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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF $v''(x) + x \sin v(x) = 0$

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Abstract. In this paper we study the asymptotic behavior of the solution $v(x)$ of initial value problem (1.1) which arises from a mathematical model describing the large deformations of a heavy cantilever by its own weight.

1. **Introduction.** In this paper we are concerned with the asymptotic behavior of the solutions of the following initial value problem:

$$(1.1)\qquad \begin{aligned} &v''(x) + x \sin v(x) = 0, \\ &v'(0) = 0, \\ &v(0) = a, \qquad 0 < a < \pi. \end{aligned}$$

The qualitative behavior of the solutions $v(x, a)$ of (1.1) is important to the studies of the following mathematical model (1.2) which describes the large deformations of a heavy cantilever by its own weight (See [2] or [3]):

$$(1.2)\qquad \begin{aligned} &v''(x) + x \sin v(x) = 0, \\ &v'(0) = 0, \qquad v(1) = \pi - \alpha, \qquad 0 \le \alpha \le \pi. \end{aligned}$$

In [2] the authors studied the two-point boundary value problem (1.2) by using shooting method. From the uniqueness of the solutions of the initial value problem (1.1), it follows that

$$\begin{aligned} &v(x, a) \equiv v(x, a + 2\pi), \\ &v(x, -a) \equiv -v(x, a), \\ &v(x, 0) \equiv 0, \qquad v(x, \pi) \equiv \pi. \end{aligned}$$

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Hence we restrict ourselves to study the case $0<a<\pi$.

First we introduce the following Liapunov function

$$V(x)=(1-\cos v(x))+\frac{1}{2}\frac{(v'(x))^2}{x}, \tag{1.3}$$

where $v(x)\equiv v(x, a)$.

It is easy to verify that

$$V'(x)=-\frac{1}{2}\left(\frac{v(x)}{x}\right)^2\leq 0. \tag{1.4}$$

Then we have

$$1-\cos v(x)\leq V(x)\leq V(0)=1-\cos a,$$

and it follows that $|v(x)|\leq a$ for all $x\geq 0$. We rewrite the equation in (1.1) as

$$v''(x)+x\left(\frac{\sin v(x)}{v(x)}\right)v(x)=0. \tag{1.5}$$

Let $0<\delta<\min_{0\leq v\leq a}(\sin v/v)$. Using Sturm's comparison theorem [1], we compare (1.5) with

$$v''(x)+\delta v(x)=0 \tag{1.6}$$

which is oscillatory over $[0, \infty)$. Thus the solution $v(x, a)$ is oscillatory over $[0, \infty)$ for $0<a<\pi$. Moreover, from (1.3) and (1.4) the solution $v(x, a)$ is oscillatory with the decreasing amplitudes. In the next section we shall prove that

$$\lim_{x\to\infty} v(x, a)=0 \quad \text{for} \quad 0<a<\pi. \tag{1.7}$$

Consequently, if we denote the zero of v by $x_1<x_2<\cdots<x_l<\cdots$, then we have $|x_l-x_{l-1}|\to 0$ as $l\to\infty$; or, more precisely,

$$x_l^{3/2}-x_{l-1}^{3/2}\to\frac{3}{2}\pi \quad \text{as} \quad l\to\infty. \tag{1.8}$$

2. **Main results.** The purpose of this section is to establish (1.7). First, we make the following change of variables:

$$y=x^{3/2}, \qquad u(y, a)=v(x, a). \tag{2.1}$$

Then we have

$$v_x = \frac{3}{2} x^{1/2} u_y$$

$$v_{xx} = \frac{9}{4} x u_{yy} + \frac{3}{4} x^{-1/2} u_y$$

$$v_{xx} + x \sin v = \frac{9}{4} x\left[u_{yy} + \frac{1}{3y} u_y + \frac{4}{9} \sin u\right].$$

Thus (1.1) becomes

$$u_{yy} + \frac{1}{3y} u_y + \frac{4}{9} \sin u = 0, \tag{2.2}$$

$$u(0) = a, \qquad 0 < a < \pi, \tag{2.3}$$

$$u_y(0) = 0. \tag{2.4}$$

We note that (2.4) follows directly from L'Hospital rule. Let $0 < c < d$ be any two real numbers. Multiplying u_y on both sides of (2.2) and integrating the resulting identity from c to d yields

$$\frac{1}{2}(u_y(d))^2 - \frac{1}{2}(u_y(c))^2 + \int_c^d \frac{1}{3y}(u_y)^2 dy + \frac{4}{9}(\cos u(c) - \cos u(d)) = 0. \tag{2.5}$$

Let $0 = \gamma_0 < \gamma_2 < \gamma_4 < \cdots < \gamma_{2k} < \cdots$ and $\gamma_1 < \gamma_3 < \cdots < \gamma_{2k+1} < \cdots$ be the local maxima and local minima of $u(y)$ respectively. Since $v(x, a)$ is oscillatory over $[0, \infty)$ with decreasing amplitudes, from (2.1) so is $u(y, a)$. Assume

$$\lim_{k\to\infty} u(\gamma_{2k}) = \xi \geq 0, \tag{2.6}$$

and

$$\lim_{k\to\infty} u(\gamma_{2k+1}) = \eta \leq 0. \tag{2.7}$$

Now we state the following lemma and we defer the proof to the end of this section.

