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STABILITY ANALYSIS FOR A CLASS OF DIFFUSIVE COUPLED SYSTEMS WITH APPLICATIONS TO POPULATION BIOLOGY

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ABSTRACT. In this paper we derive criteria for the local stability of synchronized equilibria or limit cycles for a class of lattice dynamical systems. We first prove a theorem of linear algebra then the criteria are obtained by the application of the theorem. The criteria reduce the stability of a large size problem or equivalently large matrix size eigenvalue problem to a small one. Several examples in population biology are presented to illustrate the usefulness of the criteria.

1. Introduction. In this paper we shall derive criteria for the local stability of certain synchronized solutions for a class of diffusive coupled systems. We restrict our attention to the case where the subsystems are identical and the synchronized solutions are either equilibria or limit cycles. Many important biological models involving spatial effects take the form of diffusive coupled systems, in continuous or discrete time, expressed in one- or two-dimensional lattice systems. Consider the identical subsystems of the following form

$$\frac{dx}{dt} = f(x), \quad x \in \mathbf{R}^k \tag{1.1}$$

where f is C^1, $f : \Omega \subseteq \mathbf{R}^k \to \mathbf{R}^k$, Ω is an open subset of $\mathbf{R}^k$. In this paper we first study the one-dimensional lattice system

$$\frac{dx_i}{dt} = f(x_i) + D(x_{i-1} - 2x_i + x_{i+1}), \quad i = 1, 2, \dots , N \tag{1.2}$$

where $x_i \in \mathbf{R}^k$, $D = \operatorname{diag}(d_1, d_2, \dots , d_k)$, $d_1, \dots , d_k > 0$ satisfying the periodic boundary conditions

$$x_0 = x_N, \qquad x_{N+1} = x_1, \tag{1.3$_P$}$$

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or the Neumann boundary conditions

$$(1.3)_N \qquad x_0 = x_1, \qquad x_{N+1} = x_N.$$

Then we study the two-dimensional lattice systems:

$$(1.4) \qquad \frac{dx_{ij}}{dt} = f(x_{ij}) + D(x_{i,j+1} + x_{i,j-1} + x_{i-1,j} + x_{i+1,j} - 4x_{ij}), \quad 1 \le i,j \le N$$

where $x_{ij} \in \mathbf{R}^k$, $D = \operatorname{diag}(d_1, \dots, d_k)$, $d_i > 0$, $i = 1, \dots, k$, satisfying the periodic boundary conditions

$$(1.5)_P \qquad x_{0j} = x_{Nj}, \quad x_{N+1,j} = x_{1j}, \quad x_{i0} = x_{iN}, \quad x_{i,N+1} = x_{i1} \quad \text{for} \quad 1 \le i,j \le N$$

or the Neumann boundary conditions

$$(1.5)_N \qquad x_{0j} = x_{1j}, \quad x_{N+1,j} = x_{Nj}, \quad x_{i0} = x_{i1}, \quad x_{i,N+1} = x_{iN}, \quad 1 \le i,j \le N.$$

Our method also works for the iterated maps

$$x^{n+1} = f(x^n), \quad x^n \in \mathbf{R}^k$$

with the corresponding discrete one-dimensional lattice systems

$$(1.7) \qquad x_i^{n+1} = f(x_i^n) + D(x_{i-1}^n - 2x_i^n + x_{i+1}^n)$$

and two-dimensional lattice systems

$$(1.8) \qquad x_{ij}^{n+1} = f(x_{ij}^n) + D(x_{i,j+1}^n + x_{i,j-1}^n + x_{i-1,j}^n + x_{i+1,j}^n - 4x_{ij}^n)$$

with periodic or Neumann boundary conditions.

We also discuss the following type of "disperse-reproduce" models [8], [4], [5]

$$(1.9) \qquad \begin{aligned} x_{ij}^{n+1} &= f(\tilde{x}_{ij}^n) \\ \tilde{x}_{ij}^n &= D \sum_{\substack{|r-i|\le 1 \\ |s-j|\le 1 \\ (r,s)\ne(i,j)}} (x_{rs}^n - x_{ij}^n). \end{aligned}$$

For the systems (1.7), (1.8) and (1.9), we only consider the stability of the fixed points.

In Section 2 we first present a theorem of linear algebra. Then we apply it to obtain stability criteria for the systems (1.2), (1.4), (1.7), (1.8) and (1.9) with periodic boundary conditions and Neumann boundary conditions. In Section 3 we present some examples from population biology to demonstrate the usefulness of the criteria.

2. Main results. In the following we state and prove a theorem of linear algebra which will be applied to obtain a necessary and sufficient condition for the local stability of synchronized equilibria (or limit cycles) in various lattice systems introduced in Section 1.

