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Antiperfect morse stratification

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Abstract For an equivariant Morse stratification that contains a unique open stratum, we introduce the notion of equivariant antiperfection, which means the difference of the equivariant Morse series and the equivariant Poincaré series achieves the maximal possible value (instead of the minimal possible value 0 in the equivariantly perfect case). We also introduce a weaker condition of local equivariant antiperfection. We prove that the Morse stratification of the Yang-Mills functional on the space of connections on a principal G-bundle over a connected, closed, nonorientable surface Σ is locally equivariantly $\mathbb{Q}$-antiperfect when $G = U(2), SU(2), U(3), SU(3)$; we propose that the Morse stratification is actually equivariantly $\mathbb{Q}$-antiperfect in these cases. Our proposal yields formulas of Poincaré series $P_t^G(\mathrm{Hom}(\pi_1(\Sigma), G); \mathbb{Q})$ when $G = U(2), SU(2), U(3), SU(3)$. Our $U(2), SU(2)$ formulas agree with formulas proved by T. Baird, who also verified our conjectural $U(3)$ formula.

Keywords Morse theory · Moduli space

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1 Introduction

Let f be a Morse function on a compact manifold M, so that it has finitely many isolated nondegenerate critical points. The Morse polynomial of f is defined to be

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$$M_t(f) = \sum_{p \in \mathrm{Crit}(f)} t^{\lambda_p}$$

where $\mathrm{Crit}(f)$ is the set of critical points of f and λ_p is the Morse index of p. The Morse polynomial of any Morse function satisfies the Morse inequalities

$$M_t(f) - P_t(M; K) = (1+t)R_t(K)$$

where $P_t(M; K)$ is the Poincaré polynomial of M relative to a coefficient field K and $R_t(K)$ is a polynomial with nonnegative coefficients. A Morse function is called K-perfect if $R_t(K) = 0$.

In [1], Atiyah and Bott studied Morse theory in a much more general setting: the manifold M is an infinite dimensional affine space $\mathcal{A}$ of connections on a principal G-bundle P over a Riemann surface Σ, where G is a compact connected Lie group; the functional f is the Yang-Mills functional $L : \mathcal{A} \to \mathbb{R}$, $A \mapsto \|F_A\|_{L^2}$, which is Morse-Bott instead of Morse;[1] the Yang-Mills functional is invariant under the action of the gauge group $\mathcal{G} = \mathrm{Aut}(P)$, and Atiyah and Bott consider the $\mathcal{G}$-equivariant Morse series $M_t^{\mathcal{G}}(L; K)$ of L and $\mathcal{G}$-equivariant Poincaré series $P_t^{\mathcal{G}}(\mathcal{A}; K)$ of $\mathcal{A}$ (they are infinite series instead of polynomials). The Morse inequalities in this context are encoded in the equation

$$M_t^{\mathcal{G}}(L; K) - P_t^{\mathcal{G}}(\mathcal{A}; K) = (1+t)R_t^{\mathcal{G}}(K) \tag{1}$$

where $R_t^{\mathcal{G}}(K)$ is a formal power series with nonnegative coefficients. When $K = \mathbb{Q}$, Atiyah and Bott computed $P_t^{\mathcal{G}}(\mathcal{A}; \mathbb{Q}) = P_t(B\mathcal{G}; \mathbb{Q})$, where $B\mathcal{G}$ is the classifying space of the gauge group,[2] and proved that the Morse stratification of the Yang-Mills functional is equivariantly $\mathbb{Q}$-perfect, in the sense that $R_t^{\mathcal{G}}(\mathbb{Q}) = 0$ (when $G = U(n)$, they also proved that it is equivariantly $\mathbb{Z}_p$-perfect for any prime p). This leads to a recursive formula computing the equivariant Poincaré series $P_t^{\mathcal{G}}(\mathcal{A}_{ss}; \mathbb{Q})$ of the unique open stratum $\mathcal{A}_{ss} \subset \mathcal{A}$. When the obstruction class $o_2(P) \in H^2(\Sigma; \pi_1(G)) \cong \pi_1(G)$ is torsion, the absolute minimum of the Yang-Mills functional is zero and the unique open stratum $\mathcal{A}_{ss}$ is the stable manifold of the space $\mathcal{N}_0$ of flat connections on P. We have

$$P_t^{\mathcal{G}}(\mathcal{A}_{ss}; \mathbb{Q}) = P_t^{\mathcal{G}}(\mathcal{N}_0; \mathbb{Q}) = P_t^{G}(\mathrm{Hom}(\pi_1(\Sigma), G)_P; \mathbb{Q}) \tag{2}$$

[1] Indeed, L is not Morse-Bott in the strict sense, since its critical sets $\mathcal{N}_\mu$ are singular in general, but the Morse index λ_μ of $\mathcal{N}_\mu$ is well-defined, and

$$M_t^{\mathcal{G}}(L; K) = \sum_{\lambda \in \Lambda} t^{\lambda_\mu} P_t^{\mathcal{G}}(\mathcal{N}_\mu; K) = \sum_{\lambda \in \Lambda} t^{\lambda_\mu} P_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right)$$

where $\mathcal{A}_\mu$ is the stable manifold of $\mathcal{N}_\mu$.

[2] Atiyah-Bott computed $P_t(B\mathcal{G}; \mathbb{Q})$ for $G = U(n)$ in [1, Sect. 2]; their computation can be generalized to any compact connected Lie group G [17, Theorem 3.3].

where the subscript P labels the connected component corresponding to the topological type P (which is classified by the obstruction class $o_2(P)$).

In [11,12], the authors generalized some aspects of [1] to connected, closed, *nonorientable* surfaces. Let Σ be a connected, closed nonorientable surface, so that it is the connected sum of $m > 0$ copies of $\mathbb{RP}^2$. Let $\pi : \tilde{\Sigma} \to \Sigma$ be the orientable double cover, so that $\tilde{\Sigma}$ is a Riemann surface of genus $m - 1$. Let $\mathcal{A}$ and $\tilde{\mathcal{A}}$ denote the spaces of connections on a principal G-bundle $P \to \Sigma$ and on the pull back $\pi^* P \to \tilde{\Sigma}$, respectively. Then, $A \to \pi^* A$ defines an inclusion $\mathcal{A} \hookrightarrow \tilde{\mathcal{A}}$ whose image is the fixed locus of an antiholomorphic, antisymplectic involution τ on $\tilde{\mathcal{A}}$, and the Yang-Mills functional $L : \mathcal{A} \to \mathbb{R}$ is, by definition, the restriction of the Yang-Mills functional on $\tilde{\mathcal{A}}$ to $\mathcal{A}$. The absolute minimum of the Yang-Mills functional $L : \mathcal{A} \to \mathbb{R}$ is always zero, achieved by flat connections. The normal bundles of Morse strata of $\mathcal{A}$ defined by L are real vector bundles, so a priori one can only take $K = \mathbb{Z}_2$. Together with Ramras, the authors proved that these bundles, and their associated homotopy orbit bundles, are orientable when $G = U(n)$ or $SU(n)$ [13], so we may use any field coefficient in this case. When $G = U(n)$ or $SU(n)$, the Morse stratification of $\mathcal{A}$ defined by L is not equivariantly $\mathbb{Q}$-perfect.

In this paper, we introduce the notion of equivariant K-antiperfection, which means the discrepancy $R_t^{\mathcal{G}}(K)$ in (1) achieves the maximal possible value (instead of the minimal possible value 0 in the perfect case). We also introduce a weaker condition of local equivariant K-antiperfection. We prove that the Morse stratification defined by the Yang-Mills functional on the space of connections on a principal G-bundle over a connected, closed, nonorientable surface Σ is locally equivariantly $\mathbb{Q}$-antiperfect when $G = U(2), SU(2), U(3), SU(3)$; we propose that it is actually equivariantly $\mathbb{Q}$-antiperfect in these cases. (When $G = U(1)$, there is only one stratum $\mathcal{A}_{ss} = \mathcal{A}$.) Our proposal yields formulas for the following equivariant Poincaré series when $n = 2, 3$:

$$P_t^{U(n)}(\mathrm{Hom}(\pi_1(\Sigma), U(n))_+; \mathbb{Q}), \quad P_t^{U(n)}(\mathrm{Hom}(\pi_1(\Sigma), U(n))_-; \mathbb{Q}),$$
$$P_t^{SU(n)}(\mathrm{Hom}(\pi_1(\Sigma), SU(n)); \mathbb{Q}),$$

where $+$ and $-$ label the components corresponding to the trivial and nontrivial $U(n)$-bundles over Σ, respectively. Indeed, we show that these formulas hold if and only if equivariant $\mathbb{Q}$-antiperfection holds in the rank 2 and rank 3 cases. Our rank 2 formulas (13), (14), (15) agree with formulas proved by T. Baird [3]. During the revision of this paper, Baird established equivariant $\mathbb{Q}$-antiperfection in the $U(3)$ case and thus verified our conjectural $U(3)$ formula (16) [5].

2 Preliminaries, definitions, and statements of results

2.1 Morse stratification

Let $\mathcal{A}$ be the space of connections on a principal $U(n)$-bundle or $SU(n)$-bundle P over a connected, closed, orientable or nonorientable surface Σ, and let $\mathcal{G} = \mathrm{Aut}(P)$ be the group of unitary gauge transformations. $\mathcal{A}$ is an infinite dimensional affine space, equipped with a $\mathcal{G}$-invariant Riemannian metric. The Yang-Mills functional

$L : \mathcal{A} \to \mathbb{R}$ is invariant under the action of the gauge group $\mathcal{G}$ and defines a $\mathcal{G}$-equivariant Morse stratification

$$\mathcal{A} = \bigcup_{\mu \in \Lambda} \mathcal{A}_\mu = \mathcal{A}_{ss} \cup \bigcup_{\mu \in \Lambda'} \mathcal{A}_\mu \tag{3}$$

where $\mathcal{A}_{ss}$ is the unique open stratum. When $n = 1$, there is only one stratum: $\mathcal{A} = \mathcal{A}_{ss}$. From now on, we will assume $n > 1$.

The index set Λ is partially ordered such that given $I \subset \Lambda$, $\mathcal{A}_I := \bigcup_{\lambda \in I} \mathcal{A}_\lambda$ is open if $\lambda \in I$, $\mu \leq \lambda \Rightarrow \mu \in I$; this partial ordering can be refined to a total ordering [19, Sect. 2]. In the following discussion, *we fix a total ordering on* Λ so that we have a filtration of $\mathcal{A}$ by open subsets. Given $\mu \in \Lambda'$, let $J = \{\lambda \in \Lambda \mid \lambda \leq \mu\}$ and let $I = J - \{\mu\}$, so that $\mathcal{A}_I \subset \mathcal{A}_J \subset \mathcal{A}$ are inclusions of open subsets. We have the following isomorphisms of $\mathcal{G}$-equivariant cohomology groups:

$$H^k_{\mathcal{G}}(\mathcal{A}_J, \mathcal{A}_I) \overset{\text{excision}}{\cong} H^k_{\mathcal{G}}\left(\left(\mathcal{A}_\mu\right)_\epsilon, \left(\mathcal{A}_\mu\right)_\epsilon - \mathcal{A}_\mu\right) \overset{\text{Thom isomorphism}}{\cong} H^{k-\lambda_\mu}_{\mathcal{G}}\left(\mathcal{A}_\mu\right) \tag{4}$$

where $(\mathcal{A}_\mu)_\epsilon$ is a $\mathcal{G}$-equivariant tubular neighborhood of $\mathcal{A}_\mu$ in $\mathcal{A}_J$ (see [19, Sect. 3] for a construction of $\left(\mathcal{A}_\mu\right)_\epsilon$) and λ_μ is the rank of the normal bundle $\mathbb{N}_\mu$ of $\mathcal{A}_\mu$ in $\mathcal{A}$. The normal bundle $\mathbb{N}_\mu \to \mathcal{A}_\mu$ is a $\mathcal{G}$-equivariant complex vector bundle when Σ is orientable and is a $\mathcal{G}$-equivariant orientable real vector bundle when Σ is nonorientable [13] (when Σ is the Klein bottle, we assume that $n = 2$ or 3), so the Thom isomorphism in (4) holds for any coefficient ring. We may identify the pair $\left(\left(\mathcal{A}_\mu\right)_\epsilon, \left(\mathcal{A}_\mu\right)_\epsilon - \mathcal{A}_\mu\right)$ with $\left(\mathbb{N}_\mu, \left(\mathbb{N}_\mu\right)_0\right)$, where $\left(\mathbb{N}_\mu\right)_0$ is the complement of the zero section of the vector bundle $\mathbb{N}_\mu \to \mathcal{A}_\mu$. We have the following commutative diagram for any coefficient ring:

$$\begin{array}{ccccccccc}
H^k_{\mathcal{G}}(\mathcal{A}_J, \mathcal{A}_I) & \xrightarrow{\alpha^k} & H^k_{\mathcal{G}}(\mathcal{A}_J) & \xrightarrow{\beta^k} & H^k_{\mathcal{G}}(\mathcal{A}_I) & \xrightarrow{\gamma^k} & H^{k+1}_{\mathcal{G}}(\mathcal{A}_J, \mathcal{A}_I) & \xrightarrow{\alpha^{k+1}} & \cdots \\
\downarrow\cong & & \downarrow j^k & & \downarrow i^k & & \downarrow\cong & & \\
H^k_{\mathcal{G}}(\mathbb{N}_\mu, (\mathbb{N}_\mu)_0) & \xrightarrow{\alpha^k_\epsilon} & H^k_{\mathcal{G}}(\mathbb{N}_\mu) & \xrightarrow{\beta^k_\epsilon} & H^k_{\mathcal{G}}((\mathbb{N}_\mu)_0) & \xrightarrow{\gamma^k_\epsilon} & H^{k+1}_{\mathcal{G}}(\mathbb{N}_\mu, (\mathbb{N}_\mu)_0) & \xrightarrow{\alpha^{k+1}_\epsilon} & \cdots \\
\downarrow\cong & & \downarrow s^k \cong & & & & & & \\
H^{k-\lambda_\mu}_{\mathcal{G}}(\mathcal{A}_\mu) & \xrightarrow{\cup e_{\mathcal{G}}(\mathbb{N}_\mu)} & H^k_{\mathcal{G}}(\mathcal{A}_\mu) & & & & & &
\end{array} \tag{5}$$

where i^k, j^k, s^k are induced by inclusions. From the above Diagram (5), we see that $\operatorname{Ker}\left(\alpha^k\right) \subset \operatorname{Ker}\left(\alpha^k_\epsilon\right)$ under the identification $H^k_{\mathcal{G}}(\mathcal{A}_J, \mathcal{A}_I) \cong H^k_{\mathcal{G}}(\mathbb{N}_\mu, \left(\mathbb{N}_\mu\right)_0)$.