LEMMA 2.1. *There exists* $C = C(a) > 0$ *such that*

$$|\gamma_k - \gamma_{k-1}| \leq C \quad \textit{for all} \quad k \geq 1. \tag{2.8}$$

Assume that (2.8) hold. From (2.6), (2.7), (2.8) and Cauchy-Schwarz inequality it follows that for each $k \geq 1$

$$\begin{aligned}\xi - \eta &\leq |u(\gamma_{2k}) - u(\gamma_{2k-1})| \\ &= \left| \int_{\gamma_{2k-1}}^{\gamma_{2k}} u_y(y)\, dy \right| \\ &\leq \int_{\gamma_{2k-1}}^{\gamma_{2k}} |u_y(y)|\, dy \\ &\leq (\gamma_{2k} - \gamma_{2k-1})^{1/2} \left[\int_{\gamma_{k-1}}^{\gamma_k} (u_y(y))^2\, dy \right]^{1/2} \\ &\leq C^{1/2} \left[\int_{\gamma_{k-1}}^{\gamma_k} (u_y(y))^2\, dy \right]^{1/2}\end{aligned}$$

or

$$\frac{\xi - \eta}{C^{1/2}} \leq \left[\int_{\gamma_{k-1}}^{\gamma_k} (u_y(y))^2\, dy \right]^{1/2}. \tag{2.9}$$

Letting $c = 0$, $d = \gamma_k$ in (2.5) and $k \to \infty$, we have that

$$\int_0^\infty \frac{1}{y} (u_y(y))^2\, dy < \infty. \tag{2.10}$$

Considering the following inequality

$$\int_{\gamma_{k-1}}^{\gamma_k} \frac{(u_y(y))^2}{y}\, dy \geq \frac{1}{\gamma_k} \int_{\gamma_{k-1}}^{\gamma_k} (u_y(y))^2\, dy, \tag{2.11}$$

we see that (2.9) and (2.11) imply that

$$\int_{\gamma_{k-1}}^{\gamma_k} \frac{(u_y(y))^2}{y}\, dy \geq \frac{1}{\gamma_k} \frac{(\xi - \eta)^2}{C}. \tag{2.12}$$

From Lemma 2.1 and $\lim_{k\to\infty} \gamma_k = +\infty$, there exists $k_0 > 0$ such that

$$\frac{1}{\gamma_k} \geq \frac{1}{2\gamma_{k-1}}, \qquad \text{for every } k \geq k_0. \tag{2.13}$$

From (2.12), (2.13), (2.8), we have that for $k \geq k_0$

$$\begin{aligned}\int_{\gamma_{k-1}}^{\gamma_k} \frac{(u_y(y))^2}{y}\, dy &\geq \frac{1}{2\gamma_{k-1}} \frac{(\xi - \eta)^2}{C} \\ &= \frac{1}{2} \left(\frac{\xi - \eta}{C} \right)^2 \cdot C \frac{1}{\gamma_{k-1}} \\ &\geq \frac{1}{2} \left(\frac{\xi - \eta}{C} \right)^2 \int_{\gamma_{k-1}}^{\gamma_k} \frac{1}{y}\, dy.\end{aligned} \tag{2.14}$$

Summing up (2.14) over $k \geq k_0$ yields

$$\int_{r_{k_0-1}}^{\infty} \frac{(u_y(y))^2}{y}\,dy \geq \frac{1}{2}\left(\frac{\xi-\eta}{C}\right)^2 \int_{r_{k_0-1}}^{\infty} \frac{1}{y}\,dy. \tag{2.15}$$

Therefore $\xi-\eta=0$ since otherwise (2.15) and (2.10) would lead to a contradiction. Since $\xi \geq 0$ and $\eta \leq 0$, we have that $\xi=\eta=0$ or $\lim_{y\to\infty} u(y)=0$. Thus we complete the proof of (1.7).

Proof of Lemma 2.1:

Let $w(y)=y^{1/6}u(y)$. Then (2.2) becomes

$$w_{yy}+\left(\frac{5}{36\,y^2}+\frac{4}{9}\,\frac{\sin u(y)}{u(y)}\right)w=0. \tag{2.16}$$

Since

$$\delta_0 \leq \frac{\sin u(y)}{u(y)} \leq 1 \quad \text{for all} \quad y \geq 0$$

where

$$\delta_0=\frac{\sin a}{a},$$

we compare (2.16) with

$$w_{yy}+\left(\frac{4}{9}\,\delta_0\right)w=0. \tag{2.17}$$

Let $z_1<z_2<\cdots<z_l<\cdots$ be the zeros of $u(y)$. Then from Sturm's comparison theorem it follows that

$$|z_l-z_{l-1}| \leq \frac{\pi}{\sqrt{4\,\delta_0/9}}=C/2$$

or

$$|r_k-r_{k-1}| \leq C \quad \text{for all} \quad k \geq 1,$$

and Lemma 2.1 is proved.

Since w and u have exactly the same zeros in $(0,\infty)$, it follows from (2.16), (1.7) and Sturm's comparison theorem that $|z_l-z_{l-1}| \to (3/2)\,\pi$ as $l\to\infty$. Thus (1.8) holds.

References

1. P. Hartman, *Ordinary Differential Equation*, John Wiley & Sons, Inc. (1964).
2. S. B. Hsu and S. F. Hwang, *Analysis of large deformation of a heavy cantilever*, SIAM J. Math. Analysis, to appear.
3. C. Y. Wang, *Large deformation of a heavy cantilevery*, Quart. Appl. Math. 39 (1981) 261-273.

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