Theorem 2.1. *Let $T = (t_{ij})$ be an $n \times n$ matrix with n linearly independent eigenvectors, $M = (m_{ij})$ and $L = (l_{ij})$ $k \times k$ matrices. Assume the $nk \times nk$ matrix A takes the following form*

$$A = \begin{bmatrix} M - t_{11}L & -t_{12}L & \cdots & -t_{1n}L \\ -t_{21}L & M - t_{22}L & \cdots & -t_{2n}L \\ \vdots & \vdots & \vdots & \vdots \\ -t_{n1}L & -t_{n2}L & \cdots & M - t_{nn}L \end{bmatrix}. \tag{2.1}$$

If spec $(T) = \{\lambda_1, \dots, \lambda_n\}$, *then*

$$\operatorname{spec}(A) = \bigcup_{i=1}^{n} \operatorname{spec}(M - \lambda_i L) \tag{2.2}$$

where spec (E) *denotes the set of eigenvalues of a square matrix* E.

Proof. Let $\{e_j\}_{j=1}^{nk}$ be the standard basis of $\mathbf{R}^{nk}$. Consider the permutation matrix $P \in \mathbf{R}^{nk\times nk}$, $P = [e_1, e_{k+1}, \dots, e_{(n-1)k+1}, \dots, e_k, e_{2k}, \dots, e_{nk}]$. Then, from (2.1), we have

$$B = P^T A P = \begin{bmatrix} m_{11}I_n - l_{11}T & m_{12}I_n - l_{12}T & \cdots & m_{1k}I_n - l_{1k}T \\ m_{21}I_n - l_{21}T & m_{22}I_n - l_{22}T & \cdots & m_{2k}I_n - l_{2k}T \\ \vdots & \vdots & \vdots & \vdots \\ m_{k1}I_n - l_{k1}T & m_{k2}I_n - l_{k2}T & \cdots & m_{kk}I_n - l_{kk}T \end{bmatrix}.$$

Since T has n linearly independent eigenvectors, an invertible matrix $Q \in \mathbf{R}^{n\times n}$ exists such that $Q^{-1}TQ = \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Set $U = \operatorname{diag}(Q, Q, \ldots, Q) \in \mathbf{R}^{nk\times nk}$. Then it follows that

$$C = U^{-1}BU = \begin{bmatrix} m_{11}I_n - l_{11}\Lambda & m_{12}I_n - l_{12}\Lambda & \cdots & m_{1k}I_n - l_{1k}\Lambda \\ m_{21}I_n - l_{21}\Lambda & m_{22}I_n - l_{22}\Lambda & \cdots & m_{2k}I_n - l_{2k}\Lambda \\ \vdots & \vdots & \vdots & \vdots \\ m_{k1}I_n - l_{k1}\Lambda & m_{k2}I_n - l_{k2}\Lambda & \cdots & m_{kk}I_n - l_{kk}\Lambda \end{bmatrix}$$

and it is easy to verify that

$$PCP^T = \begin{bmatrix} M - \lambda_1 L & & & \\ & M - \lambda_2 L & & \\ & & \ddots & \\ & & & M - \lambda_n L \end{bmatrix}.$$

Hence the matrix A is similar to $\operatorname{diag}(M - \lambda_1 L, \ldots, M - \lambda_n L)$ and we complete the proof of the theorem. □

Let x^* be an equilibrium of the system (1.1) and the $k \times k$ matrix M the variational matrix of f at x^*, i.e., $M = Df(x^*)$. Obviously, the one-dimensional lattice system (1.2) and the two-dimensional lattice system (1.4) has synchronized equilibrium of the form $(x^*, x^*, \ldots, x^*)$. In the following we first apply Theorem 2.1 to obtain a necessary and sufficient condition of the local stability of $(x^*, \ldots, x^*)$ for the one-dimensional lattice system (1.2) with boundary condition $(1.3)_P$ or $(1.3)_N$. Then we consider the two-dimensional lattice system (1.4) with boundary condition $(1.5)_P$ or $(1.5)_N$.

Theorem 2.2. *For one-dimensional lattice system* (1.2), *the equilibrium* $(x^*, \ldots, x^*)$ *is asymptotically stable if and only if for each* $i = 0, 1, \ldots, N-1$, *the matrix*

$$M_i = M - 2(1 - \cos i\theta)D \tag{2.3}$$

satisfies $\operatorname{Re}\lambda < 0$ *for all* $\lambda \in \operatorname{spec}(M_i)$ *where* $\theta = 2\pi/N$ *for the periodic boundary condition* $(1.3)_P$ *and* $\theta = \pi/N$ *for the Neumann boundary condition* $(1.3)_N$.