2.2 Morse inequalities

We now consider field coefficient K, so that the cohomology groups are vector spaces over K. For any $\mu \in \Lambda'$, we define

$$Z^k_{\mathcal{G}}\left(\mathcal{A}_\mu; K\right) = \operatorname{Ker}\left(H^k_{\mathcal{G}}\left(\mathcal{A}_\mu; K\right) \cong H^{k+\lambda_\mu}_{\mathcal{G}}\left(\mathcal{A}_J, \mathcal{A}_I; K\right) \xrightarrow{\alpha^{k+\lambda_\mu}} H^{k+\lambda_\mu}_{\mathcal{G}}\left(\mathcal{A}_J; K\right)\right)$$

so that $Z^k_{\mathcal{G}}\left(\mathcal{A}_\mu; K\right)$ is a subspace of $H^k_{\mathcal{G}}\left(\mathcal{A}_\mu; K\right)$. We have an exact sequence

$$0 \to \frac{H^{k-\lambda_\mu}_{\mathcal{G}}\left(\mathcal{A}_\mu; K\right)}{Z^{k-\lambda_\mu}_{\mathcal{G}}\left(\mathcal{A}_\mu; K\right)} \to H^k_{\mathcal{G}}\left(\mathcal{A}_J; K\right) \to H^k_{\mathcal{G}}\left(\mathcal{A}_I; K\right) \to Z^{k+1-\lambda_\mu}_{\mathcal{G}}\left(\mathcal{A}_\mu; K\right) \to 0. \tag{6}$$

Define a power series

$$Z^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right) = \sum_{k=0}^{\infty} t^k \dim_K Z^k_{\mathcal{G}}\left(\mathcal{A}_\mu; K\right) \in \mathbb{Z}[[t]].$$

Then, the exact sequence (6) implies

$$P^{\mathcal{G}}_t\left(\mathcal{A}_J; K\right) + (1+t)t^{\lambda_\mu - 1} Z^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right) = P^{\mathcal{G}}_t\left(\mathcal{A}_I; K\right) + t^{\lambda_\mu} P^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right). \tag{7}$$

Given two power series $p(t), q(t) \in \mathbb{Z}[[t]]$, we say $p(t) \leq q(t)$ if $q(t) - p(t)$ is a power series with nonnegative coefficients. Then,

$$0 \leq Z^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right) \leq P^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right). \tag{8}$$

Define

$$R^{\mathcal{G}}_t(K) = \sum_{\mu \in \Lambda'} t^{\lambda_\mu - 1} Z^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right), \quad \tilde{M}^{\mathcal{G}}_t(K) = \sum_{\mu \in \Lambda'} t^{\lambda_\mu - 1} P^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right).$$

Note that $\lambda_\mu - 1 \geq 0$ for $\mu \in \Lambda'$, so $R^{\mathcal{G}}_t(K)$, $\tilde{M}^{\mathcal{G}}_t(K)$ are power series in $\mathbb{Z}[[t]]$ with nonnegative coefficients. The following lemma follows from the definitions and (8).

Lemma 1

$$0 \leq R^{\mathcal{G}}_t(K) \leq \tilde{M}^{\mathcal{G}}_t(K). \tag{9}$$

Moreover,

(i) $R^{\mathcal{G}}_t(K) = 0$ *if and only if* $Z^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right) = 0$ *for all* $\mu \in \Lambda'$*;*
(ii) $R^{\mathcal{G}}_t(K) = \tilde{M}^{\mathcal{G}}_t(K)$ *if and only if* $Z^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right) = P^{\mathcal{G}}_t\left(\mathcal{A}_\mu; K\right)$ *for all* $\mu \in \Lambda'$*.*

Remark 2 A priori the definitions $Z^{\mathcal{G}}_t(K)$ and $R^{\mathcal{G}}_t(K)$ depends on the choice of the total ordering when such total ordering is not unique, since the index set $J = \{\lambda \in \Lambda \mid \lambda \leq \mu\}$ depends on the total ordering. By (11) below, $R^{\mathcal{G}}_t(K)$ does not depend on the choice. $R^{\mathcal{G}}_t(K)$ can be defined for more general equivariant Morse stratification, which contains a unique open stratum.

Define the $\mathcal{G}$-equivariant Morse series of the stratification (3) as follows.

Definition 3 (Morse series) We define the $\mathcal{G}$-equivariant Morse series of the $\mathcal{G}$-equivariant stratification (3) relative to the coefficient field K to be

$$M_t^{\mathcal{G}}(K) = \sum_{\mu\in\Lambda} t^{\lambda_\mu} P_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right) = P_t^{\mathcal{G}}(\mathcal{A}_{ss}; K) + t\tilde{M}_t^{\mathcal{G}}(K). \tag{10}$$

From (7) and (9), we obtain the following.

Lemma 4 (Morse inequalities)

$$P_t^{\mathcal{G}}(\mathcal{A}; K) + (1+t)R_t^{\mathcal{G}}(K) = M_t^{\mathcal{G}}(K) = P_t^{\mathcal{G}}(\mathcal{A}_{ss}; K) + t\tilde{M}_t^{\mathcal{G}}(K) \tag{11}$$

where

$$0 \leq R_t^{\mathcal{G}}(K) \leq \tilde{M}_t^{\mathcal{G}}(K) = \sum_{\mu\in\Lambda'} t^{\lambda_\mu - 1} P_t\left(\mathcal{A}_\mu; K\right).$$

Therefore,

$$P_t^{\mathcal{G}}(\mathcal{A}; K) - \sum_{\mu\in\Lambda'} t^{\lambda_\mu} P_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right) \leq P_t^{\mathcal{G}}(\mathcal{A}_{ss}; K) \leq P_t^{\mathcal{G}}(\mathcal{A}; K) + \sum_{\mu\in\Lambda'} t^{\lambda_\mu - 1} P_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right).$$

Remark 5 When Σ is orientable, Atiyah and Bott proved that α^k is injective for all k and for all $\mu \in \Lambda'$ when $K = \mathbb{Q}$ or $K = \mathbb{Z}_p$ (p any prime) [1]. So when $K = \mathbb{Q}$ or $K = \mathbb{Z}_p$ (p any prime), $R_t^{\mathcal{G}}(K) = 0$, and

$$P_t^{\mathcal{G}}(\mathcal{A}; K) - \sum_{\mu\in\Lambda'} t^{\lambda_\mu} P_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right) = P_t^{\mathcal{G}}(\mathcal{A}_{ss}; K).$$

2.3 Perfect stratification and antiperfect stratification

In Remark 5, the stratification is said to be equivariantly K-perfect. Motivated by the definition of K-perfect stratification in [1] and the extremal cases of the Morse inequalities (Lemma 4), we introduce the following definitions. Let α^k and α_ϵ^k be as in Diagram (5).

Definition 6 (perfect stratification and antiperfect stratification) We say the $\mathcal{G}$-equivariant stratification (3) is *equivariantly K-perfect* if

$$\alpha^k : H_{\mathcal{G}}^k(\mathcal{A}_J, \mathcal{A}_I; K) \to H_{\mathcal{G}}^k(\mathcal{A}_J; K)$$

is injective for all k and all $\mu \in \Lambda'$; we say (3) is *equivariantly K-antiperfect* if $\alpha^k = 0$ for all k and all $\mu \in \Lambda'$.

Remark 7 By Lemmas 10 and 11 below, the definitions in Definition 6 do not depend on the choice of total ordering.

Definition 8 (locally perfect stratification and locally antiperfect stratification) We say the $\mathcal{G}$-equivariant stratification (3) is *locally equivariantly K-perfect* if

$$\alpha_\epsilon^k : H_{\mathcal{G}}^k(\mathbb{N}_\mu, (\mathbb{N}_\mu)_0 ; K) \to H_{\mathcal{G}}^k(\mathbb{N}_\mu; K)$$

is injective for all k and all $\mu \in \Lambda'$; we say (3) is *locally equivariantly K-antiperfect* if $\alpha_\epsilon^k = 0$ for all k and all $\mu \in \Lambda'$.

Remark 9 Since $\text{Ker}(\alpha_k) \subset \text{Ker}(\alpha_k^\epsilon)$, it is immediate from Definition 6 and Definition 8 that

$$\text{(3) is locally equivariantly K-perfect} \Rightarrow \text{(3) is equivariantly K-perfect.}$$
$$\text{(3) is equivariantly K-antiperfect} \Rightarrow \text{(3) is locally equivariantly K-antiperfect.}$$

From Definition 6 and the discussion in Sect. 2.2, we have the following equivalent conditions of equivariant perfection and antiperfection.

Lemma 10 (reformulation of equivariant perfection) *The following conditions are equivalent:*

P1. *(3) is an equivariantly K-perfect stratification.*
P2. *For any $\mu \in \Lambda'$, the long exact sequence*

$$\begin{aligned} \cdots \to H_{\mathcal{G}}^{k-\lambda_\mu}\left(\mathcal{A}_\mu; K\right) \to H_{\mathcal{G}}^k(\mathcal{A}_J; K) \to H_{\mathcal{G}}^k(\mathcal{A}_I; K) \\ \to H_{\mathcal{G}}^{k+1-\lambda_\mu}\left(\mathcal{A}_\mu; K\right) \to \cdots \end{aligned} \tag{12}$$

breaks into short exact sequences

$$0 \to H_{\mathcal{G}}^{k-\lambda_\mu}\left(\mathcal{A}_\mu; K\right) \to H_{\mathcal{G}}^k(\mathcal{A}_J; K) \to H_{\mathcal{G}}^k(\mathcal{A}_I; K) \to 0.$$

P3. $Z_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right) = 0$ *for all* $\mu \in \Lambda'$.
P4. $R_t^{\mathcal{G}}(K) = 0$.
P5. $P_t^{\mathcal{G}}(\mathcal{A}; K) = P_t^{\mathcal{G}}(\mathcal{A}_{ss}; K) + \sum_{\mu \in \Lambda'} t^{\lambda_\mu} P_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right)$.

Lemma 11 (reformulation of equivariant antiperfection) *The following conditions are equivalent:*

A1. *(3) is an equivariantly K-antiperfect stratification.*
A2. *For any $\mu \in \Lambda'$, the long exact sequence* (12) *breaks into short exact sequences*

$$0 \to H_{\mathcal{G}}^k(\mathcal{A}_J; K) \to H_{\mathcal{G}}^k(\mathcal{A}_I; K) \to H_{\mathcal{G}}^{k+1-\lambda_\mu}\left(\mathcal{A}_\mu; K\right) \to 0.$$

A3. $Z_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right) = P_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right)$ *for all* $\mu \in \Lambda'$.

A4. $R_t^{\mathcal{G}}(K) = \tilde{M}_t^{\mathcal{G}}(K)$.
A5. $P_t^{\mathcal{G}}(\mathcal{A}_{ss}; K) = P_t^{\mathcal{G}}(\mathcal{A}; K) + \sum_{\mu\in\Lambda'} t^{\lambda_\mu - 1} P_t^{\mathcal{G}}\left(\mathcal{A}_\mu; K\right)$.

When Σ is orientable, and $K = \mathbb{Q}$ or $K = \mathbb{Z}_p$ (p any prime), Atiyah and Bott showed that $e_{\mathcal{G}}\left(\mathbb{N}_\mu\right)$ in the commutative diagram (5) is not a zero divisor[3] in $H^*_{\mathcal{G}}\left(\mathcal{A}_\mu; K\right)$. Thus, α_ϵ^k is injective. We may reformulate this result as follows.

