - \frac{1}{\varepsilon}\int_0^x c(r)h'(u(r))w(r)\int_r^x E(x,s)\,ds dr,$$

where

$$E(x,s) = \exp\left\{\frac{1}{\varepsilon}\int_s^x f'(u(\tau))\,d\tau\right\}.$$

Let $\xi = O(1)\varepsilon^{1/2}$ be such that $f'(u(x)) \leqq -k < 0$ for $\xi \leqq x \leqq 1$, for some $k > 0$ and $N = \max\{|f'(u(x))|: 0 \leqq x \leqq \xi\}$. Set $x_1 = \xi + L\varepsilon^{1/2}$, where $L = 1 + O(1)Nk^{-1}$. Then, for $0 \leqq s \leqq \xi$,

$$E(\xi, s) \leqq \exp\left\{\frac{N}{\varepsilon}(\xi - s)\right\} \leqq \exp\{MO(1)\varepsilon^{-1/2}\},$$

$$E(x,\xi) \leqq \exp\left\{\frac{-k}{\varepsilon}(x-\xi)\right\} \leqq \exp\{-kL\varepsilon^{-1/2}\},$$

$$E(x,s) \leqq \exp\{(-kL + NO(1))\varepsilon^{1/2}\} \leqq \exp\{-k\varepsilon^{-1/2}\}.$$

Thus, for $x \geqq x_1$,

$$\int_0^x E(x,s)\,ds = \int_0^\xi E(x,s)\,ds + \int_\xi^x E(x,s)\,ds$$

$$\leqq \int_0^\xi \exp\{-k\varepsilon^{-1/2}\}\,ds + \int_\xi^x \exp\left\{-\frac{k}{\varepsilon}(x-s)\right\}ds$$

$$\leqq \frac{2\varepsilon}{k}.$$

Similarly, the second term in the right-hand side of $(4.7)_2$ can be written as the sum of I_1 and I_2, where

$$I_1 = -\frac{1}{\varepsilon}\int_0^\xi c(r)h'(u(r))w(r)\left[\int_r^\xi E(x,s)\,ds + \int_\xi^x E(x,s)\,ds\right]dr,$$

$$I_2 = -\frac{1}{\varepsilon}\int_\xi^x c(r)h'(u(r))w(r)\int_r^x E(x,s)\,ds\,dr,$$

$$|I_1| \leqq \frac{n}{\varepsilon}\int_0^\xi |w(s)|\left[\exp\{-k\varepsilon^{-1/2}\xi\} + \frac{\varepsilon}{k}\right]dr \leqq \frac{2n}{k}\int_0^\xi |w(s)|\,ds,$$

$$|I_2| \leqq \frac{2n}{k}\int_\xi^x |w(s)|\,ds, \quad \text{where} \quad |c(x)h'(u(x))| \leqq n, \qquad 0 \leqq x \leqq 1.$$

Combining these estimates we obtain

$$|w(x)| \leqq \frac{2\varepsilon}{k} + \frac{2n}{k}\int_0^x |w(s)|\,ds$$

and this in turn implies

$$|w(x)| \leqq \frac{2\varepsilon}{k}\exp\left\{\frac{2nx}{k}\right\}$$

and

$$|I_1| + |I_2| \leqq \frac{2\varepsilon}{k}\left(\exp\left\{\frac{2xn}{k}\right\} - 1\right) \leqq \frac{2x\varepsilon}{k}\exp\left\{\frac{2n}{k}\right\}.$$

Thus

$$w(x) = \int_0^x E(x,s)\,ds + xO(\varepsilon).$$

As in [2], we may show that by taking care of the boundary layer,

$$w(x) = \frac{\varepsilon}{-f'(u(x))}[1 - E(x, \xi)] - \frac{O(\varepsilon^2)}{f'(u(x))} - \frac{xO(\varepsilon)}{f'(u(x))}$$

for $x \geqq x_1$, and by $(4.7)_1$ we have

$$|w'(x)| \leqq 1 + O(\varepsilon) + xO(1) + \left[\exp\left\{\frac{2nx}{k} - 1\right\}\right],$$

or

$$w'(x) = O(x + \varepsilon) \text{ for } x \geqq x_1.$$

This proves (4.6).