Proof. Let $y_i = x_i - x^*$, $i = 1, 2, \ldots, N$. Then the linearized system of (1.2) at $(x^* \ldots x^*)$ is

$$y_i' = My_i + D[y_{i-1} - 2y_i + y_{i+1}], \quad i = 1, 2, \ldots, N. \tag{2.4}$$

Let $Y = (y_1, y_2, \ldots, y_N)^T$. Then (2.4) can be written as

$$\dot{Y} = AY$$

where

$$A = \begin{bmatrix} M-2D & D & & & D \\ D & M-2D & D & & \\ & \ddots & \ddots & \ddots & \\ & & D & M-2D & D \\ D & & & D & M-2D \end{bmatrix}$$

for the periodic boundary condition $(1.3)_P$ and

$$A = \begin{bmatrix} M-D & D & & & \\ D & M-2D & D & & \\ & \ddots & \ddots & \ddots & \\ & & D & M-2D & D \\ & & & D & M-D \end{bmatrix}$$

for the Neumann boundary condition $(1.3)_N$. Since the eigenvalues of

$$T_{P1} = \begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{bmatrix} \tag{2.5}$$

and

$$T_{N1} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \tag{2.6}$$

have the same form, $2(1-\cos i\theta)$, $i = 0, 1, \dots, N-1$ with $\theta = (2\pi/N)$ for T_{P1} and $\theta = (\pi/N)$ for T_{N1} (see [**2**, p. 66], [**9**, p. 558]). Apply Theorem 2.1 with $T = T_{P1}$ and T_{N1} and $L = D$; it follows that the eigenvalues of A are the eigenvalues of M_i, $i = 0, \dots, N-1$. Thus we complete the proof of Theorem 2.2. □

Assume $\tilde{x}(t)$ is a periodic solution of (1.1) of period ω and $M(t) = Df(\tilde{x}(t))$. Obviously $(\tilde{x}(t), \dots, \tilde{x}(t))$, $0 \le t \le \omega$, is a periodic solution of (1.2). In the following we establish a criterion for the orbital stability for the ω-periodic solution $(\tilde{x}(t), \dots, \tilde{x}(t))$ by similar arguments in Theorem 2.2.

Corollary 2.1. *For the one-dimensional lattice system* (1.2), *the ω-periodic solution $(\tilde{x}(t), \dots, \tilde{x}(t))$ is asymptotically orbitally stable if and only if for each $i = 0, 1, \dots, N-1$, the periodic matrix*

$$M_i(t) = M(t) - 2(1 - \cos i\theta)D$$

has Floquet multipliers with magnitude less than one where $\theta = (2\pi/N)$ for the periodic boundary condition $(1.3)_P$ and $\theta = (\pi/N)$ for the Neumann boundary condition $(1.3)_N$.

Remark 2.1. For the one-dimensional lattice system (1.2), we may consider the general form

$$(1.2)' \qquad \frac{dx_i}{dt} = f(x_i) + D(k_{i-1}x_{i-1} - (k_{i-1} + k_i)x_i + k_i x_{i+1}).$$

Then the matrix T in Theorem 2.1 takes the form

$$T_{P1} = \begin{bmatrix} k_n + k_1 & -k_1 & & & -k_n \\ -k_1 & k_1 + k_2 & -k_2 & & \\ & \ddots & \ddots & \ddots & \\ & -k_{n-2} & k_{n-2} + k_{n-1} & & -k_{n-1} \\ -k_n & & -k_{n-1} & & k_{n-1} + k_n \end{bmatrix}$$

or

$$T_{N1} = \begin{bmatrix} k_1 & -k_1 & & & \\ -k_1 & k_1 + k_2 & -k_2 & & \\ & \ddots & \ddots & \ddots & \\ & -k_{n-2} & k_{n-2} + k_{n-1} & & -k_{n-1} \\ & & -k_{n-1} & & k_{n-1} \end{bmatrix}.$$

Corollary 2.2. *For discrete one-dimensional lattice system* (1.7), *the fixed point* $(x^*, \dots, x^*)$ *is asymptotically stable if and only if for each* $i = 0, 1, \dots, N-1$, *the matrix*

$$M_i = M - 2(1 - \cos i\theta)D$$

satisfies $|\lambda| < 1$ *for all* $\lambda \in \operatorname{spec}(M_i)$ *where* $\theta = (2\pi/N)$ *for periodic boundary condition* $(1.3)_P$ *and* $\theta = \pi/N$ *for the Neumann boundary condition* $(1.3)_N$.