Theorem 12 (Atiyah-Bott) *Let $\mathcal{A}$ be the space of connections on a principal $U(n)$-bundle or $SU(n)$-bundle over a Riemann surface. Let $K = \mathbb{Q}$ or $K = \mathbb{Z}_p$ (p any prime). Then, the stratification* (3) *is locally equivariantly K-perfect; therefore, it is equivariantly K-perfect.*

2.4 Yang-Mills theory on a closed nonorientable surface

Let Σ be a connected, closed, nonorientable surface. Let $\tilde{P}$ be the pull back of P to the orientable double cover $\tilde{\Sigma} \to \Sigma$. Recall that a stratum $\mathcal{A}_\mu \subset \mathcal{A}$ corresponds to reduction in the structure group of $\tilde{P} \to \tilde{\Sigma}$ (instead of $P \to \Sigma$) to a subgroup

$$U(n_1) \times \cdots \times U(n_r) \subset U(n)$$

or

$$(U(n_1) \times \cdots \times U(n_r)) \cap SU(n) \subset SU(n)$$

where $n_1 + \cdots + n_r = n$. We say μ contains a rank 1 factor if $n_j = 1$ for some j. In particular, when $n = 2$ or 3, every $\mu \in \Lambda'$ contains a rank 1 factor (see Sect. 3).

In Sect. 3, we prove the following:

Theorem 13 (vanishing of equivariant Euler class) *Let $\mathcal{A}$ be the space of connections on a principal $U(n)$-bundle or $SU(n)$-bundle* ($n > 1$) *over a connected, closed, nonorientable surface Σ. When $\chi(\Sigma) = 0$, so that Σ is homeomorphic to the Klein bottle, we assume in addition that $n \leq 3$. We use rational coefficient $\mathbb{Q}$.*

(i) *If $\Sigma = \mathbb{RP}^2$ then $e_{\mathcal{G}}\left(\mathbb{N}_\mu\right) = 0$ for all $\mu \in \Lambda'$*
(ii) *If Σ is not homeomorphic to $\mathbb{RP}^2$ then $e_{\mathcal{G}}\left(\mathbb{N}_\mu\right) = 0$ if μ contains a rank 1 factor.*

Therefore, $\alpha_\epsilon^k = 0$ for all k in the above two cases.

Corollary 14 *Let $\mathcal{A}$ be the space of connections on a principal $U(n)$-bundle or $SU(n)$-bundle* ($n > 1$) *over a connected, closed, nonorientable surface Σ. Then, the Morse stratification (3) is locally equivariant $\mathbb{Q}$-antiperfect in the following cases:*

[3] Indeed, it suffices to show that $e_{\mathcal{G}^0_\mu}(\mathbb{N}_\mu|_{\mathcal{B}^0_\mu})$ is not a zero divisor in $H^*_{\mathcal{G}^0_\mu}(\mathcal{B}^0_\mu; K)$, where $\mathcal{B}^0_\mu \subset \mathcal{A}_\mu$ is the subset of holomorphic structures compatible with a C^∞ direct sum decomposition of E, and $\mathcal{G}^0_\mu \subset \mathcal{G}$ is the subgroup preserving the decomposition. See [1, p. 567–569] for details.

(i) $\Sigma = \mathbb{RP}^2$, *n any positive integer greater than* 1;
(ii) Σ *is not homeomorphic to* $\mathbb{RP}^2$, $n = 2$ *or* 3.

Although local equivariant antiperfection does not imply equivariant antiperfection, it is natural to ask whether equivariant $\mathbb{Q}$-antiperfection holds in the cases listed in Corollary 14.

Notation 15 Given a principal bundle P over a connected, closed, orientable or nonorientable surface, let $\mathcal{A}(P)$ denote the space of connections on P, and let $\mathcal{N}_0(P)$ denote the space of flat connections on P. Let $\mathcal{G}(P) = \mathrm{Aut}(P)$ and $\mathcal{G}_0(P)$ be the gauge group and the based gauge group, respectively.

Let Σ be a closed, connected, nonorientable surface, so that it is the connected sum of $m > 0$ copies of $\mathbb{RP}^2$. Then, the topological type of a principal $U(n)$-bundle $P \to \Sigma$ is classified by $c_1(P) \in H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Let $P_\Sigma^{n,+}$ and $P_\Sigma^{n,-}$ denote the trivial ($c_1 = 0 \bmod 2$) and nontrivial ($c_1 = 1 \bmod 2$) principal $U(n)$-bundles over Σ, respectively, and let Q_Σ^n be a principal $SU(n)$-bundle over Σ (which must be topologically trivial).

We have

$$\mathrm{Hom}(\pi_1(\Sigma), U(n)) = \mathrm{Hom}(\pi_1(\Sigma), U(n))_{+1} \cup \mathrm{Hom}(\pi_1(\Sigma), U(n))_{-1},$$

where

$$\mathrm{Hom}(\pi_1(\Sigma), U(n))_{\pm 1} \cong \mathcal{N}_0\left(P_\Sigma^{n,\pm}\right) \Big/ \mathcal{G}_0\left(P_\Sigma^{n,\pm}\right).$$

We also have

$$\mathrm{Hom}(\pi_1(\Sigma), SU(n)) \cong \mathcal{N}_0\left(Q_\Sigma^n\right) / \mathcal{G}_0\left(Q_\Sigma^n\right).$$

When $m > 1$, $\mathrm{Hom}(\pi_1(\Sigma), U(n))_{+1}$ and $\mathrm{Hom}(\pi_1(\Sigma), U(n))_{-1}$ are the two connected components of $\mathrm{Hom}(\pi_1(\Sigma), U(n))$, and $\mathrm{Hom}(\pi_1(\Sigma), SU(n))$ is connected. When $m = 1$,

$$\mathrm{Hom}\left(\pi_1\left(\mathbb{RP}^2\right), U(n)\right)_{\pm 1} = \left\{a \in U(n) \mid a^2 = I_n, \det(a) = \pm 1\right\}.$$

$\mathrm{Hom}(\pi_1\left(\mathbb{RP}^2\right), U(n))_{+1} = \mathrm{Hom}(\pi_1\left(\mathbb{RP}^2\right), SU(n))$ is disconnected for $n \geq 2$, and $\mathrm{Hom}(\pi_1\left(\mathbb{RP}^2\right), U(n))_{-1}$ is disconnected for $n \geq 3$.

We derive the following result in Sect. 4.2.

Theorem 16 (equivariant Poincaré series, rank 2 case) *Let Σ be a connected, closed, nonorientable surface, and let $\tilde{g}$ be the genus of the orientable double cover $\tilde{\Sigma}$ of Σ. Then, the stratifications (3) on $\mathcal{A}\left(P_\Sigma^{2,+}\right)$, $\mathcal{A}\left(P_\Sigma^{2,-}\right)$, and $\mathcal{A}\left(Q_\Sigma^2\right)$ are equivariantly $\mathbb{Q}$-antiperfect if and only if the following* (13), (14), *and* (15) *hold, respectively:*

$$P_t^{U(2)}\left(\mathrm{Hom}(\pi_1(\Sigma), U(2))_{(-1)^{\tilde{g}}}; \mathbb{Q}\right) = \frac{(1+t)^{\tilde{g}}}{\left(1-t^2\right)\left(1-t^4\right)}\left(\left(1+t^3\right)^{\tilde{g}} + t^{\tilde{g}}(1+t)^{\tilde{g}}\right) \tag{13}$$

$$P_t^{U(2)}\left(\mathrm{Hom}(\pi_1(\Sigma), U(2))_{(-1)^{\tilde{g}+1}}; \mathbb{Q}\right) = \frac{(1+t)^{\tilde{g}}}{\left(1-t^2\right)\left(1-t^4\right)}\left(\left(1+t^3\right)^{\tilde{g}} + t^{\tilde{g}+2}(1+t)^{\tilde{g}}\right) \tag{14}$$

$$P_t^{SU(2)}\left(\mathrm{Hom}(\pi_1(\Sigma), SU(2)); \mathbb{Q}\right) = \begin{cases} \dfrac{\left(1+t^3\right)^{\tilde{g}} + t^{\tilde{g}}(1+t)^{\tilde{g}}}{1-t^4}, & \tilde{g} \text{ is even}, \\ \dfrac{\left(1+t^3\right)^{\tilde{g}} + t^{\tilde{g}+2}(1+t)^{\tilde{g}}}{1-t^4}, & \tilde{g} \text{ is odd}. \end{cases} \tag{15}$$

The formulas in Theorem 16 have been proved by T. Baird [3]:

Theorem 17 (Baird) (13), (14), *and* (15) *hold for any* $\tilde{g} \geq 0$.

From Theorem 16 and Theorem 17, we conclude that

Corollary 18 *Let $\mathcal{A}$ be the space of connections on a principal $U(2)$-bundle or $SU(2)$-bundle over a connected, closed, nonorientable surface. Then, the Morse stratification* (3) *on $\mathcal{A}$ is equivariantly $\mathbb{Q}$-antiperfect.*

We derive the following result in Sect. 4.3.

Theorem 19 (equivariant Poincaré series, rank 3 case) *Let Σ be a connected, closed, nonorientable surface, and let $\tilde{g}$ be the genus of the orientable double cover $\tilde{\Sigma}$ of Σ. Then, the stratifications* (3) *on $\mathcal{A}\left(P_\Sigma^{3,\pm}\right)$ and $\mathcal{A}\left(Q_\Sigma^3\right)$ are equivariantly $\mathbb{Q}$-antiperfect if and only if the following* (16) *and* (17) *hold, respectively.*

$$\begin{aligned} P_t^{U(3)}\left(\mathrm{Hom}(\pi_1(\Sigma), U(3))_{+1}; \mathbb{Q}\right) &= P_t^{U(3)}\left(\mathrm{Hom}(\pi_1(\Sigma), U(3))_{-1}; \mathbb{Q}\right) \\ &= \frac{(1+t)^{\tilde{g}}}{\left(1-t^2\right)\left(1-t^4\right)\left(1-t^6\right)}\left(\left(1+t^3\right)^{\tilde{g}}\left(1+t^5\right)^{\tilde{g}}\right. \\ &\quad \left. + t^{3\tilde{g}}(1+t)^{2\tilde{g}}\left(1+t^2+t^4\right)\right) \end{aligned} \tag{16}$$

$$P_t^{SU(3)}\left(\mathrm{Hom}(\pi_1(\Sigma), SU(3)); \mathbb{Q}\right) = \frac{\left(1+t^3\right)^{\tilde{g}}\left(1+t^5\right)^{\tilde{g}} + t^{3\tilde{g}}(1+t)^{2\tilde{g}}\left(1+t^2+t^4\right)}{\left(1-t^4\right)\left(1-t^6\right)} \tag{17}$$

Motivated by Theorem 13 and 19, we make the following conjecture.

Conjecture 20 (16) *and* (17) *hold for any* $\tilde{g} \geq 0$.

During the revision of this paper, T. Baird showed that the stratifications (3) on $\mathcal{A}(P_\Sigma^{n,+})$ and $\mathcal{A}(P_\Sigma^{n,-})$ are equivariantly $\mathbb{Q}$-antiperfect for $n = 3$, but not equivariantly $\mathbb{Q}$-antiperfect for $n \geq 4$ [5]. Therefore, (16) holds for any $\tilde{g} \geq 0$. Using a different approach, Baird verified our conjectural rank 3 formulas (16) and (17) when Σ is the real projective plane ($\tilde{g} = 0$) or the Klein bottle ($\tilde{g} = 1$) [4].

3 Equivariant euler class

Let Σ be a connected, closed, nonorientable surface, so that it is the connected sum of $m > 0$ copies of $\mathbb{RP}^2$, and let $\pi : \tilde{\Sigma} \to \Sigma$ be the orientable double cover, so that $\tilde{\Sigma}$ is a Riemann surface of genus $\tilde{g} = m - 1$. Let $P_{\Sigma}^{n,+}$ and $P_{\Sigma}^{n,-}$ be defined as in Sect. 2.3, and let $P_{\tilde{\Sigma}}^{n,k}$ denote the degree k principal $U(n)$-bundle on $\tilde{\Sigma}$. Then, $\pi^* P_{\Sigma}^{n,\pm} \cong P_{\tilde{\Sigma}}^{n,0} \cong U(n) \times \tilde{\Sigma}$ is a trivial $U(n)$-bundle over $\tilde{\Sigma}$.

Let $\mathcal{A}(P), \mathcal{N}_0(P), \mathcal{G}(P)$, and $\mathcal{G}_0(P)$ be defined as in Notation 15. There is an inclusion $\mathcal{A}(P_{\Sigma}^{n,\pm}) \hookrightarrow \mathcal{A}(P_{\tilde{\Sigma}}^{n,0})$ defined by $A \mapsto \pi^* A$, and the image is the fixed locus of an antisymplectic, antiholomorphic involution $\tau^{\pm}$ on $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})$. The Yang-Mills functional on $\mathcal{A}(P_{\Sigma}^{n,\pm})$ is, by definition, restriction of the Yang-Mills functional on $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})$ to $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})^{\tau^{\pm}}$. The Yang-Mills functional on $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})$ and the metric on $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})$ are invariant under the involutions τ^+, τ^-. The Morse strata of $\mathcal{A}(P_{\Sigma}^{n,\pm})$ are of the form $\mathcal{A}_{\mu} = \tilde{\mathcal{A}}_{\mu} \cap \mathcal{A}(P_{\tilde{\Sigma}}^{n,0})^{\tau^{\pm}}$, where $\tilde{\mathcal{A}}_{\mu}$ is a Morse stratum of $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})$. The Yang-Mills functional is invariant under the action of the gauge group, and each Morse stratum is preserved by the action of the gauge group. Since the arguments for $\mathcal{A}(P_{\Sigma}^{n,-})$ and $\mathcal{A}(P_{\Sigma}^{n,+})$ are the same, we will use the notation $\mathcal{A}$ instead of $\mathcal{A}(P_{\Sigma}^{n,\pm})$ when there is no confusion.