Let λ be a small positive constant to be determined later and set

$$w(x) \equiv v(x)\exp\left\{-\frac{\lambda x}{\varepsilon}\right\}.$$

We have

$$w'(x) = \frac{\lambda}{\varepsilon}w(x) + v'(x)\exp\left\{-\frac{\lambda x}{\varepsilon}\right\}.$$

From (4.6) we have

$$v(x_1) > 0, \qquad v'(x_1) > 0,$$

provided that $\lambda \geqq C\varepsilon^{1/2}$ for some large C. By (4.3) and the above it follows that

$$\begin{aligned} &\varepsilon v'' + (-f'(u) + 2\lambda)v' \\ &+ \left[\frac{\lambda^2 - \lambda f'(u)}{\varepsilon} + f''(u)u' + c(x)h'(u)\right]v = 0. \end{aligned} \tag{4.8}$$

In view of (4.7) we know that

$$|f''(u)u' + c(x)h'(u)| = O(1),$$

for $x_1 \leqq x \leqq 1$. By Theorem 3.8, outside the layer, $x > x_1$, $u(x)$ is close to the inviscid stationary solution which is subsonic, $f'(u(x)) < 0$. Thus we can choose λ small such that the bracket in (4.8) is positive. One such choice is $\lambda = C\varepsilon^{1/2}$ for some large C. Thus the lemma is proved by the maximum principle.

Lemma 4.4. *Suppose that $u(x)$ is a solution of (1.1), (1.2) and has only one boundary layer at $x = 0$ or $x = 1$, but not both, and has no interior layer. Then the* *solution* $w(x)$ *of* (4.3) *with either* $w(0) = 0, w'(0) = 1$ *or* $w(1) = 0, w'(1) = 1$ *satisfies either* $w(1) > 0, w'(1) > -\lambda\varepsilon^{-1}w(1)$ *or* $w(0) < 0, w'(0) < \lambda\varepsilon^{-1}w(0)$ *for some* $\lambda = C\varepsilon^{1/2}$.

Proof: Suppose that $w(0) = 0, w'(0) = 1$. The lemma is reduced to the previous one if the boundary layer is located at $x = 0$. Suppose that the boundary layer is located at $x = 1$. Then by the same argument as in the last proof we can show that the solution $\bar{w}(x)$ of (4.3) with $\bar{w}(1) = 0, \bar{w}'(1) = 1$ satisfies $\bar{w}(0) < 0$. Since the Wronskian $w(x)\bar{w}'(x) - \bar{w}(x)w'(x)$ does not change sign and is positive at $x = 0$, we see that $w(1) > 0$. Other parts of the lemma are proved similarly.

LEMMA 4.5. *Suppose that* $u(x)$ *is a solution of* (1.1), (1.2) *with only an interior layer, that is, there exists* x_0 *and* $x^0, 0 < x_0 < x^0 < 1$, *such that* $|u'(x)| = O(1)$ *independent of* ε *for* $x \notin (x_0, x^0)$, $|x^0 - x_0| = O(1)\varepsilon$, $f(u(x^0)) = f(u(x_0)) + O(1)\varepsilon$, $u(x_0) > 0 > u(x^0)$. *Then the solution* $w(x)$ *of* (4.3), (4.4) *satisfies* $w(x) > 0$ *for* $x < x_1$, $w(x_1) = 0$, $w(x) > 0$ *for* $x > x_1$ *for some* x_1 *not in the interior layer,* $x_1 > x_0$ *with* $|x_1 - x_0| = O(1)\varepsilon|\log\varepsilon|$. *Moreover,* $w'(x_1) < 0$, *and for some constants* $C, D > 0$, $w'(x) < \lambda\varepsilon^{-1}w(x)$ *or* $x > x_1 + D\varepsilon^{1/2}$ *and* $\lambda = C\varepsilon^{1/2}$.

Proof: The proof consists of three steps investigating the behavior of $w(x)$ before the layer, in the layer, and after the layer.