Next we consider the two-dimensional lattice system (1.4) with periodic boundary condition $(1.5)_P$ or Neumann boundary condition $(1.5)_N$.

Theorem 2.4. *For two-dimensional lattice system* (1.4), *the equilibrium* $(x^*, \dots, x^*)$ *is asymptotically stable if and only if for each* $i, j = 0, 1, \dots, N-1$, *the matrix*

$$M_{ij} = M - 2(2 - \cos i\theta - \cos j\theta)D \tag{2.7}$$

satisfies $\operatorname{Re}\lambda < 0$ *for all* $\lambda \in \operatorname{spec}(M_{ij})$ *where* $\theta = 2\pi/N$ *for the periodic boundary condition* $(1.5)_P$ *and* $\theta = \pi/N$ *for the Neumann boundary condition* $(1.5)_N$.

Proof. Let $y_{ij} = x_{ij} - x^*$, $i, j = 1, 2, \dots, N$. Then the linearized system of (1.4) at $(x^*, \dots, x^*)$ is

$$\begin{aligned} y'_{ij} = My_{ij} + D(y_{i,j+1} + y_{i,j-1} + y_{i-1,j} + y_{i+1,j} - 4y_{ij}),\\ 1 \le i, j \le N. \end{aligned} \tag{2.8}$$

Let

$$y = (y_{11}, \dots, y_{1N}, y_{21}, \dots, y_{2N}, \dots, y_{N1}, \dots, y_{NN})^T \in \mathbf{R}^{N^2k}. \tag{2.9}$$

Under the periodic boundary $(1.5)_P$, the system (2.8) can be rewritten as

$$y' = Ay, \quad A \in \mathbf{R}^{N^2k \times N^2k} \tag{2.10}$$

where

$$A = \begin{bmatrix} G & H & & H \\ H & G & H & \\ & \ddots & \ddots & \ddots \\ & H & G & H \\ H & & H & G \end{bmatrix}, \tag{2.11}$$

$$G = \begin{bmatrix} M-4D & D & & D \\ D & M-4D & D & \\ & \ddots & \ddots & \ddots \\ & D & M-4D & D \\ D & & D & M-4D \end{bmatrix}, \tag{2.12}$$

$$H = \operatorname{diag}(D, D, \cdots, D) \in \mathbf{R}^{Nk \times Nk}. \tag{2.13}$$

From (2.11), (2.12) and (2.13), the matrix A in (2.11) takes the form of (2.1) in Theorem 2.1 with $L = D$ and the corresponding matrix T is

$$T_P = \begin{bmatrix} F & -I_N & & -I_N \\ -I_N & F & -I_N & \\ & \ddots & \ddots & \ddots \\ & -I_N & F & -I_N \\ -I_N & & -I_N & F \end{bmatrix} \tag{2.14}$$

where

$$F = \begin{bmatrix} 4 & -1 & & -1 \\ -1 & 4 & -1 & \\ & \ddots & \ddots & \ddots \\ & -1 & 4 & -1 \\ -1 & & -1 & 4 \end{bmatrix}. \tag{2.15}$$

From the following Remark 2.2, the eigenvalues of T_P are

$$\lambda_{ij} = 2(2 - \cos i\theta - \cos j\theta), \quad 0 \le i, j \le N-1, \quad \theta = \frac{2\pi}{N}.$$

Apply Theorem 2.1, and we complete the proof for the periodic case. For the Neumann boundary condition $(1.5)_N$, the system (2.8) can be rewritten as (2.10) with

$$A = \begin{bmatrix} G_1 & H & & & \\ H & \tilde{G} & H & & \\ & \ddots & \ddots & \ddots & \\ & & H & \tilde{G} & H \\ & & & H & G_1 \end{bmatrix} \in \mathbf{R}^{N^2k \times N^2k} \tag{2.16}$$

where

(2.17)

$$G_1 = \begin{bmatrix} M-2D & D & & & \\ D & M-3D & D & & \\ & \ddots & \ddots & \ddots & \\ & & D & M-3D & D \\ & & & D & M-2D \end{bmatrix} \in \mathbf{R}^{Nk \times Nk},$$

(2.18)

$$\tilde{G} = \begin{bmatrix} M-3D & D & & & \\ D & M-4D & D & & \\ & \ddots & \ddots & \ddots & \\ & & D & M-4D & D \\ & & & D & M-3D \end{bmatrix} \in \mathbf{R}^{Nk \times Nk}.$$