3.1 Harder-Narasimhan types

The Morse strata on $\mathcal{A}(P_{\tilde{\Sigma}}^{n,k})$ are labeled by the Harder-Narasimhan types $\mu \in I_{n,k}$, where

$$I_{n,k} = \left\{ \mu = (\mu_1, \ldots, \mu_n) = \left(\underbrace{\frac{k_1}{n_1}, \ldots, \frac{k_1}{n_1}}_{n_1}, \ldots, \underbrace{\frac{k_m}{n_m}, \ldots, \frac{k_m}{n_m}}_{n_m} \right) \right.$$
$$\left. \middle| \, n_j \in \mathbb{Z}_{>0}, k_j \in \mathbb{Z}, \sum_{j=1}^{m} n_j = n, \sum_{j=1}^{m} k_j = k, \frac{k_1}{n_1} > \cdots > \frac{k_m}{n_m} \right\}$$

The Morse stratification on $\mathcal{A}\left(P_{\tilde{\Sigma}}^{n,k}\right)$ is given by

$$\mathcal{A}\left(P_{\tilde{\Sigma}}^{n,k}\right) = \bigcup_{\mu \in I_{n,k}} \tilde{\mathcal{A}}_{\mu}.$$

The unique open stratum is

$$\mathcal{A}\left(P_{\tilde{\Sigma}}^{n,k}\right)_{ss} = \tilde{\mathcal{A}}_{\frac{k}{n}, \ldots, \frac{k}{n}}.$$

The partial ordering on $I_{n,k}$ is described by Shatz [20]:

$$\mu \geq \nu \quad \text{iff} \quad \sum_{j \leq i} \mu_j \geq \sum_{j \leq i} \nu_j, \quad \forall i = 1, \ldots, n-1.$$

The involution $\tau^{\pm}$ acts on strata by $\mathcal{A}_\mu \mapsto \mathcal{A}_{\tau_0(\mu)}$, where

$$\tau_0 : I_{n,0} \to I_{n,0}, \quad (\mu_1, \ldots, \mu_n) \mapsto (-\mu_n, \ldots, -\mu_1).$$

Using the same notation as in [11, Sect. 7.1], denote $I_n = I_{n,0}^{\tau_0}$ the fixed point set of τ_0 on $I_{n,0}$. Then, any $\mu \in I_n$ is of the form

$$\mu = \left(\underbrace{\frac{k_1}{n_1}, \ldots, \frac{k_1}{n_1}}_{n_1}, \ldots, \underbrace{\frac{k_r}{n_r}, \ldots, \frac{k_r}{n_r}}_{n_r}, \underbrace{0, \ldots, 0}_{n_0}, \underbrace{-\frac{k_r}{n_r}, \ldots, -\frac{k_r}{n_r}}_{n_r}, \ldots, \underbrace{-\frac{k_1}{n_1}, \ldots, -\frac{k_1}{n_1}}_{n_1} \right) \tag{18}$$

where

$$\frac{k_1}{n_1} > \cdots > \frac{k_r}{n_r} > 0, \quad n_0 \geq 0, \quad n_i > 0, \quad 2(n_1 + \cdots + n_r) + n_0 = n,$$

Define

$$I_n^0 = \{\mu \in I_n \mid \mu_i = 0 \text{ for some } i\}.$$

For $\mu \in I_n^0$, $\tilde{\mathcal{A}}_\mu$ intersects both $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})^{\tau^+}$ and $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})^{\tau^-}$. Note that $I_n = I_n^0$ when n is odd.

When $n = 2n'$ is even, any $\mu \in I_n \setminus I_n^0$ is of the form

$$\mu = (\nu, \tau_0(\nu)), \quad \nu \in I_{n',k}, \quad \nu_1 > \ldots > \nu_{n'} > 0. \tag{19}$$

By [11, Sect. 7.1], $\tilde{\mathcal{A}}_\mu$ intersect $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})^{\tau^+}$ (resp. $\mathcal{A}(P_{\tilde{\Sigma}}^{n,0})^{\tau^-}$) if and only if $n'\chi(\Sigma)+k$ is even (resp. odd). Here, $\chi(\Sigma)$ is the Euler characteristic of the nonorientable surface Σ; if Σ is the connected sum of m copies of $\mathbb{RP}^2$, then $\chi(\Sigma) = 2 - m$.

When $n = 2n'$ is even, we define

$$I_n^{\pm}(\Sigma) = \left\{ (\nu, \tau_0(\nu)) \in I_n \setminus I_n^0 \mid \nu \in I_{n',k}, (-1)^{n'\chi(\Sigma)+k} = \pm 1 \right\}.$$

When n is odd, we define $I_n^{\pm}(\Sigma)$ to be empty sets. Then,

$$\mathcal{A}\left(P_\Sigma^{n,\pm}\right) = \bigcup_{\mu \in I_n^0 \cup I_n^{\pm}(\Sigma)} \mathcal{A}_\mu.$$

By the discussion in [13, Sect. 3.3], there is an inclusion $\iota : \mathcal{A}(Q^n_\Sigma) \hookrightarrow \mathcal{A}(P^{n,+}_\Sigma)$, and the Morse stratification on $\mathcal{A}(Q^n_\Sigma)$ is given by

$$\mathcal{A}\left(Q^n_\Sigma\right) = \bigcup_{\mu \in I^0_n \cup I^+_n(\Sigma)} \mathcal{A}'_\mu$$

where $\mathcal{A}'_\mu = \mathcal{A}_\mu \cap \mathcal{A}\left(Q^n_\Sigma\right)$. Let

$$\mathcal{G}' = \mathrm{Aut}\left(Q^n_\Sigma\right) = \mathrm{Map}(\Sigma, SU(n)), \quad \mathcal{G} = \mathrm{Aut}\left(P^{n,+}_\Sigma\right) = \mathrm{Map}(\Sigma, U(n)),$$

and let $\mathbb{N}_\mu$ (resp. $\mathbb{N}'_\mu$) be the normal bundle of $\mathcal{A}_\mu$ (resp. $\mathcal{A}'_\mu$) in $\mathcal{A}(P^{n,+}_\Sigma)$ (resp. $\mathcal{A}(Q^n_\Sigma)$). Then, there are continuous maps

$$\left(\mathcal{A}'_\mu\right)_{h\mathcal{G}'} \overset{\iota_\mu}{\hookrightarrow} \left(\mathcal{A}_\mu\right)_{h\mathcal{G}'} \overset{q_\mu}{\to} \left(\mathcal{A}_\mu\right)_{h\mathcal{G}}$$

and the vector bundle $(\mathbb{N}'_\mu)_{h\mathcal{G}'}$ over $\left(\mathcal{A}'_\mu\right)_{h\mathcal{G}'}$ is the pullback of the vector bundle $\left(\mathbb{N}_\mu\right)_{h\mathcal{G}}$ over $\left(\mathcal{A}_\mu\right)_{h\mathcal{G}}$ under $q_\mu \circ \iota_\mu$. So if $e_{\mathcal{G}}\left(\mathbb{N}_\mu\right) = 0$ then $e_{\mathcal{G}'}\left(\mathbb{N}'_\mu\right) = 0$. Therefore, to prove the vanishing of the equivariant Euler class (Theorem 13), it suffices to consider the $U(n)$ case.

3.2 Decomposition of the normal bundle

Let $E = P^{n,k}_{\tilde{\Sigma}} \times_\rho \mathbb{C}^n$ be the complex vector bundle associated with the fundamental representation $\rho : U(n) \to GL(n, \mathbb{C})$. Then, $E \to \tilde{\Sigma}$ is a rank n, degree k complex vector bundle equipped with a Hermitian metric h, and $\mathcal{A}(P^{n,k}_{\tilde{\Sigma}})$ can be identified with $\mathcal{A}(E, h)$, the space of Hermitian connections on (E, h) (i.e. connections on E that are compatible with the Hermitian metric h). Let $\mathcal{C}(E)$ denote the space of holomorphic structures on E. Then, there is an isomorphism $\mathcal{A}(P^{n,k}_{\tilde{\Sigma}}) \overset{\cong}{\to} \mathcal{C}(E)$ of complex affine spaces, given by $\nabla \mapsto \nabla^{0,1}$ (see e.g. [1, p. 570] or [16, Chap. VII § 1]). Let $\mathcal{E}$ denote E equipped with a (0,1)-connection (holomorphic structure), so that $\mathcal{E}$ can be viewed as a point in $\mathcal{C}(E)$ and thus a point in $\mathcal{A}(P^{n,k}_{\tilde{\Sigma}})$.

Let $\mu \in I^0_n \cup I^\pm_n(\Sigma)$ be as in (18), so that $\mathcal{A}_\mu$ is a stratum of $\mathcal{A}(P^{n,\pm}_\Sigma)$, and

$$\mathcal{A}_\mu = \tilde{\mathcal{A}}_\mu \cap \mathcal{A}\left(P^{n,0}_{\tilde{\Sigma}}\right)^{\tau^\pm}$$

where $\tilde{\mathcal{A}}_\mu$ is the corresponding stratum of $\mathcal{A}(P^{n,0}_{\tilde{\Sigma}})$ labeled by the same Harder-Narasimhan type μ. Let $\mathcal{N}_\mu$ be the critical set of $\mathcal{A}_\mu$, and let $i : \mathcal{N}_\mu \hookrightarrow \mathcal{A}_\mu$ be the inclusion map. There is a gauge equivariant deformation retraction $r : \mathcal{A}_\mu \to \mathcal{N}_\mu$ (cf: [6,18]), so $e_{\mathcal{G}}\left(\mathbb{N}_\mu\right) = 0$ if and only if $e_{\mathcal{G}}\left(i^*\mathbb{N}_\mu\right) = 0$. We have the following equivalences of equivariant pairs:

$$\left(\mathcal{A}_\mu, \mathcal{G}\left(P_\Sigma^{n,\pm}\right)\right) \sim \left(\mathcal{N}_\mu, \mathcal{G}\left(P_\Sigma^{n,\pm}\right)\right)$$
$$\sim \left(\mathcal{N}_0\left(P_\Sigma^{n_0,\pm}\right), \mathcal{G}\left(P_\Sigma^{n_0,\pm}\right)\right) \times \prod_{j=1}^{r} \left(\mathcal{N}_{ss}\left(P_{\tilde{\Sigma}}^{n_j,k_j}\right), \mathcal{G}\left(P_{\tilde{\Sigma}}^{n_j,k_j}\right)\right).$$

When $\mu \in I_n^0$ so that $n_0 > 0$, the parity of $P_\Sigma^{n_0,\pm}$ can either agree or disagree with that of $P_\Sigma^{n,\pm}$.

A point in $\mathcal{N}_\mu$ corresponds to a holomorphic vector bundle $\mathcal{E}$ of the form

$$\mathcal{E} = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_r \oplus \mathcal{D}_0 \oplus \tau_{\mathcal{C}}(\mathcal{D}_r) \oplus \ldots \oplus \tau_{\mathcal{C}}(\mathcal{D}_1)$$

where $\mathcal{D}_j$ is a degree k_j, rank n_j polystable vector bundle, $\mathcal{D}_0$ is a degree 0, rank n_0 polystable vector bundle, $\tau_{\mathcal{C}}\left(\mathcal{D}_j\right) = \tau^*\overline{\mathcal{D}_j^\vee}$ and $\tau_{\mathcal{C}}(\mathcal{D}_0) \cong \mathcal{D}_0$ (see [13, Sect. 3] for more details).