Step 1. Let λ be a positive constant to be determined later and set

$$w(x) = v(x)\exp\left\{\frac{\lambda x}{\varepsilon}\right\}. \tag{4.9}$$

We have from (4.3), (4.4) that

$$w'(x) = \frac{\lambda}{\varepsilon}w(x) + v'(x)\exp\left\{\frac{\lambda x}{\varepsilon}\right\}, \tag{4.10}$$

$$\begin{gathered}\varepsilon v'' + (2\lambda - f'(u))v' + \left[\frac{\lambda^2}{\varepsilon} - f'(u)\frac{\lambda}{\varepsilon} + m\right]v = 0,\\ v(0) = 0, \qquad v'(0) = 1.\end{gathered} \tag{4.11}$$

From Theorem 3.8 we know that $u(x)$ is close to a supersonic inviscid stationary wave for $0 < x \leqq x_0$. Thus we may choose a small λ such that $(\lambda^2 - f'(u(x))\lambda)\varepsilon^{-1} \ll -1$ for $0 < x \leqq x_0$. Since $m(x)$ is bounded over $0 \leqq x \leqq x_0$,

$$\frac{\lambda^2}{\varepsilon} - f'(u)\frac{\lambda}{\varepsilon} + m \ll -1, \qquad 0 \leqq x \leqq x_0.$$

Thus, by the maximum principle, applied to (4.11), (4.9)–(4.11) imply that

$v(x) > 0$, $v'(x) > 0$ and

$$w' > \frac{\lambda}{\varepsilon} w \quad \text{for} \quad 0 < x < x_0.$$

This and (4.3) yield

$$\varepsilon w'' = f'(u)w' - mw > \left[\frac{\lambda f'(u)}{2\varepsilon} - m\right]w + \frac{1}{2}f'(u)w',$$

and so, for small ε,

$$\varepsilon w'' > \frac{1}{2}f'(u)w' \quad \text{for} \quad 0 \leqq x \leqq x_0.$$

This and the initial data (4.4) for w yield

$$(4.12) \qquad \begin{aligned} w'(x) &> \exp\left\{\frac{K_1 x}{\varepsilon}\right\}, \\ w(x) &> \frac{\varepsilon}{K_1}\left(\exp\left\{\frac{K_1 x}{\varepsilon}\right\} - 1\right), \qquad 0 \leqq x \leqq x_0, \end{aligned}$$

for some $K_1 > 0$.

Step 2. Define x^* by $u(x^*) = 0$, $x_0 < x^* < x^0$. Integrate (4.5) from 0 to x^*, to obtain

$$\varepsilon w'(x^*) = \varepsilon - \int_0^{x^*} c(x)h'(u(x))w(x)\,dx.$$

Since x^* is within the layer we have from Lemma 4.4 that $w(x) > 0$ for $0 < x \leqq x^*$. Moreover, we have the estimate (4.12). Thus the above yields $w'(x^*) < 0$ and we conclude that

$$(4.13) \qquad \begin{aligned} w'(\hat{x}) &= 0 \quad \text{for some} \quad \hat{x}, \qquad x_0 < \hat{x} < x^*, \\ w'(x) &< 0 \quad \text{for} \quad \hat{x} < x \leqq x^*. \end{aligned}$$

Step 3. We want to show that there exists $x_1 > \hat{x}$, $|x_1 - \hat{x}|$ small, such that $w(x_1) = 0$. For the moment we assume that

$$(4.14) \qquad w(x) > 0 \quad \text{for} \quad 0 < x < x_1,$$

for some $x_1 > \hat{x}$. We have either $x_1 = 1$ or $x_1 < 1$ and $w(x_1) = 0$. From Lemma 4.4 we know that x_1 does not lie in the interior layer. Integrate (4.5) from $\hat{x}$ to

x, $\hat{x} < x < x_1$, then

$$\varepsilon w'(x) = f'(u(x))w(x) - f'(u(\hat{x}))w(\hat{x}) - \int_{\hat{x}}^{x} c(y)h'(u(y))w(y)\,dy. \tag{4.15}$$

From (4.14), (4.15) and $u(\hat{x}) > 0$ we obtain

$$\varepsilon w'(x) < f'(u(x))w(x) \quad \text{for} \quad \hat{x} < x < x_1. \tag{4.16}$$

Then (4.13), (4.16) yield

$$w'(x) < 0 \quad \text{for} \quad \hat{x} < x < x_1. \tag{4.17}$$

Let $\bar{x} = \hat{x} + D\varepsilon^{1/4}$ for some large positive constant D. Then $\bar{x}$ is outside the layer and $\bar{x} > \tilde{x}$, where $\tilde{x}$ is characterized by $u'(\tilde{x}) = 0$. Suppose (4.14) holds for $x_1 > \bar{x} + D_1\varepsilon|\log \varepsilon|$ for some large positive constant D_1. Integrate (4.3) from $\bar{x}$ to x:

$$\begin{aligned} w'(x)E(x) &= w'(\bar{x}) - \frac{1}{\varepsilon}\int_{\bar{x}}^{x} m(s)E(s)w(s)\,ds, \\ E(x) &= \exp\left\{-\frac{1}{\varepsilon}\int_{\bar{x}}^{x} f'(u(y))\,dy\right\}, \\ m(x) &= c(x)h'(u(x)) - f''(u(x))u'(x). \end{aligned} \tag{4.18}$$