From (2.16), (2.17), (2.18) and (2.13), the matrix A in (2.16) takes the form of (2.1) in Theorem 2.1 with $L = D$ and the corresponding matrix T is

$$T_N = \begin{bmatrix} F_1 & -I_N & & & \\ -I_N & \tilde{F} & -I_N & & \\ & \ddots & \ddots & \ddots & \\ & & -I_N & \tilde{F} & -I_N \\ & & & -I_N & F_1 \end{bmatrix} \in \mathbf{R}^{N^2 \times N^2}, \tag{2.19}$$

where

$$F_1 = \begin{bmatrix} 2 & -1 & & & \\ -1 & 3 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 3 & -1 \\ & & & -1 & 2 \end{bmatrix}, \tag{2.20}$$

and

$$\tilde{F} = \begin{bmatrix} 3 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 3 \end{bmatrix}. \tag{2.21}$$

From the following Remark 2.3, the eigenvalues of T_N are

$$\lambda_{ij} = 2(2 - \cos i\theta - \cos j\theta), \quad 0 \leq i, j \leq N-1, \quad \theta = \frac{\pi}{N}.$$

Apply Theorem 2.1 and the proof for Neumann case follows.

Remark 2.2. To obtain the eigenvalues of the matrix T_P in (2.14), we first rewrite the matrix F in (2.15) as

$$F = 2I + T_{P1},$$

where T_{P1} is the matrix in (2.5). Applying Theorem 2.1 with $A = T_P$, $M = T_{P1}$, $L = I$ and the corresponding matrix $T = -T_{P1}$, then it follows that

$$\operatorname{spec}(T_P) = \bigcup_{i=1}^{N} \operatorname{spec}(T_{P1} + \lambda_i I)$$

where $\operatorname{spec}(T_{P1}) = \{\lambda_1, \ldots, \lambda_N\}$. Since $\lambda_i = 2(1 - \cos(i-1)\theta)$, $\theta = (2\pi/N)$, $i = 1, \ldots, N$, it follows that the eigenvalues λ_{ij} of T_P are

$$\lambda_{ij} = \lambda_i + \lambda_j = 2(2 - \cos i\theta - \cos j\theta), \quad 0 \leq i, j \leq N-1.$$

Remark 2.3. To obtain the eigenvalues of the matrix T_N in (2.19), we rewrite the matrices F_1 and $\tilde{F}$ in (2.20) and (2.21) as

$$F_1 = I + T_{N1},$$
$$\tilde{F} = 2I + T_{N1},$$

where T_{N1} is the matrix in (2.6). Apply Theorem 2.1 with $A = T_N$, $M = T_{N1}$, $L = I$ and the corresponding matrix $T = -T_{N1}$, then it follows that

$$\operatorname{spec}(T_N) = \bigcup_{i=1}^{N} \operatorname{spec}(T_{N1} + \lambda_i I)$$

where $\operatorname{spec}(T_{N1}) = \{\lambda_1, \dots, \lambda_N\}$. Since $\lambda_i = 2(1 - \cos(i-1)\theta)$, $\theta = (\pi/N)$, it follows that the eigenvalues λ_{ij} of T_N are

$$\lambda_{ij} = \lambda_i + \lambda_j = 2(2 - \cos i\theta - \cos j\theta), \quad 0 \le i, j \le N-1. \qquad \square$$

Corollary 2.3. *For the two-dimensional lattice system (1.4), the ω-periodic solution $(\bar{x}(t), \dots, \bar{x}(t))$ is asymptotically stable if and only if for each $i, j = 0, 1, \dots, N-1$, the periodic matrix*

$$M_{ij}(t) = M(t) - 2(2 - \cos i\theta - \cos j\theta)D$$

has Floquet multipliers with magnitude less than one where $\theta = (2\pi/N)$ for periodic boundary condition $(1.5)_P$ and $\theta = (\pi/N)$ for Neumann boundary condition $(1.5)_N$.

Corollary 2.4. *For the discrete two-dimensional lattice system (1.7), the fixed point $(x^*, \dots, x^*)$ is asymptotically stable if and only if for each $i, j = 0, 1, \dots, N-1$ the matrix*

$$M_{ij} = M - 2(2 - \cos i\theta - \cos j\theta)D$$

satisfies $|\lambda| < 1$ for all $\lambda \in \operatorname{spec}(M_{ij})$ where $\theta = (2\pi/N)$ for periodic boundary condition $(1.5)_P$ and $\theta = (\pi/N)$ for Neumann boundary condition $(1.5)_N$.