Let $\tilde{\mathbb{N}}_\mu$ be the normal bundle of $\tilde{\mathcal{A}}_\mu$ in $\mathcal{A}\left(P_{\tilde{\Sigma}}^{n,0}\right)$. Then, the fiber of $\tilde{\mathbb{N}}_\mu$ at $\mathcal{E}$ is

$$\begin{aligned}\left(\tilde{\mathbb{N}}_\mu\right)_{\mathcal{E}} &= H^1\left(\tilde{\Sigma}, \mathrm{End}''(\mathcal{E})\right)\\ &= \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_i, \mathcal{D}_j\right)\right) \oplus \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\tau_{\mathcal{C}}\left(\mathcal{D}_j\right), \tau_{\mathcal{C}}(\mathcal{D}_i)\right)\right)\\ &\quad \oplus \bigoplus_{0<i,j} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_i, \tau_{\mathcal{C}}\left(\mathcal{D}_j\right)\right)\right)\\ &\quad \oplus \bigoplus_{i>0} H^1\left(\tilde{\Sigma}, \mathrm{Hom}(\mathcal{D}_i, \mathcal{D}_0)\right) \oplus \bigoplus_{i>0} H^1\left(\tilde{\Sigma}, \mathrm{Hom}(\mathcal{D}_0, \tau_{\mathcal{C}}(\mathcal{D}_i))\right).\end{aligned}$$

As explained in [13, Sect. 4.1], τ induces conjugate linear maps of complex vector spaces:

$$H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_i, \mathcal{D}_j\right)\right) \to H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\tau_{\mathcal{C}}\left(\mathcal{D}_j\right), \tau_{\mathcal{C}}(\mathcal{D}_i)\right)\right), \text{ and its inverse,}$$
$$H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_i, \tau_{\mathcal{C}}\left(\mathcal{D}_j\right)\right)\right) \to H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_j, \tau_{\mathcal{C}}(\mathcal{D}_i)\right)\right),$$
$$H^1(\tilde{\Sigma}, \mathrm{Hom}(\mathcal{D}_i, \mathcal{D}_0)) \to H^1(\tilde{\Sigma}, \mathrm{Hom}(\mathcal{D}_0, \tau_{\mathcal{C}}(\mathcal{D}_i))), \text{ and its inverse.}$$

Let $\mathbb{N}_\mu$ be the normal bundle of $\mathcal{A}_\mu$ in $\mathcal{A}\left(P_\Sigma^{n,\pm}\right) = \mathcal{A}\left(P_{\tilde{\Sigma}}^{n,0}\right)^{\tau^\pm}$. Then, the fiber of $\mathbb{N}_\mu$ at $\mathcal{E}$ is

$$\begin{aligned}\left(\mathbb{N}_\mu\right)_{\mathcal{E}} &= H^1\left(\tilde{\Sigma}, \mathrm{End}''(\mathcal{E})\right)^\tau\\ &\cong \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_i, \mathcal{D}_j\right)\right) \oplus \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_i, \tau_{\mathcal{C}}\left(\mathcal{D}_j\right)\right)\right)\\ &\quad \oplus \bigoplus_{j>0} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_j, \mathcal{D}_0\right)\right) \oplus \bigoplus_{j>0} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_j, \tau_{\mathcal{C}}\left(\mathcal{D}_j\right)\right)\right)^\tau \end{aligned} \quad (20)$$

Therefore, $i^*\mathbb{N}_\mu = \mathbb{N}_\mu^{\mathbb{C}} \oplus \mathbb{N}_\mu^{\mathbb{R}}$, where

$$\left(\mathbb{N}_\mu^{\mathbb{C}}\right)_{\mathcal{E}} = \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_i, \mathcal{D}_j\right)\right) \oplus \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathrm{Hom}(\mathcal{D}_i, \tau_{\mathcal{C}}\left(\mathcal{D}_j\right)\right) \\ \oplus \bigoplus_{j>0} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_j, \mathcal{D}_0\right)\right),$$

$$\left(\mathbb{N}_\mu^{\mathbb{R}}\right)_{\mathcal{E}} = \bigoplus_{j>0} H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_j, \tau_{\mathcal{C}}\left(\mathcal{D}_j\right)\right)\right)^{\tau}.$$

Note that $\mathbb{N}_\mu^{\mathbb{C}}$ is a complex vector bundle over $\mathcal{N}_\mu$ and $\mathbb{N}_\mu^{\mathbb{R}}$ is a real vector bundle over $\mathcal{N}_\mu$. We have

$$e_{\mathcal{G}}\left(i^*\mathbb{N}_\mu\right) = e_{\mathcal{G}}\left(\mathbb{N}_\mu^{\mathbb{C}}\right) \cup e_{\mathcal{G}}\left(\mathbb{N}_\mu^{\mathbb{R}}\right). \tag{21}$$

Let

$$\lambda_\mu = \mathrm{rank}_{\mathbb{R}}\mathbb{N}_\mu, \quad \lambda_\mu^{\mathbb{C}} = \mathrm{rank}_{\mathbb{C}}\mathbb{N}_\mu^{\mathbb{C}}, \quad \lambda_\mu^{\mathbb{R}} = \mathrm{rank}_{\mathbb{R}}\mathbb{N}_\mu^{\mathbb{R}}.$$

Then,

$$\lambda_\mu = 2\lambda_\mu^{\mathbb{C}} + \lambda_\mu^{\mathbb{R}}.$$

Lemma 21 *Let $K = \mathbb{Q}$ or $K = \mathbb{Z}_p$ (p any prime). Then, $e_{\mathcal{G}}\left(\mathbb{N}_\mu^{\mathbb{C}}\right)$ is not a zero divisor in $H_{\mathcal{G}}^*(\mathcal{N}_\mu; K)$.*

Proof Let $U(1)_j$ be the center of $U(n_j)$, the group of *constant* gauge transformation on $P_{\tilde{\Sigma}}^{n_j,k_j}$. Let $T^r = U(1)_1 \times \cdots \times U(1)_r$. Then, $T^r \subset \mathcal{G}\left(P_{\Sigma}^{n_0,\pm}\right) \times \prod_{j=1}^r \mathcal{G}\left(P_{\tilde{\Sigma}}^{n_j,k_j}\right)$ acts trivially on $\mathcal{N}_0\left(P_{\Sigma}^{n_0,\pm}\right) \times \prod_{j=1}^r \mathcal{N}_{ss}\left(P_{\tilde{\Sigma}}^{n_j,k_j}\right)$, and the weights of the T^r-action on $\mathbb{N}_\mu^{\mathbb{C}}$ are given by

$$\begin{aligned} & t_j t_i^{-1} \quad \text{on} \quad H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_i, \mathcal{D}_j\right)\right), \quad i < j, \\ & t_j t_i^{-1} \quad \text{on} \quad H^1\left(\tilde{\Sigma}, \mathrm{Hom}(\mathcal{D}_i, \tau_{\mathcal{C}}\left(\mathcal{D}_j\right)\right), \quad i < j, \\ \text{and} \quad & t_j^{-1} \quad \text{on} \quad H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_j, \mathcal{D}_0\right)\right), \quad j > 0. \end{aligned}$$

where $(t_1, \ldots, t_r) \in T^r$ (cf: [1, p. 569]). So the representation of T^r on the fiber of $\mathbb{N}_\mu^{\mathbb{C}}$ is primitive. By [1, Proposition 13.4]), $e_{\mathcal{G}}\left(\mathbb{N}_\mu^{\mathbb{C}}\right)$ is not a zero divisor in $H_{\mathcal{G}}^*(\mathcal{N}_\mu; K)$. □

By (21) and Lemma 21, $e_{\mathcal{G}}(i^*\mathbb{N}_\mu) = 0$ if and only if $e_{\mathcal{G}}\left(\mathbb{N}_\mu^{\mathbb{R}}\right) = 0$. To study $\mathbb{N}_\mu^{\mathbb{R}}$, we reduce it to bundles over representation varieties, which we recall in the next subsection.

3.3 Representation varieties for Yang-Mills connections

Let Σ_0^ℓ be the closed, compact, connected, orientable surface with $\ell \geq 0$ handles. Let Σ_1^ℓ be the connected sum of Σ_0^ℓ and $\mathbb{RP}^2$, and let Σ_2^ℓ be the connected sum of Σ_0^ℓ and a Klein bottle. Any connected, closed, nonorientable surface is of the form Σ_i^ℓ, where ℓ is a nonnegative integer and $i = 1, 2$. Note that Σ_i^ℓ is the connected sum of $(2\ell + i)$-copies of $\mathbb{RP}^2$ and that the orientable double cover of Σ_i^ℓ is $\Sigma_0^{\tilde{g}}$, where $\tilde{g} = 2\ell + i - 1$.

A Yang-Mills G-connection on Σ gives rise to a homomorphism $\Gamma_{\mathbb{R}}(\Sigma) \to G$ where $\Gamma_{\mathbb{R}}(\Sigma)$ is the *super central extension* introduced in [11, Sect. 4.6]. Given $V = (a_1, b_1, \ldots, a_\ell, b_\ell) \in G^{2\ell}$, define

$$\mathfrak{m}(V) = \prod_{i=1}^{\ell} [a_i, b_i], \quad \mathfrak{r}(V) = (b_\ell, a_\ell, \ldots, b_1, a_1).$$

In [11], the authors introduced the following symmetric representation varieties of Yang-Mills connections on the orientable double cover $\widetilde{\Sigma_i^\ell} = \Sigma_0^{\tilde{g}}$:

$$\begin{aligned}
Z_{\mathrm{YM}}^{\ell,1}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} = \Big\{ \Big(V, c, V', c', -2\sqrt{-1}\pi \frac{k}{n} I_n \Big) \mid V, V' \in U(n)^{2\ell},\ c, c' \in U(n), \\
\mathfrak{m}(V) = e^{-\pi\sqrt{-1}k/n} I_n c c',\ \mathfrak{m}\left(V'\right) = e^{\pi\sqrt{-1}k/n} I_n c' c \Big\}, \\
Z_{\mathrm{YM}}^{\ell,2}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} = \Big\{ \Big(V, d, c, V', d', c', -2\sqrt{-1}\pi \frac{k}{n} I_n \Big) \mid V, V' \in U(n)^{2\ell}, \\
d, c, d', c' \in U(n), \mathfrak{m}(V) = e^{-\pi\sqrt{-1}k/n} I_n c d' c^{-1} d, \\
\mathfrak{m}\left(V'\right) = e^{\pi\sqrt{-1}k/n} I_n c' d \left(c'\right)^{-1} d' \Big\}.
\end{aligned}$$

We also have the following representation variety of Yang-Mills connections on $\Sigma_0^{\tilde{g}}$:

$$\begin{aligned}
X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} &= \Big\{ \Big(V, -2\sqrt{-1}\pi \frac{k}{n} I_n \Big) \mid V \in U(n)^{2\tilde{g}},\ \mathfrak{m}(V) = e^{-2\pi\sqrt{-1}k/n} I_n \Big\} \\
&\cong \mathcal{N}_{ss}\left(P^{n,k}_{\Sigma_0^{\tilde{g}}}\right) \Big/ \mathcal{G}_0\left(P^{n,k}_{\Sigma_0^{\tilde{g}}}\right).
\end{aligned}$$

Note that $X_{\mathrm{YM}}^{0,0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$ is empty unless $\frac{k}{n} \in \mathbb{Z}$ and $X_{\mathrm{YM}}^{0,0}(U(n))_{d,\ldots,d}$ consists of a point if $d \in \mathbb{Z}$.

The surjective maps $\Phi^{\ell,i} : Z_{\mathrm{YM}}^{\ell,i}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} \to X_{\mathrm{YM}}^{2\ell+i-1,0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$ are given by

$$\Phi^{\ell,1}\Big(V, c, V', c', -2\sqrt{-1}\pi \frac{k}{n} I_n \Big) = \Big(V, c\mathfrak{r}\left(V'\right) c^{-1}, -2\sqrt{-1}\pi \frac{k}{n} I_n \Big)$$

$$\Phi^{\ell,2}\Big(V, d, c, V', d', c', -2\sqrt{-1}\pi \frac{k}{n} I_n \Big) = \Big(V, d^{-1} c\mathfrak{r}\left(V'\right) c^{-1} d, d^{-1}, cc', -2\sqrt{-1}\pi \frac{k}{n} I_n \Big)$$

In particular, when $n = 1$, $k \in \mathbb{Z}$, we have

$$\begin{aligned}
Z^{\ell,1}_{\mathrm{YM}}(U(1))_k &= \left\{\left(V, c, V', (-1)^k c^{-1}, -2\sqrt{-1}\pi k\right) \mid V, V \in U(1)^{2\ell},\ c \in U(1)\right\} \\
&\cong U(1)^{4\ell+1}, \\
Z^{\ell,2}_{\mathrm{YM}}(U(1))_k &= \left\{(V, d, c, V', (-1)^k d^{-1}, c', -2\sqrt{-1}\pi k) \mid V, V' \in U(1)^{2\ell},\ d, c, c' \in U(1)\right\} \\
&\cong U(1)^{4\ell+3}, \\
X^{\tilde{g},0}_{\mathrm{YM}}(U(1))_k &= \left\{\left(V, -2\sqrt{-1}\pi k\right) \mid V \in U(1)^{2\tilde{g}}\right\} \cong U(1)^{2\tilde{g}}.
\end{aligned}$$

The maps $\Phi^{\ell,i} : Z^{\ell,i}_{\mathrm{YM}}(U(1))_k \cong U(1)^{4\ell+2i-1} \to X^{2\ell+i-1,0}_{\mathrm{YM}}(U(1))_k \cong U(1)^{4\ell+2i-2}$, $i = 1, 2$, are given by

$$\begin{aligned}
\Phi^{\ell,1}\left(V, c, V', (-1)^k c^{-1}, -2\sqrt{-1}\pi k\right) &= \left(V, \mathfrak{r}\left(V'\right), -2\sqrt{-1}\pi k\right) \\
\Phi^{\ell,2}\left(V, d, c, V', (-1)^k d^{-1}, c', -2\sqrt{-1}\pi k\right) &= \left(V, \mathfrak{r}\left(V'\right), d^{-1}, cc', -2\sqrt{-1}\pi k\right)
\end{aligned}$$

3.4 Vanishing of equivariant Euler class

Let $\mathbb{V}_{n_j,k_j} \to \mathcal{N}_{ss}\left(P^{n_j,k_j}_{\tilde{\Sigma}}\right)$ be the real vector bundle whose fiber over $\mathcal{D}_j \in \mathcal{N}_{ss}\left(P^{n_j,k_j}\right)$ is $H^1\left(\tilde{\Sigma}, \mathrm{Hom}\left(\mathcal{D}_j, \tau_{\mathcal{C}}\left(\mathcal{D}_j\right)\right)\right)^{\tau}$. Then, $\mathbb{V}_{n_j,k_j}$ is a $\mathcal{G}_j$-equivariant real vector bundle of rank $2n_j k_j + n_j^2(\tilde{g} - 1)$, where $\mathcal{G}_j = \mathcal{G}\left(P^{n_j,k_j}_{\tilde{\Sigma}}\right)$ and $\tilde{g}$ is the genus of $\tilde{\Sigma}$.