Divide (4.18) by $E(x)$ and integrate to obtain

$$w(x) = w(\bar{x}) + w'(\bar{x})\int_{\bar{x}}^{x} E(y)^{-1}\,dy - \frac{1}{\varepsilon}\int_{\bar{x}}^{x} E(y)^{-1}\,dy\int_{\bar{x}}^{y} m(s)E(s)w(s)\,ds. \tag{4.19}$$

From (4.14), (4.15), (4.17) we obtain

$$w'(\bar{x}) < \frac{1}{\varepsilon}(f'(u(\bar{x})) - O(1)(\bar{x} - \hat{x}))w(\bar{x}).$$

Thus (4.19) and the above yield

$$\begin{aligned} w(x) < w(\bar{x})\left[1 + \frac{f'(u(\bar{x})) - O(1)(\bar{x} - \hat{x})}{\varepsilon}\int_{\bar{x}}^{x} E(y)^{-1}\,dy\right] \\ - \frac{1}{\varepsilon}\int_{\bar{x}}^{x} E(y)^{-1}\,dy\int_{\bar{x}}^{y} m(s)E(s)w(s)\,ds, \end{aligned} \tag{4.20}$$

for $\bar{x} \leqq x \leqq x_1$. Integrate (4.16) to obtain

$$w(s) \leqq w(\bar{x})\exp\left\{\int_{\bar{x}}^{s} \frac{f'(u(\tau))}{\varepsilon} d\tau\right\} \quad \text{for} \quad \bar{x} \leqq s \leqq x_1.$$

Since $m(s)$ is bounded, the above estimate yields

$$|m(s)E(s)w(s)| \leqq Cw(\bar{x}) \quad \text{for} \quad \bar{x} \leqq x \leqq x_1,$$

for some C independent of ε. This and (4.20) imply

$$w(x) < w(\bar{x})\left[1 + \frac{f'(u(\bar{x})) - O(1)(\bar{x} - \hat{x})}{\varepsilon} \int_{\bar{x}}^{x} E(y)^{-1}\, dy + \frac{O(1)}{\varepsilon} \int_{\bar{x}}^{x} E(y)^{-1}(y - \bar{x})\, dy\right], \tag{4.21}$$

Take $x = \bar{x} + 2\varepsilon|\log \varepsilon|$ in (4.21). Then we obtain

$$\begin{aligned} w(x) &< w(\bar{x})\left[1 + \frac{f'(u(\bar{x}))}{\varepsilon} \int_{\bar{x}}^{x} E(y)^{-1}\, dy + \frac{1}{\varepsilon}\left(\int_{\bar{x}}^{x} E(y)^{-1}\, dy\right)\right. \\ &\qquad \left. \times (O(1)2\varepsilon|\log \varepsilon| - O(1)\varepsilon^{1/4}\right] \\ &< w(\bar{x})\left[1 + \frac{f'(u(\bar{x}))}{\varepsilon} \int_{\bar{x}}^{x} E(y)^{-1}\, dy\right], \end{aligned} \tag{4.22}$$

for $\varepsilon > 0$ sufficiently small. It remains to show that the above bracket is negative and thus contradicts (4.14). For this we expand $E(y)^{-1}$ by Taylor expansion, taking note of the fact that $u'(\hat{x}) = 0$, $\hat{x}$ is in the interior layer, and $\bar{x} > \hat{x}$ is outside the layer,

$$f'(u(x)) = f'(u(\bar{x})) + (x - \bar{x})u'(\zeta)f''(u(\zeta)), \qquad \bar{x} < \zeta < x. \tag{4.23}$$