At the end of this section we consider a discrete two-dimensional lattice system which describes the "disperse-reproduce" mechanism in the population models [8], [4]. Consider the following iterated map:

$$\bar{x}_{i,j} = f(\hat{x}_{ij})$$

(2.22)

$$\hat{x}_{ij} = D\left(\sum_{\substack{|r-i|\le 1\\ |s-j|\le 1\\ (r,s)\ne(i,j)}} (x_{rs} - x_{ij})\right)$$

where $f : \mathbf{R}^k \to \mathbf{R}^k$, $D = \operatorname{diag}(d_1, \dots, d_k)$, $d_i > 0$, $i = 1, \dots, k$.

Let x^* be a fixed point of f, $M = Df(x^*)$ and $y_{ij} = x_{ij} - x^*$, $1 \le i, j \le N$. Then the linearized system of (2.22) at $(x^*, \dots, x^*)$ is

$$\bar{y}_{ij} = M\left(y_{ij} + D\left(\sum_{\substack{|r-i|\le 1\\ |s-j|\le 1\\ (r,s)\neq(i,j)}} (y_{rs} - y_{ij})\right)\right). \tag{2.23}$$

Let

$$y = (y_{11}, \dots, y_{1N}, y_{21}, \dots, y_{2N}, \dots, y_{N1}, \dots, y_{NN}) \in \mathbf{R}^{N^2 k}.$$

Under the Neumann boundary condition $(1.5)_N$, the system (2.23) takes the form

$$\bar{y} = Ay$$

where

$$A = \begin{bmatrix} \hat{G}_1 & \hat{H} & & \\ \hat{H} & \hat{G} & \hat{H} & \\ & \ddots & \ddots & \ddots \\ & \hat{H} & \hat{G} & \hat{H} \\ & & \hat{H} & \hat{G}_1 \end{bmatrix}, \tag{2.24}$$

$L = MD$,

$$\hat{G}_1 = \begin{bmatrix} M-5L & 2L & & \\ 2L & M-7L & 2L & \\ & \ddots & \ddots & \ddots \\ & 2L & M-7L & 2L \\ & & 2L & M-5L \end{bmatrix},$$

$$\hat{G} = \begin{bmatrix} M-7L & L & & \\ L & M-8L & L & \\ & \ddots & \ddots & \ddots \\ & L & M-8L & L \\ & & L & M-7L \end{bmatrix}$$

$$\hat{H} = \begin{bmatrix} 2L & L & & & \\ L & L & L & & \\ & \ddots & \ddots & \ddots & \\ & & L & L & L \\ & & & L & 2L \end{bmatrix}$$

The corresponding matrix T in Theorem 2.1 is

$$T_N = \begin{bmatrix} \hat{F}_1 & J & & & \\ J & \hat{F} & J & & \\ & \ddots & \ddots & \ddots & \\ & & J & \hat{F} & J \\ & & & J & \hat{F}_1 \end{bmatrix},$$

where

$$\hat{F}_1 = \begin{bmatrix} 5 & -2 & & & \\ -2 & 7 & -2 & & \\ & \ddots & \ddots & \ddots & \\ & & -2 & 7 & -2 \\ & & & -2 & 5 \end{bmatrix},$$

$$\hat{F} = \begin{bmatrix} 7 & -1 & & & \\ -1 & 8 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 8 & -1 \\ & & & -1 & 7 \end{bmatrix}$$

$$J = \begin{bmatrix} -2 & -1 & & & \\ -1 & -1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & -1 & -1 \\ & & & -1 & -2 \end{bmatrix}.$$

Since $\hat{F}_1 = 9I_N + 2J$, $\hat{F} = 9I_N + J$, we apply Theorem 2.1 with $A = T_N$, $L = J$, $M = 9I_N$, $T = J$, then the eigenvalues of T_N are the set of eigenvalues of $9I_N - \lambda_i J$ where $\lambda_i \in \operatorname{spec}(J)$, $i = 0, \ldots, N-1$. Since $J = T_{N1} - 3I_N$, then the eigenvalues of J are $\lambda_i = 2(1 - \cos i\theta) - 3 = -(1 + 2\cos i\theta)$, $i = 0, \ldots, N-1$, $\theta = \pi/N$. Then it follows that the eigenvalues α_{ij} of T_N are $9-(1+2\cos i\theta)(1+2\cos j\theta)$, $i,j=0,1,2,\dots,N-1$. Again, by Theorem 2.1, the eigenvalues of A in (2.24) are the set of eigenvalues of

$$M_{ij} = M - \alpha_{ij}L = M(I - \alpha_{ij}D),$$

$i,j=0,1,\dots,N-1$. Thus we have the following theorem.