For $j = 1, \ldots, r$, let

$$p_j : \mathcal{N}_{ss}\left(P^{n_0,\pm}_{\Sigma}\right) \times \prod_{i=1}^{r} \mathcal{N}_{ss}(P^{n_i,k_i}_{\tilde{\Sigma}}) \longrightarrow \mathcal{N}_{ss}\left(P^{n_j,k_j}_{\tilde{\Sigma}}\right)$$

be the natural projection. Under the isomorphism of equivariant pairs

$$\left(\mathcal{N}_{\mu}, \mathcal{G}\left(P^{n,\pm}_{\Sigma}\right)\right) \cong \left(\mathcal{N}_{ss}\left(P^{n_0,\pm}_{\Sigma}\right), \mathcal{G}(P^{n_0,\pm}_{\Sigma}\right) \times \prod_{j=1}^{r}\left(\mathcal{N}_{ss}\left(P^{n_j,k_j}_{\tilde{\Sigma}}\right), \mathcal{G}_j\right)$$

the $\mathcal{G}$-equivariant vector bundle $\mathbb{N}^{\mathbb{R}}_{\mu}$ over $\mathcal{N}_{\mu}$ is isomorphic to the $\prod_{j=1}^{r} \mathcal{G}_j$-equivariant vector bundle $\bigoplus_{j=1}^{r} p_j^* \mathbb{V}_{n_j,k_j}$ over $\prod_{j=1}^{r} \mathcal{N}_{ss}\left(P^{n_j,k_j}_{\tilde{\Sigma}}\right)$. In other words, there is a homeomorphism of the total spaces of vector bundles

$$\left(\mathbb{N}^{\mathbb{R}}_{\mu}\right)^{h\mathcal{G}} \cong \bigoplus_{j=1}^{r} p_j^* \mathbb{V}^{h\mathcal{G}_j}_{n_j,k_j}$$

which covers the homeomorphism of the bases

$$\mathcal{N}_{\mu}^{h\mathcal{G}} \cong \mathcal{N}_{ss}\left(P_{\Sigma}^{n_0,\pm}\right)^{h\mathcal{G}\left(P_{\Sigma}^{n_0,\pm}\right)} \times \prod_{j=1}^{r} \mathcal{N}_{ss}\left(P_{\tilde{\Sigma}}^{n_j,k_j}\right)^{h\mathcal{G}_j}.$$

So

$$e_{\mathcal{G}}\left(\mathbb{N}_{\mu}^{\mathbb{R}}\right) = \prod_{j=1}^{r} e_{\mathcal{G}_j}\left(\mathbb{V}_{n_j,k_j}\right),$$

The $\mathcal{G}_j$-equivariant vector bundle $\mathbb{V}_{n_j,k_j} \to \mathcal{N}_{ss}\left(P_{\tilde{\Sigma}}^{n_j,k_j}\right)$ descends to a $U(n_j)$-equivariant vector bundle V_{n_j,k_j} over $X_{\mathrm{YM}}^{\tilde{g},0}\left(U\left(n_j\right)\right)_{\frac{k_j}{n_j},\ldots,\frac{k_j}{n_j}}$, and $e_{\mathcal{G}_j}\left(\mathbb{V}_{n_j,k_j}\right)$ descends to $e_{U(n_j)}\left(V_{n_j,k_j}\right)$.

In the remainder of this subsection, we use rational coefficient $\mathbb{Q}$.

Lemma 22 *When $n = 1, k > 0$, the $U(1)$-action on $V_{1,k}$ is trivial, and*

$$e_{U(1)}(V_{1,k}) = e(V_{1,k}) = 0$$

Proof The $U(1)$-action is similar to that in Lemma 21.

We first review some discussion in [13, Sect. 6.2]. Given $c \in U(1)$, let $\bar{c} = c^{-1}$ denote the complex conjugate. Given $V = (a_1, b_1, \ldots, a_\ell, b_\ell) \in U(1)^{2\ell}$ and $V' = \left(a_1', b_1', \ldots, a_\ell', b_\ell'\right)$, let

$$\bar{V} = (\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_\ell, \bar{b}_\ell), \quad VV' = \left(a_1a_1', b_1b_1', \ldots, a_\ell a_\ell', b_\ell b_\ell'\right).$$

The map $\mathcal{L} \mapsto \mathrm{Hom}(\mathcal{L}, \tau_{\mathcal{C}}(\mathcal{L})) = \mathcal{L}^\vee \otimes \tau^*\overline{\mathcal{L}^\vee}$, where $\mathcal{L}$ is a degree $k > 0$ holomorphic line bundle over $\tilde{\Sigma}$, induces a map $\phi_Z : Z_{\mathrm{YM}}^{\ell,i}(U(1))_k \longrightarrow Z_{\mathrm{YM}}^{\ell,i}(U(1))_{-2k}$ given by

$$\begin{aligned}
&\left(V, c, V', (-1)^k\bar{c}, -2\sqrt{-1}\pi k\right)\\
&\quad\mapsto \left(\bar{V}V', (-1)^k\bar{c}^2, \bar{V}'V, (-1)^k c^2, 4\sqrt{-1}\pi k\right), \quad i = 1,\\
&\left(V, d, c, V', (-1)^k\bar{d}, c', -2\sqrt{-1}\pi k\right)\\
&\quad\mapsto \left(\bar{V}V', (-1)^k\bar{d}^2, \bar{c}c', \bar{V}'V, (-1)^k d^2, \bar{c}'c, 4\sqrt{-1}\pi k\right), \quad i = 2.
\end{aligned}$$

It descends to a map $\phi_X : X_{\mathrm{YM}}^{2\ell+i-1,0}(U(1))_k \longrightarrow X_{\mathrm{YM}}^{2\ell+i-1,0}(U(1))_{-2k}$ given by

$$\begin{aligned}
\left(V_1, V_2, -2\sqrt{-1}k\right) &\mapsto \left(\mathfrak{r}(V_2)\bar{V}_1, \mathfrak{r}(V_1)\bar{V}_2, 4\sqrt{-1}\pi k\right), \quad i = 1\\
\left(V_1, V_2, d, c, -2\sqrt{-1}k\right) &\mapsto \left(\mathfrak{r}(V_2)\bar{V}_1, \mathfrak{r}(V_1)\bar{V}_2, (-1)^k\bar{d}^2, 1, 4\sqrt{-1}\pi k\right), \quad i = 2.
\end{aligned}$$

The map $\mathcal{M} \mapsto \tau^*\overline{\mathcal{M}}$, where $\mathcal{M}$ is a degree $-2k$ holomorphic line bundle over $\tilde{\Sigma}$, induces an involution $\hat{\tau}_Z : Z^{\ell,i}_{\mathrm{YM}}(U(1))_{-2k} \to Z^{\ell,i}_{\mathrm{YM}}(U(1))_{-2k}$ given by

$$\left(V, c, V', \bar{c}, 4\sqrt{-1}\pi k\right) \mapsto \left(\bar{V}', c, \bar{V}, \bar{c}, 4\sqrt{-1}\pi k\right), \quad i = 1,$$
$$\left(V, d, c, V', \bar{d}, c', 4\sqrt{-1}\pi k\right) \mapsto \left(\bar{V}', d, \bar{c}', \bar{V}, \bar{d}, \bar{c}, 4\sqrt{-1}\pi k\right), \quad i = 2.$$

It descends to an involution $\hat{\tau}_X : X^{2\ell+i-1,0}_{\mathrm{YM}}(U(1))_{-2k} \longrightarrow X^{2\ell+i-1,0}_{\mathrm{YM}}(U(1))_{-2k}$ given by

$$\left(V_1, V_2, 4\sqrt{-1}\pi k\right) \mapsto \left(\mathfrak{r}(\bar{V}_2), \mathfrak{r}(\bar{V}_1), 4\sqrt{-1}\pi k\right), \quad i = 1$$
$$\left(V_1, V_2, d, c, 4\sqrt{-1}\pi k\right) \mapsto \left(\mathfrak{r}(\bar{V}_2), \mathfrak{r}(\bar{V}_1), d, \bar{c}, 4\sqrt{-1}\pi k\right), \quad i = 2.$$

We have

$$\mathrm{Im}\,\phi_Z = Z^{\ell,i}_{\mathrm{YM}}(U(1))^{\hat{\tau}_Z}_{-2k} \cong U(1)^{2\ell+i}, \ \ \mathrm{Im}\,\phi_X = X^{2\ell+i-1,0}_{\mathrm{YM}}(U(1))^{\hat{\tau}_X}_{-2k} \cong U(1)^{2\ell+i-1}.$$

Let $U_k \to Z^{\ell,i}_{\mathrm{YM}}(U(1))_{-2k}$ and $F_k \to X^{2\ell+i-1,0}_{\mathrm{YM}}(U(1))_{-2k}$ be the vector bundles whose fiber over $\mathcal{M}$ is $H^1(\tilde{\Sigma}, \mathcal{M})$. Then, the involution $\hat{\tau}_Z$ (resp. $\hat{\tau}_X$) lifts to an involution on U_k (resp. F_k):

$$(U_k)_{\mathcal{M}} = (F_k)_{\mathcal{M}} = H^1(\tilde{\Sigma}, \mathcal{M}) \longrightarrow (U_k)_{\tau^*\overline{\mathcal{M}}} = (F_k)_{\tau^*\overline{\mathcal{M}}} = H^1\left(\tilde{\Sigma}, \tau^*\overline{\mathcal{M}}\right).$$

The fixed locus $U_k^{\hat{\tau}_Z}$ (resp. $F_k^{\hat{\tau}_X}$) is a real vector bundle over $Z^{\ell,i}_{\mathrm{YM}}(U(1))^{\hat{\tau}_Z}_{-2k}$ (resp. $X^{2\ell+i-1,0}_{\mathrm{YM}}(U(1))^{\hat{\tau}_X}_{-2k}$). Let $W_k \to Z^{\ell,i}_{\mathrm{YM}}(U(1))_k$ be the vector bundle whose fiber over $\mathcal{L}$ is $H^1(\tilde{\Sigma}, \mathrm{Hom}(\mathcal{L}, \tau_{\mathcal{C}}(\mathcal{L}))^{\tau}$. Then,

$$\phi_Z^* U_k^{\hat{\tau}_Z} = W_k, \quad \phi_X^* F_k^{\hat{\tau}_X} = V_{1,k},$$
$$\mathrm{rank}_{\mathbb{R}} V_{1,k} = \mathrm{rank}_{\mathbb{R}} F_k^{\hat{\tau}_X} = \mathrm{rank}_{\mathbb{C}} F_k = 2k + 2\ell + i - 2.$$

The $U(1)$-action on $\mathrm{Hom}(\mathcal{L}, \tau_{\mathcal{C}}(\mathcal{L}))$ is given by $t \cdot t^{-1}$, and thus the weights of the $U(1)$-action on $(V_{1,k})_{\mathcal{L}} = H^1(\tilde{\Sigma}, \mathrm{Hom}(\mathcal{L}, \tau_{\mathcal{C}}(\mathcal{L})))^{\tau}$ are also given by $t \cdot t^{-1}$, which is trivial. So $e_{U(1)}(V_{1,k}) = e(V_{1,k})$.

We have

$$\mathrm{rank}_{\mathbb{R}} F_k^{\hat{\tau}_X} = 2k + 2\ell + i - 2 > 2\ell + i - 1 = \dim_{\mathbb{R}} X^{2\ell+i-1,0}_{\mathrm{YM}}(U(1))^{\hat{\tau}_X}_{-2k}$$

since $k > 0$. So $e\left(F_k^{\hat{\tau}_X}\right) = 0$. Therefore,

$$e(V_{1,k}) = \phi_X^* e(F_k^{\hat{\tau}_X}) = 0.$$

□

Proof of Theorem 13 Part (ii) follows from Lemma 22. For part (i), recall that $X^{0,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$ is empty unless $\frac{k}{n} \in \mathbb{Z}$ and $X^{0,0}_{\mathrm{YM}}(U(n))_{d,\cdots,d}$ consists of a point if $d \in \mathbb{Z}$. We need to prove that for any positive integers $n, d > 0$,

$$e_{U(n)}(V_{n,nd}) \in H^*_{U(n)}\left(X^{0,0}_{\mathrm{YM}}(U(n))_{d,\ldots,d};\mathbb{Q}\right)$$

is zero. Since $Y_{n,d} := X^{0,0}_{\mathrm{YM}}(U(n))_{d,\ldots,d}$ is a point, the inclusion of the maximal torus $T = U(1)^n \subset U(n)$ induces an injective ring homomorphism

$$\beta : H^*_{U(n)}(Y_{n,d};\mathbb{Q}) \cong \mathbb{Q}[u_1,\ldots,u_n]^{S_n} \to H^*_T(Y_{n,d};\mathbb{Q}) \cong \mathbb{Q}[u_1,\ldots,u_n].$$