Since $|\varphi(x)| \leq C_1\varepsilon^{1/4}$ for $x \geq x^0$,

$$u'(x) = \frac{-C(x)h(u(x)) - \varphi(x)}{-f'(u(x))} \geqq C_2,$$

for some positive C_2 and $x > \bar{x}$. Thus we have from (4.23) and (1.3), $f'' > 0$, that

$$f'(u(x)) > f'(u(\bar{x})) + \bar{C}(x - \bar{x}), \qquad \bar{x} < x < x_1,$$

for some $\bar{C}$ independent of ε. Consequently,

$$E(y)^{-1} = \exp\left\{\frac{1}{\varepsilon}\int_{\bar{x}}^{y} f'(u(x))\,dx\right\}$$

$$(4.24) \qquad > \exp\left\{\frac{1}{\varepsilon}\int_{\bar{x}}^{y} f'(u(\bar{x})) + \bar{C}(x-\bar{x}))\,dx\right\}$$

$$= \exp\left\{-\frac{1}{\varepsilon}|f'(u(x))|(y-\bar{x})\right\}\exp\left\{\frac{C}{2\varepsilon}(y-\bar{x})^2\right\}, \quad \bar{x} < y < x_1.$$

Let $M = |f'(u(\bar{x}))|$. Then

$$\bar{M}\int_{\bar{x}}^{x} E(y)^{-1}\,dy > \bar{M}\int_{\bar{x}}^{x}\exp\left\{-\frac{\bar{M}}{\varepsilon}(y-\bar{x})\right\}\exp\left\{\frac{C}{2\varepsilon}(y-\bar{x})^2\right\}dy$$

$$> \bar{M}\int_{\bar{x}}^{x}\exp\left\{-\frac{\bar{M}}{\varepsilon}(y-\bar{x})\right\}\left(1+\frac{\bar{C}}{2\varepsilon}(y-\bar{x})^2\right)dy$$

$$= \varepsilon\left[1-\exp\left\{-\frac{\bar{M}}{\varepsilon}(x-\bar{x})\right\}\right] + \frac{\bar{C}\varepsilon^2}{2\bar{M}^2}\int_0^{\bar{M}(x-\bar{x})/\varepsilon} z^2 e^{-z}\,dz > \varepsilon.$$

Thus the bracket in (4.22) is negative.

To finish the proof we observe from (4.15) and $w(x_1) = 0$ that $w'(x_1) < 0$. Since x_1 is on the right side of the interior layer, we may apply Lemma 4.3 to $-w(x)/w'(x_1)$ to show that $w(x) < 0$ for $x > x_1$.

LEMMA 4.6. *Suppose that $u(x)$ is a solution of (1.1) which either has an interior layer not located within $O(1)\varepsilon$ of $x = 0$ or $x = 1$, or has no interior layer. In the former case, the corresponding solution $w(x)$ of (4.3), (4.4) has the property that $w(1) < 0$, and in the latter case, $w(1) > 0$.*

Proof: Using Lemmas 4.3–4.5 and Theorem 3.8, it remains to treat the case where $u(x)$ has either an interior layer and a rarefaction layer at $x = 1$, cf. (ii) of Case E1 in Section 2, or a boundary layer at $x = 0$ and a rarefaction layer at $x = 1$, cf. (iii) of Case E1 and Case D1 in Section 2. In all these cases the solution is subsonic before the rarefaction layer, which connects the sonic state zero to the positive state u_r around $x = 1$. We treat the case without interior layer; the one with the interior layer is similar. From Lemma 4.3 we have, before the rarefaction layer,

$$w(x) > 0, \qquad w(x_2) > w(x_1)\exp\left\{-\frac{\lambda}{\varepsilon}(x_2 - x_1)\right\}, \qquad x_2 > x_1,$$

$$w'(x) > -\lambda\varepsilon^{-1}w(x), \tag{4.25}$$

$$\lambda = C\varepsilon^{1/2} \quad \text{for} \quad x, x_1, x_2 \quad \text{in} \quad \left(\frac{1}{2}, 1 - \delta\right),$$

for some positive constant C and x, x_1, x_2 outside the layers, $0 < \delta < \frac{1}{2}$. By Theorem 3.8, at x_0 the solution $u(x)$ is close to the inviscid stationary solution $v(x)$, which is sonic at $x = 1$. The equation (2.2) which $v(x)$ satisfies is singular at sonic. Thus $v'(x)$ is large for x close to 1. Consequently we may choose $x_0 = 1 - 2\delta$, so that, for δ small, $u'(x_0)$ is large and thus

$$m(x) = c(x)h'(u(x)) - f''(u(x))u'(x) < -m_0, \quad x_0 < x < 1, \tag{4.26}$$

for some positive constant m_0 independent of ε. For any given $\delta > 0$ we have