Theorem 2.5. *For the discrete two-dimensional lattice system (2.22), with Neumann boundary condition $(1.5)_N$, the equilibrium $(x^*,\dots,x^*)$ is asymptotically stable if and only if for each $i,j = 0,1,\dots,N-1$, the matrix*

$$M_{ij} = M(I - \alpha_{ij}D),$$

where

$$\alpha_{ij} = 9 - (1+2\cos i\theta)(1+2\cos j\theta), \quad \theta = \frac{\pi}{N}$$

satisfying $|\lambda| < 1$ for all $\lambda \in \operatorname{spec}(M_{ij})$.

Remark 2.4. For the system (2.22) with periodic boundary condition $(1.5)_P$, it is easy to verify that Theorem 2.5 holds with $\theta = (2\pi/N)$.

3. Applications.

Predator-prey systems. Consider the following predator-prey system with logistic growth for prey x in the absence of predation and Holling type II functional response for predator y.

$$\begin{aligned} \frac{dx}{dt} &= \gamma x\left(1-\frac{x}{K}\right) - \frac{mx}{a+x}y = f(x,y) \\ \frac{dy}{dt} &= \left(\frac{mx}{a+x} - d\right)y = g(x,y) \\ &x(0)>0, \quad y(0)>0 \end{aligned} \tag{3.1}$$

where $\gamma, m, d, K, a > 0$. Let $m > d$ and $\lambda = a/[(m/d)-1]$. If $[(K-a)/2] < \lambda < K$, then the equilibrium (x^*, y^*) is global asymptotically stable [6] where $x^* = \lambda$, $y^* = (\gamma/m)(a+x^*)[1-(x^*/K)]$. When

$0 < \lambda < [(K-a)/2]$, the equilibrium (x^*, y^*) becomes an unstable focus, and a unique limit cycle exists for (3.1) [3]. Consider the two-dimensional lattice system

$$(3.2)\quad \begin{aligned} \frac{dx_{ij}}{dt} &= f(x_{ij}, y_{ij}) + d_1(x_{i,j+1} + x_{i,j-1} + x_{i-1,j} + x_{i+1,j} - 4x_{ij}) \\ \frac{dy_{ij}}{dt} &= g(x_{ij}, y_{ij}) + d_2(y_{i,j+1} + y_{i,j-1} + y_{i-1,j} + y_{i+1,j} - 4y_{ij}) \end{aligned}$$

$1 \le i,\ j \le N$, with the Neumann boundary condition

$$\begin{pmatrix} x_{0j} \\ y_{0j} \end{pmatrix} = \begin{pmatrix} x_{1j} \\ y_{1j} \end{pmatrix}, \qquad \begin{pmatrix} x_{N+1,j} \\ y_{N+1,j} \end{pmatrix} = \begin{pmatrix} x_{Nj} \\ y_{Nj} \end{pmatrix},$$
$$\begin{pmatrix} x_{i0} \\ y_{i0} \end{pmatrix} = \begin{pmatrix} x_{i1} \\ y_{i1} \end{pmatrix}, \qquad \begin{pmatrix} x_{i,N+1} \\ y_{i,N+1} \end{pmatrix} = \begin{pmatrix} x_{iN} \\ y_{iN} \end{pmatrix},$$
$$1 \le i, j \le N.$$

The variational matrix M of (3.1) at (x^*, y^*) is

$$M = \begin{pmatrix} (2\gamma x^*/(K(a+x^*)))[((K-a)/2) - x^*] & -(mx^*/(a+x^*)) \\ [ma/(a+x^*)^2]y^* & 0 \end{pmatrix}.$$

It is easy to verify that if $(K-a)/2 < x^*$, then for any nonnegative, diagonal matrix E, the eigenvalues of $M - E$ have negative real parts. Thus, from Theorem 2.4, the synchronized equilibrium $(z^*, \dots, z^*)$ of (3.2), where $z^* = (x^*, y^*)$, is asymptotically stable.

If $(K-a)/2 > x^*$, then from [3] the system (3.1) has a unique limit cycle $\Gamma = \{\tilde{z}(t) = (\tilde{x}(t), \tilde{y}(t)) : 0 \le t \le \omega\}$. To verify the orbital stability of the periodic solution $(\tilde{z}(t), \dots, \tilde{z}(t))$ from Corollary 2.3, we need to compute the Floquet's multipliers of the periodic matrix

$$M_{ij}(t) = M(t) - 2(2 - \cos i\theta - \cos j\theta)D,$$

where $D = \operatorname{diag}(d_1, d_2), d_1, d_2 > 0,\ i, j = 0, 1, \dots, N-1$. Since

$$M(t) = \begin{pmatrix} \gamma[1 - (2x/K)] - (may/(a+x)^2) & -(mx/(a+x)) \\ (may/(a+x)^2) & (mx/(a+x)) - d \end{pmatrix}_{\substack{x=\tilde{x}(t) \\ y=\tilde{y}(t)}},$$

from [3] it follows that $\int_0^\omega [(m\tilde{x}(t)/(a+\tilde{x}(t))) - d]\, dt = 0$ and

$$\int_0^w \left(\gamma\left(1 - \frac{2\tilde{x}(t)}{K}\right) - \frac{ma\tilde{y}(t)}{(a+\tilde{x}(t))^2}\right) dt < 0.$$

