§2. Linear system with constant coefficients

Consider $\begin{cases} \dot{x} = Ax, & A = (a_{ij})_{i,j=1,\dots,d} \text{ const. matrix} \\ x(\tau) = \xi \end{cases}$ (LC)

When $d=1$, $A=a$, the sol. is $x(t) = e^{a(t-\tau)}\xi$.
This suggests sol. of (LC) are of the form $e^{A(t-\tau)}\xi$. Need to define "exponential of matrices".

Begin w/ some observations: $\|\cdot\|$ matrix norm.

<u>Lemma 1</u>. $(\mathbb{R}^{d\times d}, \|\cdot\|)$ is a Banach space.

pf. $\{A_n\}$ Cauchy seq.

$\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\|A_n - A_m\| < \varepsilon$ $\forall n, m \geq N$.

$\Rightarrow \|A_n x - A_m x\| < \varepsilon$ $\forall n, m \geq N$. $\forall x$ w/ $|x| = 1$.

$\Rightarrow \{A_n x\}$ is Cauchy

$\Rightarrow \{A_n x\}$ conv.

Define $A: \mathbb{R}^d \to \mathbb{R}$ by $Ax = \lim_{n\to\infty} A_n x$.

$\Rightarrow A$ is linear since A_n and "lim" are linear.

$\because \|A_n x - A_m x\| < \varepsilon$ $\forall n, m \geq N$. $\forall |x| = 1$.

$\therefore \|A_n x - Ax\| \leq \varepsilon$ $\forall n \geq N$. $\forall |x| = 1$.

$\therefore \|A_n - A\| \leq \varepsilon$ $\forall n \geq N$ <u>QED</u>.

<u>Lemma 2</u>. $\|AB\| \leq \|A\|\|B\|$.

pf. $\|AB\| = \sup_{|x|\leq 1} \|ABx\| = \|B\| \sup_{|x|\leq 1} \left\| A\left(\frac{Bx}{\|B\|}\right)\right\|$

$\leq \|A\|\|B\|$ since $|Bx| \leq \|B\|$ $\forall |x| \leq 1$ <u>QED</u>.

Def. Given a series $\sum_{n=1}^{\infty} A_n$ in $\mathbb{R}^{d\times d}$, we say it converges to $A \in \mathbb{R}^{d\times d}$ if

$$\left\| \sum_{k=1}^{n} A_k - A \right\| \to 0 \quad \text{as } n \to \infty.$$

We say $\sum_{n=1}^{\infty} A_n$ converges absolutely if $\sum_{n=1}^{\infty} \|A_n\|$ converges.

Lemma 3. Absolute convergence $\Rightarrow$ Convergence.

pf. Let $B_n = \sum_{k=1}^{n} A_k$. $\{A_n\}$ abs. conv.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\sum_{k=n}^{\infty} \|A_k\| < \varepsilon \quad \forall n \geq N$

$\Rightarrow \|B_n - B_m\| < \varepsilon \quad \forall n, m \geq N$

$\Rightarrow \{B_n\}$ is Cauchy $\Rightarrow$ conv., by Lemma 1.

QED

Def. The Cauchy product of two infinite series $\sum_{n=0}^{\infty} A_n$, $\sum_{n=0}^{\infty} B_n$ is defined by $\sum_{n=0}^{\infty} C_n$, where

$$C_n = \sum_{k=0}^{n} A_k B_{n-k}.$$

Lemma 4. (Merten's Theorem)

Suppose $\sum_{n=0}^{\infty} A_n = A$, $\sum_{n=0}^{\infty} B_n = B$ are conv., and one of them conv. absolutely. Then their Cauchy product conv. to AB.

pf. Similar to the Merten thm. in advanced calculus. (exercise)

Given power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ w/ radius of convergence $R > 0$.

Define f as a function on $\mathbb{R}^{d \times d}$ by

$$f(A) = \sum_{k=0}^{\infty} a_k A^k$$

<u>Lemma 5.</u> If $0 \leq \|A\| < R$, then $f(A)$ conv.

pf. $\|A\| < R \Rightarrow \sum_{k=0}^{\infty} a_k \|A\|^k$ conv. absolutely

$\because \sum_{k=0}^{\infty} |a_k| \|A^k\| \leq \sum_{k=0}^{\infty} |a_k| \|A\|^k$. by Lemma 2.

$\therefore f(A)$ conv. absolutely $\therefore$ conv., by Lemma 3.