$$f'(u(x)) < -C_1, \qquad \tfrac{1}{2} < x < 1 - \delta, \tag{4.27}$$

for some $C_1 > 0$ depending only on δ. Integrate (4.3) to get

$$w'(x)E(x) = w'(x_0) + \int_{x_0}^{x}\frac{1}{\varepsilon}(-m(s))E(s)w(s)\,ds,$$

$$E(x) \equiv \exp\left\{-\int_{x_0}^{x}\frac{f'(u(\xi))}{\varepsilon}\,d\xi\right\}. \tag{4.28}$$

From (4.27) we have

$$E(s) \geqq \exp\left\{\frac{C_1(s - x_0)}{\varepsilon}\right\}, \qquad x_0 < s < 1 - \delta.$$

This, (4.25) and (4.26) yield

$$\int_{x_0}^{1-\delta}\frac{1}{\varepsilon}(-m(s))E(s)w(s)\,ds$$

$$\geqq \frac{m_0 w(x_0)}{\varepsilon}\int_{x_0}^{1-\delta}\exp\left\{-\frac{\lambda}{\varepsilon}(s - x_0)\right\}\exp\left\{\frac{C_1(s - x_0)}{\varepsilon}\right\}ds$$

$$= \frac{m_0 w(x_0)}{\varepsilon}\int_{0}^{\delta}\exp\left\{\frac{(C_1 - \lambda)s}{\varepsilon}\right\}ds$$

$$= \frac{m_0 w(x_0)}{C_1 - \lambda}\left(\exp\left\{\frac{(C_1 - \lambda)\delta}{\varepsilon}\right\} - 1\right).$$

Since C_1 and δ are positive constants independent of ε and $\lambda = C\varepsilon^{1/2}$, the above estimate yields, for small ε,

$$\int_{x_0}^{1-\delta} \frac{1}{\varepsilon}(-m(s))E(s)w(s)\,ds \gg \frac{w(x_0)}{\varepsilon}.$$

This, (4.25) and (4.28) imply

$$w'(1-\delta)E(1-\delta) > -\lambda\varepsilon^{-1}w(x_0) + \int_{x_0}^{1-\delta}\frac{1}{\varepsilon}(-m(s))E(s)w(s)\,ds > 0.$$

Thus we have

$$w'(x) > 0 \quad \text{for} \quad x = 1-\delta. \tag{4.29}$$

This and (4.25), $w(x) > 0$ for $x = 1-\delta$, implies that $w'(x) > 0$ for $1-\delta \leqq x \leqq 1$. Indeed, from (4.3) if $w'(\bar{x}) = 0$ for some $\bar{x}, 1-\delta < \bar{x} \leqq 1$ and $w(\bar{x}) > 0$, then

$$\varepsilon w''(\bar{x}) = -m(\bar{x})w(\bar{x}),$$

which is positive by (4.26), a contradiction. Thus $w'(x) > 0$ for $1-\delta < x \leqq 1$. In particular, $w(1) > w(1-\delta) > 0$. This proves the lemma.

THEOREM 4.7. *Suppose that* $c(x)h'(u) > 0$. *Then, for sufficiently small* ε, (1.1), (1.2) *have one or three solutions which tend to the corresponding inviscid solutions as* $\varepsilon \to 0_+$. *Moreover*, (1.1), (1.2) *have three solutions for small* ε *in Case* E1 *and* (iii) *of Case* E3 *(in Section* 2 *for inviscid classification), and in other cases there exists only one solution.*

Proof: That there exist at most one or three solutions in each respective case follows from the a priori estimate in Section 3 in linking the solutions to inviscid solutions and the local uniqueness theorem as a consequence of Lemmas 4.4–4.6. It remains to show the existence of a solution which is close to any given inviscid solution. This is done by the shooting method. We shall carry out the analysis for the case where there is a shock layer at $x = 0$ and a rarefaction layer at $x = 1$; other cases can be treated by similar arguments. For simplicity, we assume that $\bar{u}_* > u_l > 0$ and $u_r > 0$. The inviscid solution consists of a shock layer (u_l, u_*) at $x = 0$, an inviscid stationary solution $(u_*, 0)$ satisfying $f(u)_x = c(x)h(u)$ and a rarefaction layer $(0, u_r)$ at $x = 1$. We want to find a value $\alpha = \alpha_0$ so that the solution $u(x,\alpha)$ of (1.1), with