Then the Floquet's multiplier λ_1, λ_2 of $M_{ij}(t)$ satisfies $0 < \lambda_1\lambda_2 < 1$. Obviously, if λ_1, λ_2 are complex, then $|\lambda_1| = |\lambda_2| < 1$. If we conjecture that if λ_1, λ_2 are real, $|\lambda_1|, |\lambda_2| < 1$ for any $d_1, d_2 > 0$. That is, the periodic orbit is orbital stable.

Brusselator. Consider the equations of "Brusselator" which is a simple model of a hypothetical chemical oscillator,

$$\begin{aligned} \frac{dx}{dt} &= a - (b+1)x + x^2 y \\ \frac{dy}{dt} &= bx - x^2 y. \end{aligned} \tag{3.3}$$

It is easy to verify that the equilibrium $(a, b/a)$ is locally stable if $b < a^2 + 1$ and is an unstable focus if $b > a^2 + 1$. In [**7**, page 107], the authors consider the case $a = 2$, $b = 5.9$ and $N = 2$ for the system (1.2) with Neumann boundary condition $(1.3)_N$, i.e.,

$$\begin{aligned} x_1' &= a - (b+1)x_1 + x_1^2 y_1 + d_1(x_2 - x_1) \\ y_1' &= bx_1 - x_1^2 y_1 + d_2(y_2 - y_1) \\ x_2' &= a - (b+1)x_2 + x_2^2 y_2 + d_1(x_1 - x_2) \\ y_2' &= bx_2 - x_2^2 y_2 + d_2(y_1 - y_2). \end{aligned} \tag{3.4}$$

With the fixed ratio $d_1/d_2 = 0.1$, the authors varied the parameter d_1 from 1.16 to 1.26 and found that the behavior of solutions changes from periodic doubling cascade to chaos. Let $\tilde{z}(t) = (\tilde{x}(t), \tilde{y}(t))$ be an ω-periodic solution of (3.2). Applying Corollary 2.1, we may find the ω-periodic solution $(\tilde{z}(t), \tilde{z}(t))$ of (3.3) becomes orbitally unstable as we increase the parameter d_1.

Host-parasitoids models. Consider the following extension of the familiar Nicholson-Bailey host-parasite equation which describes the interaction between a population of herbivorous arthropods and their insect parasitoids [**1**],

$$\begin{aligned} H_{n+1} &= H_n \exp(r - (1 - H_n/K) - aP_n) = f_1(H_n, P_n) \\ P_{n+1} &= \alpha H_n[1 - \exp(aP_n)] = f_2(H_n, P_n). \end{aligned} \tag{3.5}$$

Let $E^* = (H^*, P^*)$ be the fixed point of (3.5) and $q = H^*/K$. From [1], (H^*, P^*) is a stable fixed point for some range of q. Now we consider the spatial dynamics of host-parasitoid system. Let

$$(\hat{H}_{ij})_n = (1 - \mu_H)(H_{ij})_n + \mu_H(\overline{H}_{ij})_n,$$
$$(\hat{P}_{ij})_n = (1 - \mu_P)(P_{ij})_n + \mu_p(\overline{P}_{ij})_n,$$
$$(\overline{H}_{ij})_n = \frac{1}{8} \sum_{\substack{(r,s)\neq(i,j)\\ |r-i|\leq 1,\\ |s-j|\leq 1}} H_{rs},$$
$$(\overline{P}_{ij})_n = \frac{1}{8} \sum_{\substack{|r-i|\leq 1,\\ |s-j|\leq 1\\ (r,s)\neq(i,j)}} P_{rs}$$

$$\text{(3.6)} \qquad \begin{aligned} (H_{ij})_{n+1} &= f_1((\hat{H}_{ij})_n, (\hat{P}_{ij})_n) \\ (P_{ij})_{n+1} &= f_2((\hat{H}_{ij})_n, (\hat{P}_{ij})_n). \end{aligned}$$

Then we may apply Theorem 2.5 with $D = \operatorname{diag}[(\mu H/8), (\mu P/8)]$ to check the stability property of synchronized equilibrium $(E^*, \dots, E^*)$ of the system (3.5) with Neumann boundary condition.

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