<u>QED</u>

<u>Def.</u> The exponential of matrix A is defined by $\exp(A) = e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

The function $\exp : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ is called the matrix exponential.

By Lemma 5, exp. is well-defined on $\mathbb{R}^{d \times d}$, and $\exp(A)$ conv. absolutely $\forall A \in \mathbb{R}^{d \times d}$.

<u>Theorem 1.</u> $\forall A . B \in \mathbb{R}^{d \times d}$

(a) $\|e^A\| \leq e^{\|A\|}$ (b) $e^0 = I$

(c) $AB = BA \Rightarrow e^{A+B} = e^A \cdot e^B$.

(d) e^A is invertible and $(e^A)^{-1} = e^{-A}$.

(e) Q is invertible $\Rightarrow e^{Q^{-1} A Q} = Q^{-1} e^A Q$.

pf. (a) $\|e^A\| = \|\sum_{k=0}^{\infty} \frac{1}{k!} A^k\| \le \sum_{k=0}^{\infty} \frac{1}{k!} \|A^k\|$

$$\le \sum_{k=0}^{\infty} \frac{1}{k!} \|A\|^k = e^{\|A\|}.$$

(b) $e^0 = \sum_{k=0}^{\infty} \frac{1}{k!} 0^k = I$.

(c) $AB = BA \Rightarrow (A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$

$\therefore e^{A+B} = \sum_{n=0}^{\infty} \frac{1}{n!} (A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} A^k \cdot \frac{1}{(n-k)!} B^{n-k}$$

This is precisely the Cauchy product of $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$. $\sum_{k=0}^{\infty} \frac{1}{k!} B^k$.

$\therefore$ By Merten's theorem, $e^{A+B} = e^A \cdot e^B$.

(d) $A \cdot (-A) = (-A) \cdot A$

$\therefore I = e^0 = e^{A+(-A)} = e^A \cdot e^{-A}$

$\therefore e^{-A} = (e^A)^{-1}$.

(e) $(Q^{-1} A Q)^k = Q^{-1} A^k Q$

$\Rightarrow \sum_{k=0}^{n} \frac{1}{k!} (Q^{-1} A Q)^k = \sum_{k=0}^{n} \frac{1}{k!} Q^{-1} A^k Q$

$$= Q^{-1} \left(\sum_{k=0}^{n} \frac{1}{k!} A^k \right) Q$$

Let $n \to \infty$, then $e^{Q^{-1}AQ} = Q^{-1} e^A Q$. <u>QED</u>.

<u>Theorem 2</u>. e^{At} is a fundamental matrix of $\dot{x} = Ax$.

pf. $\forall t, s \in \mathbb{R}$. At, As commute.

$\therefore e^{A(t+h)} - e^{At} = e^{At} \cdot e^{Ah} - e^{At}$

$= e^{At}(e^{Ah} - I) = e^{At}(hA + \frac{h^2}{2!}A^2 + \frac{h^3}{3!}A^3 + \cdots)$

$= e^{At}(hA + O(h^2))$ as $h \to 0$.

$\therefore (e^{At})' = Ae^{At}$.

$\because e^{At}$ is nonsingular $\forall t$.

$\therefore e^{At}$ is a fundamental matrix. (by Thm 4.3.1).

QED.

Corollary. $\begin{cases} \dot{x} = Ax + h(t). & h \in C(\mathbb{R}, \mathbb{R}^d). \\ x(\tau) = \xi. \end{cases}$

The sol. is

$$\varphi(t; \tau, \xi) = e^{A(t-\tau)}\xi + \int_\tau^t e^{A(t-s)} h(s)\, ds$$

(By Variation of Const. Formula.)

Some special cases:

(1) A is diagonal matrix $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$.

$$A^n = \begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_d^n \end{pmatrix}.$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$= \begin{pmatrix} 1 + \lambda_1 t + \frac{1}{2!}\lambda_1 t^2 + \cdots & & \\ & \ddots & \\ & & 1 + \lambda_d t + \frac{1}{2!}\lambda_d t^2 + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_d t} \end{pmatrix}$$

If A is diagonalizable, $Q^{-1} A Q = D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$

then $e^{At} = e^{QDQ^{-1}t} = Q e^{Dt} Q^{-1}$

$$= Q \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_d t} \end{pmatrix} Q^{-1}$$

(2) A is block diagonal. $A = \begin{pmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{pmatrix}$

$$\Rightarrow e^A = \begin{pmatrix} e^{J_1} & & & \\ & e^{J_2} & & \\ & & \ddots & \\ & & & e^{J_k} \end{pmatrix}$$

(3) $A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ $\Rightarrow e^A = e^{\alpha} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

$\alpha, \beta \in \mathbb{R}$.

pf. Eigenvalues of A are $\alpha \pm i\beta$.