$$u(0,\alpha) = u_l, \qquad u'(0,\alpha) = \alpha,$$

satisfies $u(1,\alpha_0) = u_r$. For $|\alpha| \gg 1, \alpha < 0$, i.e., $f'(u_l)\alpha - c(0)h(u_l) \leqq -C_1\varepsilon^{1/4}$, we have from (ii) of Lemma 3.4, that $u(x, \alpha)$ has a boundary layer at $x = 0$. From (1.1), $u'' > 0$ so long as $u < 0$ and $u' < 0$. Thus $u'(x_0, \alpha) = 0$ for some $x_0 > 0$. By Remark 3.9 on the thickness of boundary layer, we have $x_0 = O(1)\varepsilon$. Integrate (1.1) over $0 \leqq x \leqq x_0$ to get

$$-\varepsilon\alpha = f(u(x_0, \alpha)) - f(u_l) - \int_0^{x_0} c(x)h(u(x, \alpha))\,dx,$$

or

$$f(u(x_0, \alpha)) + O(1)\varepsilon \max\{h(u) : u(x_0, \alpha) < u < u_l\} = \varepsilon|\alpha| + f(u_l).$$

Thus from (2.7) we see that, as $\alpha \to -\infty$, $u(x_0, \alpha) \to -\infty$. For the boundary layer to exist at $x = 0$ it is sufficient to have $f'(u_l)\alpha \leqq c(0)h(u_l) - C_1\varepsilon^{1/4}$. For $\alpha = \alpha_1$ satisfying $f'(u_l)\alpha_1 = c(0)h(u_l) - C_1\varepsilon^{1/4}$ the above identity yields

$$\begin{aligned} f(u(x_0, \alpha_1)) &= f(u_l) - \varepsilon\alpha_1 + O(1)\varepsilon \\ &= f(u_l) - \varepsilon(f'(u_l))^{-1}(c(0)h(u_l) - C_1\varepsilon^{1/4}) + O(1)\varepsilon \\ &= f(u_l) + O(1)\varepsilon. \end{aligned}$$

Thus, as $\varepsilon \to 0$, $u(x_0, \alpha_1) \to \bar{u}_l > u_*$, and so for small ε, $u(x_0, \alpha_1) > u_*$. Since $u(x, \alpha)$ is close to being inviscid for $x > x_0$ so long as $u(x, \alpha)$ is not close to the sonic state zero, (i) of Lemma 3.4, it follows that, for small ε, $u(x, \alpha)$ stays subsonic for $u(x_0, \alpha) < u_*$ and becomes sonic at $x = x(\alpha) < 1$ for $u(x_0, \alpha) > u_*$. The latter holds for $\alpha = \alpha_1$. Moreover, once $u(x(\alpha), \alpha) = 0$ for $x(\alpha) < 1$, which is the case for $\alpha = \alpha_1$, $u(x, \alpha)$ has a boundary layer at $x = 1$, Theorem 3.3, and is strictly increasing for $x_0 < x < 1$. Moreover, from Lemma 3.5, if $1 - x(\alpha) > O(1)\varepsilon^{1/2}$, then $u(x, \alpha)$ becomes $+\infty$ before reaching $x = 1$. Since $u(1, \alpha)$ is a strictly increasing function of α for $\alpha < \alpha_1$, Lemma 4.3, and $u(1, \alpha_1) = \infty$, $u(1, \alpha) \to -\infty$ as $\alpha \to -\infty$, we see that there exists an $\alpha_0 < \alpha_1$ with $u(1, \alpha_0) = u_r$. Moreover, since $\alpha_0 < \alpha_1$, $u(x, \alpha_0)$ has a shock layer at $x = 0$. By Theorem 3.8, $u(x, \alpha_0)$ also has a rarefaction layer $x = 1$. We have thus constructed the designated solution of (1.1). The proof of the theorem is complete.