So it suffices to show that $e_T(V_{n,nd}) = \beta\left(e_{U(n)}(V_{n,nd})\right)$ is zero. We have

$$\begin{aligned} V_{n,nd} &= H^1\left(\mathbb{P}^1, \mathrm{Hom}\left(\bigoplus_{i=1}^n \mathcal{L}_i, \bigoplus_{j=1}^n \tau_{\mathcal{C}}\left(\mathcal{L}_j\right)\right)\right)^\tau \\ &\cong \bigoplus_{i<j} H^1\left(\mathbb{P}^1, \mathcal{L}_i^\vee \otimes \tau_{\mathcal{C}}\left(\mathcal{L}_j\right)\right) \oplus \bigoplus_{i=1}^n H^1\left(\mathbb{P}^1, \mathcal{L}_i^\vee \otimes \tau_{\mathcal{C}}(\mathcal{L}_i)\right)^\tau \end{aligned}$$

where $\mathcal{L}_i = \mathcal{O}_{\mathbb{P}^1}(d)$ for $i = 1,\ldots,n$ and $\tau_{\mathcal{C}}\left(\mathcal{L}_j\right) = \mathcal{O}_{\mathbb{P}^1}(-d)$ for $j = 1,\ldots,n$. The weights of T-action on $H^1\left(\mathbb{P}^1, \mathcal{L}_i^\vee \otimes \tau_{\mathcal{C}}\left(\mathcal{L}_j\right)\right)$ is $t_j t_i^{-1}$, where $(t_1,\ldots,t_n) \in U(1)^n = T$. Let

$$V_{\mathbb{C}} = \bigoplus_{i<j} H^1\left(\mathbb{P}^1, \mathcal{L}_i^{-1} \otimes \tau_{\mathcal{C}}\left(\mathcal{L}_j\right)\right), \quad V_{\mathbb{R}} = \bigoplus_{i=1}^n H^1\left(\mathbb{P}^1, \mathcal{L}_i^{-1} \otimes \tau_{\mathcal{C}}(\mathcal{L}_i)\right)^\tau.$$

Then,

$$V_{n,nd} = V_{\mathbb{C}} \oplus V_{\mathbb{R}}$$

where $V_{\mathbb{C}}$ is a complex vector space, $V_{\mathbb{R}}$ is a real vector space on which T-acts trivially, and

$$\dim_{\mathbb{R}} V_{n,nd} = n^2(2d-1), \quad \dim_{\mathbb{C}} V_{\mathbb{C}} = \frac{n(n-1)}{2}(2d-1), \quad \dim_{\mathbb{R}} V_{\mathbb{R}} = n(2d-1).$$

We have

$$e_T(V_{n,nd}) = e_T(V_{\mathbb{C}}) e_T(V_{\mathbb{R}}),$$

where

$$e_T(V_{\mathbb{C}}) = \pm \prod_{i<j}(u_i - u_j)^{2d-1}, \quad e_T(V_{\mathbb{R}}) = 0,$$

since $\mathrm{rank}_{\mathbb{R}} V_{\mathbb{R}} = \dim_{\mathbb{R}} V_{\mathbb{R}} > 0 = \dim_{\mathbb{R}} Y_{n,d}$. Therefore, $e_T(V_{n,nd}) = 0$. □

4 Equivariant Poincaré series

By P5 of Lemma 10, the stratification is equivariantly $\mathbb{Q}$-perfect if and only if

$$P_t^{\mathcal{G}}(\mathcal{A}_{ss}; \mathbb{Q}) = P_t^{\mathcal{G}}(\mathcal{A}; \mathbb{Q}) - \sum_{\mu \in \Lambda'} t^{\lambda_\mu} P_t^{\mathcal{G}}(\mathcal{A}_\mu; \mathbb{Q}) \tag{22}$$

By A5 of Lemma 11, the stratification is equivariantly $\mathbb{Q}$-antiperfect if and only if

$$P_t^{\mathcal{G}}(\mathcal{A}_{ss}; \mathbb{Q}) = P_t^{\mathcal{G}}(\mathcal{A}; \mathbb{Q}) + \sum_{\mu \in \Lambda'} t^{\lambda_\mu - 1} P_t^{\mathcal{G}}(\mathcal{A}_\mu; \mathbb{Q}). \tag{23}$$

4.1 Representation varieties for flat connections

A flat G-connection on Σ gives rise to a homomorphism $\pi_1(\Sigma) \to G$. Recall that

$$\pi_1\left(\Sigma_1^\ell\right) = \langle A_1, B_1, \ldots, A_\ell, B_\ell, C \mid \prod_{i=1}^{\ell}[A_i, B_i] = C^2\rangle,$$

$$\pi_1\left(\Sigma_2^\ell\right) = \langle A_1, B_1, \ldots, A_\ell, B_\ell, D, C \mid \prod_{i=1}^{\ell}[A_i, B_i] = CDC^{-1}D\rangle.$$

Representation varieties of flat $U(n)$-connections and $SU(n)$-connections on Σ_1^ℓ and Σ_2^ℓ are given by

$$\begin{aligned}
X_{\mathrm{flat}}^{\ell,1}(U(n)) &= \left\{(V, c) \mid V \in U(n)^{2\ell}, c \in U(n), \mathfrak{m}(V) = c^2\right\}\\
X_{\mathrm{flat}}^{\ell,1}(U(n))_{\pm 1} &= \left\{(V, c) \in X_{\mathrm{flat}}^{\ell,1}(U(n)) \mid \det c = \pm 1\right\}\\
X_{\mathrm{flat}}^{\ell,1}(SU(n)) &= \left\{(V, c) \mid V \in SU(n)^{2\ell}, c \in SU(n), \mathfrak{m}(V) = c^2\right\}\\
X_{\mathrm{flat}}^{\ell,2}(U(n)) &= \left\{(V, d, c) \mid V \in U(n)^{2\ell}, d, c \in U(n), \mathfrak{m}(V) = cdc^{-1}d\right\}\\
X_{\mathrm{flat}}^{\ell,2}(U(n))_{\pm 1} &= \left\{(V, d, c) \in X_{\mathrm{flat}}^{\ell,1}(U(n)) \mid \det d = \pm 1\right\}\\
X_{\mathrm{flat}}^{\ell,2}(SU(n)) &= \left\{(V, d, c) \mid V \in SU(n)^{2\ell}, d, c \in SU(n), \mathfrak{m}(V) = cdc^{-1}d\right\}
\end{aligned}$$

For $i=1,2$,

$$\mathrm{Hom}\left(\pi_1\left(\Sigma_i^\ell\right), U(n)\right)_{\pm 1} = X_{\mathrm{flat}}^{\ell,i}(U(n))_{\pm 1},$$
$$\mathrm{Hom}\left(\pi_1\left(\Sigma_i^\ell\right), SU(n)\right) = X_{\mathrm{flat}}^{\ell,i}(SU(n)).$$

4.2 Rank 2 case

Proof of Theorem 16 There are two possible principal $U(2)$-bundles $P^{2,+}_{\Sigma_i^\ell}$, $P^{2,-}_{\Sigma_i^\ell}$ over the nonorientable surface Σ_i^ℓ. In notation in Sect. 3.1,

$$\begin{aligned} &I_2^0 = \{(0,0)\} \\ &I_2^+\left(\Sigma_1^\ell\right) = I_2^-\left(\Sigma_2^\ell\right) = \{(2r-1, 1-2r) \mid r \in \mathbb{Z}_{>0}\}, \\ &I_2^-\left(\Sigma_1^\ell\right) = I_2^+\left(\Sigma_2^\ell\right) = \{(2r, -2r) \mid r \in \mathbb{Z}_{>0}\}. \end{aligned}$$

So when $\mathcal{A} = \mathcal{A}\left(P_\Sigma^{2,\pm}\right)$, $\Lambda' = I_2^\pm(\Sigma)$.

Let $\tilde{g} = 2\ell + i - 1$ be the genus of the oriented double cover of Σ_i^ℓ. From [11, Example 7.5], The codimension of each stratum is

$$d_{r,-r} = 2r + \tilde{g} - 1,$$

and the equivariant Poincaré series for stratum $\mu = (r, -r)$ is

$$\begin{aligned} P_t^{\mathcal{G}}\left(\mathcal{A}\left(\Sigma_i^\ell\right)_{r,-r}; \mathbb{Q}\right) &= P_t^{U(2)}\left(X_{\mathrm{YM}}^{\ell,i}(U(2))_{r,-r}; \mathbb{Q}\right) = P_t^{U(1)}\left(X_{\mathrm{YM}}^{\tilde{g},0}(U(1))_r; \mathbb{Q}\right) \\ &= P_t^{U(1)}(U(1)^{2\tilde{g}}) = \frac{(1+t)^{2\tilde{g}}}{1-t^2}. \end{aligned}$$

By [12, Theorem 2.5],

$$P_t^{\mathcal{G}}(\mathcal{A}; \mathbb{Q}) = P_t(B\mathcal{G}; \mathbb{Q}) = \frac{(1+t)^{\tilde{g}}\left(1+t^3\right)^{\tilde{g}}}{\left(1-t^2\right)\left(1-t^4\right)}.$$

We have

$$\sum_{r \text{ odd}} t^{d_{r,-r}-1} = \frac{t^{\tilde{g}}}{1-t^4}, \quad \sum_{r \text{ even}} t^{d_{r,-r}-1} = \frac{t^{\tilde{g}+2}}{1-t^4}.$$

Therefore, (23) is equivalent to the following identities

$$P_t^{U(2)}\left(X_{\text{flat}}^{\ell,i}(U(2))_{(-1)^i};\mathbb{Q}\right)=P_t(B\mathcal{G};\mathbb{Q})+\sum_{r\text{ even}}t^{d_{r,-r}-1}P_t^{\mathcal{G}}\left(\mathcal{A}\left(\Sigma_i^\ell\right)_{r,-r};\mathbb{Q}\right)$$
$$=\frac{(1+t)^{\tilde g}}{\left(1-t^2\right)\left(1-t^4\right)}\left(\left(1+t^3\right)^{\tilde g}+t^{\tilde g+2}(1+t)^{\tilde g}\right),$$
$$P_t^{U(2)}\left(X_{\text{flat}}^{\ell,i}(U(2))_{(-1)^{i+1}};\mathbb{Q}\right)=P_t(B\mathcal{G};\mathbb{Q})+\sum_{r\text{ odd}}t^{d_{r,-r}-1}P_t^{\mathcal{G}}\left(\mathcal{A}\left(\Sigma_i^\ell\right)_{r,-r};\mathbb{Q}\right)$$
$$=\frac{(1+t)^{\tilde g}}{\left(1-t^2\right)\left(1-t^4\right)}\left(\left(1+t^3\right)^{\tilde g}+t^{\tilde g}(1+t)^{\tilde g}\right).$$

We now consider the principal $SU(2)$-bundles $Q^2_{\Sigma_i^\ell}\cong\Sigma_i^\ell\times SU(2)$ over the non-orientable surface Σ_i^ℓ together with the gauge group $\mathcal{G}'=\text{Aut}\left(Q^2_{\Sigma_i^\ell}\right)$ action. The set of Harder-Narasimhan types is $I_2^0\cup I_2^+\left(\Sigma_i^\ell\right)$, so

$$\Lambda'=\{(r,-r)\mid r\in\mathbb{Z}_{>0},\ r=i\ (\text{mod }2)\ \}.$$

The codimension of $\mathcal{A}'_{r,-r}$ in $\mathcal{A}(Q^2_{\Sigma_i^\ell})$ is the same as the codimension of $\mathcal{A}_{r,-r}$ in $\mathcal{A}\left(P^{2,+}_{\Sigma_i^\ell}\right)$, which is $d_{r,-r}=2r+\tilde g-1$.

We now derive the reduction formula for each stratum $\mu=(r,-r), r>0$. The corresponding representation varieties are

$$X_{\text{YM}}^{\ell,1}(SU(2))_\mu=\Big\{(V,c,X)\in SU(2)^{2\ell+1}\times C_{\mu/2}|\ V\in(SU(2)_X)^{2\ell},$$
$$\text{Ad}(c)X=-X,\mathfrak{m}(V)=\exp(X)c^2\Big\},$$
$$X_{\text{YM}}^{\ell,2}(SU(2))_\mu=\Big\{(V,d,c,X)\in SU(2)^{2\ell+2}\times C_{\mu/2}|\ (V,d)\in(SU(2)_X)^{2\ell+1},$$
$$\text{Ad}(c)(X)=-X,\ \mathfrak{m}(V)=\exp(X)cdc^{-1}d\Big\}.$$

where $C_{\mu/2}$ is the orbit of $X_\mu/2=-\pi\sqrt{-1}\text{diag}(r,-r)\in\mathfrak{su}(2)$ under the Adjoint action of $SU(2)$ on $\mathfrak{su}(2)$. Let

$$\epsilon=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Then, $\text{Ad}(\epsilon)(X_\mu)=-X_\mu$. Note that

$$SU(2)_{X_\mu}=\Big\{\text{diag}(u,u^{-1})\mid u\in U(1)\Big\}\cong U(1),\quad \exp(X_\mu/2)=(-1)^rI_2.$$

For $\mu \in \Lambda' = \{(r, -r) \mid r > 0, r = i \pmod 2\}$, define $V^{\ell,i}(SU(2))_\mu$ as follows:

$$\begin{aligned}
V^{\ell,1}(SU(2))_\mu &= \Big\{(V, c) \in SU(2)^{2\ell+1} \mid V \in \left(SU(2)_{X_\mu}\right)^{2\ell}, \\
&\qquad \mathrm{Ad}(c)X_\mu = -X_\mu, c^2 = -I_2\Big\}, \\
&\overset{c=c'\epsilon}{\cong} \Big\{\left(V, c'\right) \mid V \in \left(SU(2)_{X_\mu}\right)^{2\ell}, c' \in SU(2)_{X_\mu}\Big\} \cong U(1)^{2\ell+1} \\
V^{\ell,2}(SU(2))_\mu &= \Big\{(V, d, c) \in SU(2)^{2\ell+2} \mid (V, d) \in \left(SU(2)_{X_\mu}\right)^{2\ell+1}, \\
&\qquad \mathrm{Ad}(c)(X_\mu) = -X_\mu, cdc^{-1}d = I_2\Big\} \\
&\overset{c=c'\epsilon}{\cong} \Big\{\left(V, d, c'\right) \mid V \in \left(SU(2)_{X_\mu}\right)^{2\ell}, d, c' \in SU(2)_{X_\mu}\Big\} \cong U(1)^{2\ell+2}
\end{aligned}$$