$A - (\alpha + i\beta) I = \begin{pmatrix} -i\beta & -\beta \\ \beta & -i\beta \end{pmatrix}$. e.vector $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

$A - (\alpha - i\beta) I = \begin{pmatrix} i\beta & -\beta \\ \beta & i\beta \end{pmatrix}$. e.vector $\begin{pmatrix} i \\ -1 \end{pmatrix}$.

Let $Q = \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}$, $Q^{-1} = \frac{1}{2} \begin{pmatrix} -i & 1 \\ -i & -1 \end{pmatrix}$.

$$\Rightarrow Q^{-1} A Q = \begin{pmatrix} \alpha + i\beta & \\ & \alpha - i\beta \end{pmatrix}$$

$$Q^{-1}e^{A}Q = \exp\begin{pmatrix} \alpha+i\beta & \\ & \alpha-i\beta \end{pmatrix} = \begin{pmatrix} e^{\alpha+i\beta} & \\ & e^{\alpha-i\beta} \end{pmatrix}$$

$$= e^{\alpha}\begin{pmatrix} e^{i\beta} & \\ & e^{-i\beta} \end{pmatrix} = e^{\alpha}\begin{pmatrix} \cos\beta + i\sin\beta & 0 \\ 0 & \cos\beta - i\sin\beta \end{pmatrix}$$

$$\therefore\ e^{A} = e^{\alpha}\, Q \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} Q^{-1} = e^{\alpha}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

$$\therefore\ e^{At} = e^{\alpha t}\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}.$$

(4) $\underset{(d\times d)}{J} = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \Rightarrow e^{Jt} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{d-1}}{(d-1)!} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & t^2/2! \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}$

pf. Let $D = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$. $N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$

$\Rightarrow D+N = J$ and $DN = ND$

$$\therefore\ e^{Jt} = e^{Dt}\cdot e^{Nt} = e^{\lambda t}e^{Nt}$$

$N^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix}$, $N^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 1 \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & 0 \\ & & & & & 0 \end{pmatrix}$, ...

$N^{d-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix}$. $N^d = O$.

$$\therefore\ e^{Nt} = \sum_{k=0}^{d-1}\frac{1}{k!}N^k t^k = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{d-1}}{(d-1)!} \\ & \ddots & t & \cdots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}$$

QED.

Review (from linear algebra).

Let $v_1, \dots, v_m$ be linearly indep. eigenvectors of $A \in \mathbb{R}^{d\times d}$. $\lambda_1, \dots, \lambda_m$ are corresponding eigenvalues.

$$(A - \lambda_i I) v_i = 0.$$

Generalized eigenvectors: $v_i = v_i^{(1)}$. $v_i^{(k)} \neq 0$ given by:

$$(A - \lambda_i I)^2 v_i^{(2)} = 0 \quad \& \quad (A - \lambda_i I) v_i^{(2)} = v_i^{(1)}.$$

$$(A - \lambda_i I)^3 v_i^{(3)} = 0 \quad \& \quad (A - \lambda_i I) v_i^{(3)} = v_i^{(2)}.$$

$$\vdots$$

$$Q = \left(v_1^{(1)}, \dots, v_1^{(l_1)}, v_2^{(1)}, \dots, v_2^{(l_2)}, \dots, v_m^{(1)}, \dots, v_m^{(l_m)}\right).$$

l_i — algebraic multiplicity of λ_i.

$\{v_i^{(1)}, \dots, v_i^{(l_i)}\}$: basis for $\ker(A - \lambda_i I)^{l_i}$

— generalized eig. space

$\Rightarrow l_1 + \dots + l_m = d.$

$\mathbb{C}^d$ = Direct sum of generalized eigenspaces.

$$Q^{-1} A Q = J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{pmatrix}. \quad J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

J_i — Jordan blocks of A. J — Jordan form of A.

$$\therefore e^{At} = Q e^{Jt} Q^{-1} = Q \begin{pmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_m t} \end{pmatrix} Q^{-1}$$

Notation: $\sigma(A)$ = spectrum of A
(eig. values)