5. Asymptotic Stability and Instability

In this section, we study nonlinear stability and instability of stationary solutions of

$$\text{(5.1)} \qquad \begin{aligned} &u_t + f(u)_x = \varepsilon u_{xx} + c(x)h(u), \\ &u(0, t) = u_l, \quad u(1, t) = u_r, \quad u(x, 0) = u_0(x), \qquad 0 \leqq x \leqq 1. \end{aligned}$$

Let $U(x) = U(x, \varepsilon)$ be a stationary solution,

$$(5.2)\qquad \begin{gathered} f(U)_x = \varepsilon U_{xx} + c(x)h(U), \\ U(0) = u_l,\ U(1) = u_r. \end{gathered}$$

Set $w(x, t) = u(x, t) - U(x)$. From (5.1), (5.2) we have

$$(5.3)\qquad \begin{gathered} w_t + f'(w + U)(w_x + U_x) = \varepsilon w_{xx} + f'(U)U_x - c(x)h(U) + c(x)h(w + U), \\ w(0, t) = 0, \qquad w(1, t) = 0. \end{gathered}$$

The linearized equation is

$$(5.4)\qquad \begin{gathered} w_t + f'(U)w_x + f''(U)U_x w = \varepsilon w_{xx} + c(x)h'(U)w, \\ w(0, t) = 0, \qquad w(1, t) = 0. \end{gathered}$$

We shall use spectral analysis. Set $w(x, t) = e^{\lambda t}q(x)$ and obtain from (5.4)

$$(5.5)\qquad \begin{gathered} \varepsilon q'' - (f'(U)q)' + c(x)h'(U)q = \lambda q, \\ q(0) = 0, \qquad q(1) = 0. \end{gathered}$$

THEOREM 5.1. *Suppose* (1.6) *holds*, $c(x)h'(u) < 0$, *then every steady state* $U(x)$ *of* (5.1) *is asymptotically stable. When* (1.7) *holds*, $c(x)h'(u) > 0$, *then steady states containing no interior layer are asymptotically stable; those containing interior layer are asymptotically unstable.*

Proof: A steady state $U(x)$ is stable if the largest eigenvalue λ is negative. From linear Sturm-Liouville theory (see [1]) the eigenfunction $q(x)$ corresponding to the largest eigenvalue λ is of one sign. We may therefore assume that

$$(5.6)\qquad q(x) > 0,\quad 0 < x < 1,\quad q'(0) = 1,\quad q'(1) < 0.$$

When $c(x)h'(u) < 0$, we have, integrating (5.5), that

$$(5.7)\qquad \varepsilon(q'(1) - q'(0)) + \int_0^1 c(x)h'(U(x))q(x)\,dx = \lambda\int_0^1 q(x)\,dx.$$

From (5.6) and (5.7) it is clear that $\lambda < 0$ and so $U(x)$ is stable.

When $c(x)h'(u) > 0$, we have two cases: (i) $U(x)$ contains no interior layer, (ii) $U(x)$ contains an interior layer. In case (i), for $\lambda = \lambda(\varepsilon)$ is bounded uniformly in ε, then we may rewrite (5.5) as

$$(5.8)\qquad \begin{gathered} \varepsilon q'' - (f'(U)q)' + (c(x)h'(U) - \lambda)q = 0, \\ q(0) = 0, \qquad q'(0) = 1, \end{gathered}$$

which is of the same form as (4.5). Thus it follows from Lemmas 3.3 and 3.4 that $q(1) > 0$, which contradicts (5.5), $q(1) = 0$. In case (i) if $\lambda = \lambda(\varepsilon)$ becomes large $\lambda \gg 1$, then it follows from integrating (5.5) that

$$(q(x)E(x))' = E(x) - \frac{1}{\varepsilon}E(x)\int_0^x (c(x)h'(U) - \lambda)q(\tau)\,d\tau,$$

$$E(x) \equiv \exp\left\{-\frac{1}{\varepsilon}\int_0^x f'(U)(\tau)\right\} d\tau;$$

whence (by $ch'(U) - \lambda < 0$ as $\lambda \gg 1$) we have

$$q(x) > \frac{1}{E(x)}\int_0^x E(\tau)\,d\tau.$$

In particular, $q(1) > 0$, again a contradiction. Thus, in (i), $\lambda < 0$ and $U(x)$ is stable.

Finally we consider the case where $c(x)h'(u) > 0$ and $U(x)$ contains an interior layer. We want to prove by contradiction that $\lambda > 0$. If not, then

$$c(x)h'(U) - \lambda > 0$$

and so from adapting the proof of Lemma 4.5 to (5.8) we have $q(1) < 0$ which contradicts (5.5), $q(1) = 0$. Thus $\lambda > 0$ and $U(x)$ is unstable. This proves the theorem.

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