Let $SU(2)_{X_\mu}$ act on $V^{\ell,i}(SU(2))_\mu$ by

$$u \cdot \left(V, c'\right) = \left(uVu^{-1}, \epsilon^{-1}u\epsilon c'u^{-1}\right) = \left(V, u^{-2}c'\right), \quad u \cdot \left(V, d, c'\right) = \left(V, d, u^{-2}c'\right)$$

Then, the following equivariant pairs are equivalent

$$\left(X^{\ell,i}_{\mathrm{YM}}(SU(2))_\mu, SU(2)\right) \cong \left(V^{\ell,i}(SU(2))_\mu, SU(2)_{X_\mu}\right) \cong \left(U(1)^{2\ell+i}, U(1)\right).$$

Thus, the $\mathcal{G}'$-equivariant Poincaré series for stratum $\mathcal{A}'_{r,-r}$ is

$$P^{\mathcal{G}'}_t\left(\mathcal{A}'_{r,-r}; \mathbb{Q}\right) = P^{SU(2)}_t\left(X^{\ell,i}_{\mathrm{YM}}(SU(2))_{r,-r}; \mathbb{Q}\right) = P_t\left(U(1)^{\tilde{g}}; \mathbb{Q}\right) = (1+t)^{\tilde{g}},$$

where $\tilde{g} = 2\ell + i - 1$. By [12, Theorem 2.5],

$$P^{\mathcal{G}'}_t\left(\mathcal{A}\left(Q^2_{\Sigma^\ell_i}\right); \mathbb{Q}\right) = P_t\left(B\mathcal{G}'; \mathbb{Q}\right) = \frac{\left(1+t^3\right)^{\tilde{g}}}{1-t^4}.$$

Therefore, (23) is equivalent to the following identities

$$\begin{aligned}
P^{SU(2)}_t\left(X^{\ell,1}_{\mathrm{flat}}(SU(2)); \mathbb{Q}\right) &= P_t\left(B\mathcal{G}'; \mathbb{Q}\right) + \sum_{r \text{ odd}} t^{d_{r,-r}-1}(1+t)^{\tilde{g}} \\
&= \frac{\left(1+t^3\right)^{\tilde{g}} + t^{\tilde{g}}(1+t)^{\tilde{g}}}{1-t^4} \\
P^{SU(2)}_t\left(X^{\ell,2}_{\mathrm{flat}}(SU(2)); \mathbb{Q}\right) &= P_t\left(B\mathcal{G}'; \mathbb{Q}\right) + \sum_{r \text{ even}} t^{d_{r,-r}-1}(1+t)^{\tilde{g}} \\
&= \frac{\left(1+t^3\right)^{\tilde{g}} + t^{\tilde{g}+2}(1+t)^{\tilde{g}}}{1-t^4}.
\end{aligned}$$

□

4.3 Rank 3 case

Proof of Theorem 19 There are two possible principal $U(3)$-bundles $P^{3,+}_{\Sigma^\ell_i}$, $P^{3,-}_{\Sigma^\ell_i}$ over the nonorientable surface Σ^ℓ_i. In the notation of Sect. 3.1,

$$I_3 = I_3^0 = \{(0,0,0)\} \cup \{(r,0,-r) \mid r \in \mathbb{Z}_{>0}\}$$

So when $\mathcal{A} = \mathcal{A}\left(P^{3,\pm}_{\Sigma}\right)$, $\Lambda' = \{(r,0,-r)\} \mid r \in \mathbb{Z}_{>0}\}$.

Let $\tilde{g} = 2\ell + i - 1$ be the genus of the oriented double cover of Σ^ℓ_i. From [11, Example 7.6], the codimension of each stratum is

$$d_{r,0,-r} = 4r + 3(\tilde{g} - 1),$$

and the equivariant Poincaré series for stratum $\mu = (r,0,-r)$ is

$$\begin{aligned} P_t^{\mathcal{G}}\left(\mathcal{A}\left(\Sigma^\ell_i\right)_{r,0,-r};\mathbb{Q}\right) &= P_t^{U(3)}\left(X^{\ell,i}_{\mathrm{YM}}(U(3))_{r,0,-r};\mathbb{Q}\right) = P_t^{U(1)\times U(1)}\left(U(1)^{3\tilde{g}};\mathbb{Q}\right) \\ &= \frac{(1+t)^{3\tilde{g}}}{\left(1-t^2\right)^2}. \end{aligned}$$

By [12, Theorem 2.5],

$$P_t^{\mathcal{G}}(\mathcal{A};\mathbb{Q}) = P_t(B\mathcal{G};\mathbb{Q}) = \frac{(1+t)^{\tilde{g}}\left(1+t^3\right)^{\tilde{g}}\left(1+t^5\right)^{\tilde{g}}}{\left(1-t^2\right)\left(1-t^4\right)\left(1-t^6\right)}.$$

Therefore, (23) is equivalent to the following identity

$$\begin{aligned} P_t^{U(3)}\left(X^{\ell,i}_{\mathrm{flat}}(U(3))_{\pm 1};\mathbb{Q}\right) &= P_t(B\mathcal{G};\mathbb{Q}) + \sum_{r>0} t^{d_{r,0,-r}-1} P_t^{\mathcal{G}}\left(\mathcal{A}\left(\Sigma^\ell_i\right)_{r,0,-r};\mathbb{Q}\right) \\ &= \frac{(1+t)^{\tilde{g}}}{\left(1-t^2\right)\left(1-t^4\right)\left(1-t^6\right)}\left(\left(1+t^3\right)^{\tilde{g}}\left(1+t^5\right)^{\tilde{g}} \right. \\ &\quad \left. + t^{3\tilde{g}}(1+t)^{2\tilde{g}}\left(1+t^2+t^4\right)\right). \end{aligned}$$

We now consider the principal $SU(3)$-bundles $Q^3_{\Sigma^\ell_i} \cong \Sigma^\ell_i \times SU(3)$ over the nonorientable surface Σ^ℓ_i together with the gauge group $\mathcal{G}' = \mathrm{Aut}\left(Q^3_{\Sigma^\ell_i}\right)$ action. The set of Harder-Narasimhan types is I^0_3, so $\Lambda' = \{(r,0,-r) \mid r \in \mathbb{Z}_{>0}\}$. The codimension of $\mathcal{A}'_{r,0,-r}$ in $\mathcal{A}\left(Q^3_{\Sigma^\ell_i}\right)$ is the same as the codimension of $\mathcal{A}_{r,0,-r}$ in $\mathcal{A}(P^{3,+}_{\Sigma^\ell_i})$, which is $d_{r,0,-r} = 4r + 3(\tilde{g} - 1)$.

We now derive the reduction formula for each stratum $\mu = (r, 0, -r)$. The corresponding representation varieties are

$$X^{\ell,1}_{\mathrm{YM}}(SU(3))_{r,0,-r} = \Big\{(V, c, X) \in SU(3)^{2\ell+1} \times C_{\mu/2} |\, V \in (SU(3)_X)^{2\ell}, \\ Ad(c)X = -X, \mathfrak{m}(V) = \exp(X)c^2\Big\},$$

$$X^{\ell,2}_{\mathrm{YM}}(SU(3))_{r,0,-r} = \Big\{(V, d, c, X) \in SU(3)^{2\ell+2} \times C_{\mu/2} |\, (V, d) \in (SU(3)_X)^{2\ell+1}, \\ \mathrm{Ad}(c)(X) = -X,\ \mathfrak{m}(V) = \exp(X)cdc^{-1}d\Big\}.$$

where $C_{\mu/2}$ is the orbit of $X_\mu/2 = -\pi\sqrt{-1}\mathrm{diag}(r, 0, -r) \in \mathfrak{su}(3)$ under the Adjoint action of $SU(3)$ on $\mathfrak{su}(3)$. Let

$$\epsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in SU(3).$$

Then, $\mathrm{Ad}(\epsilon)X_\mu = -X_\mu$. Note that

$$SU(3)_{X_\mu} = \{\mathrm{diag}(u_1, u_2, u_3) \mid u_1, u_2, u_3 \in U(1), u_1u_2u_3 = 1\} \cong U(1) \times U(1), \\ \exp(X_\mu/2) = \mathrm{diag}\left((-1)^r, 1, (-1)^r\right).$$

Given $\mu \in \Lambda' = \{(r, 0, -r) \mid r \in \mathbb{Z}_{>0}\}$, define $V^{\ell,i}(SU(3))_\mu$ as follows:

$$V^{\ell,1}(SU(3))_\mu \\ = \left\{\left(V, c'\right) \in \left(SU(3)_{X_\mu}\right)^{2\ell+1} |\, \mathfrak{m}(V) = \exp(X_\mu/2)\left(\epsilon c'\right)^2\right\} \\ = \left\{\left(V, c'\right) \in \left(SU(3)_{X_\mu}\right)^{2\ell+1} \mid c' = \mathrm{diag}\left(c_1, (-1)^{r+1}, (-1)^{r+1}c_1^{-1}\right), c_1 \in U(1)\right\}$$

$$V^{\ell,2}(SU(3))_\mu \\ = \left\{\left(V, d, c'\right) \in \left(SU(3)_{X_\mu}\right)^{2\ell+2} |\, \mathfrak{m}(V) = \exp(X_\mu/2)\epsilon c' d\left(\epsilon c'\right)^{-1} d\right\} \\ = \left\{\left(V, d, c'\right) \in \left(SU(3)_{X_\mu}\right)^{2\ell+2}, d = \mathrm{diag}\left(d_1, (-1)^r, (-1)^r d_1^{-1}\right), d_1 \in U(1)\right\}$$

Let $SU(3)_{X_\mu}$ act on $V^{\ell,i}(SU(3))_\mu$ by

$$u \cdot \left(V, c'\right) = \left(uVu^{-1}, \epsilon^{-1}u\epsilon c'u^{-1}\right) = \left(V, \rho(u)c'\right), \quad u \cdot \left(V, d, c'\right) = \left(V, d, \rho(u)c'\right)$$

where

$$\rho\left(\mathrm{diag}\left(u_1, u_2, u_1^{-1}u_2^{-1}\right)\right) = \mathrm{diag}\left(u_1^{-2}u_2^{-1}, 1, u_1^2u_2\right) \in SU(3)_{X_\mu}, \\ \rho\left(SU(3)_{X_\mu}\right) \cong U(1).$$

Then, the following equivariant pairs are equivalent:

$$\begin{aligned}\left(X_{\mathrm{YM}}^{\ell,i}(SU(3))_\mu, SU(3)\right) &\cong \left(V^{\ell,i}(SU(3))_\mu, SU(3)_{X_\mu}\right)\\ &\cong \left(U(1)^{2\tilde{g}+1}, U(1)\times U(1)\right) \cong \left(U(1)^{2\tilde{g}}, U(1)\right)\end{aligned}$$

where $U(1)$ acts trivially on $U(1)^{2\tilde{g}}$.

Thus, the $\mathcal{G}'$-equivariant Poincaré series for stratum $\mathcal{A}'_{r,0,-r}$ is

$$\begin{aligned}P_t^{\mathcal{G}'}\left(\mathcal{A}'_{r,0,-r};\mathbb{Q}\right) &= P_t^{SU(3)}\left(X_{\mathrm{YM}}^{\ell,i}(SU(3))_{r,0,-r};\mathbb{Q}\right) = P_t^{U(1)}\left(X_{\mathrm{YM}}^{\tilde{g},0}(U(1))_r;\mathbb{Q}\right)\\ &= \frac{(1+t)^{2\tilde{g}}}{1-t^2}.\end{aligned}$$

By [12, Theorem 2.5],

$$P_t^{\mathcal{G}'}\left(\mathcal{A}\left(Q^3_{\Sigma_i^\ell}\right);\mathbb{Q}\right) = P_t\left(B\mathcal{G}';\mathbb{Q}\right) = \frac{\left(1+t^3\right)^{\tilde{g}}\left(1+t^5\right)^{\tilde{g}}}{\left(1-t^4\right)\left(1-t^6\right)}.$$

Therefore, (23) is equivalent to the following identity

$$\begin{aligned}P_t^{SU(3)}\left(X_{\mathrm{flat}}^{\ell,i}(SU(3));\mathbb{Q}\right) &= P_t\left(B\mathcal{G}';\mathbb{Q}\right) + \sum_{r>0} t^{4r+3(\tilde{g}-1)-1}\frac{(1+t)^{2\tilde{g}}}{1-t^2}\\ &= \frac{\left(1+t^3\right)^{\tilde{g}}\left(1+t^5\right)^{\tilde{g}}}{\left(1-t^4\right)\left(1-t^6\right)} + \frac{(1+t)^{2\tilde{g}}t^{3\tilde{g}}}{\left(1-t^2\right)\left(1-t^4\right)}\end{aligned}$$

□

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