Chapter 1. Introduction

§1. What is ODE

A system of ordinary differential equations (ODE) in Euclidean space $\mathbb{R}^d$ is a differential equation of the form

$$\dot{x} = f(t, x)$$

where $\dot{x} = \frac{dx}{dt}$, $(t, x) \in I \times D$ for some interval $I \subset \mathbb{R}$ and region $D \subset \mathbb{R}^d$, and $f : I \times D \to \mathbb{R}^d$ is a function.

A solution is a differentiable function $x(t)$ satisfying the diff. eqn.

If f does not depend on t, we say the ODE is autonomous, o.w. we say it is non-autonomous.

Typically, minimum assumptions:

$$D \text{ is open. } f \in C(I \times D, \mathbb{R}^d).$$

Usually ODEs arise w/ additional conditions:

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$ — Initial-value problem (IVP)

$$\begin{cases} \dot{x} = f(t, x) \\ x(a) = x_a, \ x(b) = x_b \end{cases}$$ — Boundary-value problem (BVP)

We say the ODE is linear if

$$f(t,x) = A(t)\,x + g(t),$$

where $A: I \to \mathbb{R}^{d\times d}$, $g: I \to \mathbb{R}^d$. Here g is called the non-homogeneous term.

If $g \equiv 0$, then we say the linear system is homogeneous.

ODE may appear in the form of n^{th} order ODE:

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right), \quad x \in C^n(I, \mathbb{R}).$$

Any n^{th} order ODE can be written as system of 1^{st} order ODE:

let $x_1 = x$, $x_2 = \frac{dx}{dt}$, ..., $x_n = \frac{d^{n-1}x}{dt^{n-1}}$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = f(t, x_1, x_2, \dots, x_n) \end{cases}$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad F(t, X) = \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ f(t, x_1, \dots, x_n) \end{pmatrix}$$

$$\Rightarrow \dot{X} = F(t, X).$$

Theory of ODE is the study of behavior of solutions.

Behavior in short time (local theory)

— linearization, local stability. (Chap 3.4.6)

Behavior in long time. (global theory)

— global stability, asymp. behavior, global structure, etc. (Chap 5.6, 7.8)

Further Generalizations:

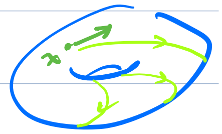
(1) More general underlying space.

eg. $x \in$ manifold M. $f(t,x) \in T_x M$ (tangent space)

$\to$ ODE on manifolds or surfaces

eg. $x \in$ Banach space X.

$\to$ ODE on abstract spaces.



(2) Weaker sense of solutions.

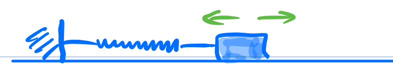
eg. x is AC (absolutely cont.), BV (functions of bounded variations), etc.

eg. sol. in the sense of distributions (Chap 7)

§2. Where do ODE arise.

ODE arise in all areas of applications:
physics, biology, chemistry, economics, social sciences
.....

Example 1. Spring vibration



displacement from equil. position $= x$, mass $= m$.
Force acting upon the object : restoration force
+ friction or damping.

By Newton's law & Hooke's law,

$$m\ddot{x} = -kx - c\dot{x}. \qquad k, c > 0 \text{ are const.}$$

Set $y = \dot{x} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\frac{k}{m}x - \frac{c}{m}y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

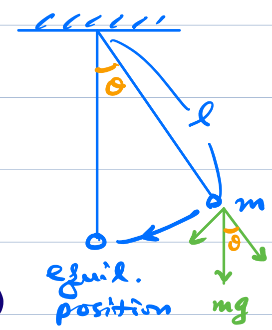
— homogeneous linear ODE.

If $\exists$ external force $F(t)$, then

$$m\ddot{x} = -kx - c\dot{x} + F(t)$$

or $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ F(t) \end{pmatrix}$ — non-homogeneous.

Example 2. Simple pendulum.



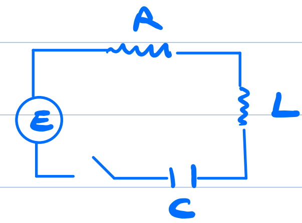
Length of rod $= l$. mass $= x$
angle from equil. position $= \theta$.

$-m\frac{d^2}{dt^2}(l\theta) = mg\sin\theta$ (Newton's law)

i.e. $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$.

Let $w = \dot{\theta} \Rightarrow \begin{pmatrix} \dot{\theta} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} w \\ -\frac{g}{\ell} \sin\theta \end{pmatrix}$ — nonlinear system.

Example 3. Electrical circuit.



Resistor $= R$. Charge $= Q$.

Inductor $= L$. Current $= I$.

Capacitor $= C$.

Electromotive force $= E$ (eg. battery. generator)

Kirchhoff's law: sum of voltages drops = supplied voltage.

$$L \frac{dI}{dt} + RI + \frac{Q}{C} = E(t).$$

$\because I = \frac{dQ}{dt}$. $\therefore$ $L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = E(t)$.

Or:
$$\begin{pmatrix} \dot{Q} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} I \\ -\frac{Q}{LC} - \frac{RI}{L} + \frac{E}{L} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L} E(t) \end{pmatrix}$$

— non-homogeneous linear ODE.

Example 4. Van der Pol oscillator

Oscillatory electrical circuit w/ nonlinear damping

$$\ddot{x} + \varepsilon \dot{x}(x^2-1) + x = 0. \qquad 0 < \varepsilon \ll 1.$$

w/ external force $\Rightarrow \ddot{x} + \varepsilon \dot{x}(x^2-1) + x = F(t)$.

(forced van der Pol oscillator)

Example 5. N-body problem.

m_k = mass of k-th body.

x_k = position " " (in $\mathbb{R}^3$). $k=1,\dots,N$.

By Newton's law of universal gravitation.

$$m_k \ddot{x}_k = \sum_{i\neq k} \frac{G m_i m_k (x_i - x_k)}{|x_i - x_k|^3}.$$

G – gravitational const. ($\approx 6.67\times 10^{-11}\ m^3/s^2\cdot kg$)

– highly nonlinear & singular

Example 6. Population growth

Model for population growth of single species

Population $= x$.

$$\dot{x} = r x \left(1 - \frac{x}{K}\right).$$

– logistic equation.

r: intrinsic growth rate.

K: carrying capacity.

When $\exists$ 2 species, one is predator (y), the other is prey (x).

$$\begin{cases} \dot{x} = r x \left(1 - \frac{x}{K}\right) - b x y \\ \dot{y} = (c x - d) y \end{cases}$$

$r, K, b, c > 0$ are const.

or

$$\begin{cases} \dot{x} = r x \left(1 - \frac{x}{K}\right) - \alpha x y \\ \dot{y} = s y \left(1 - \frac{y}{h}\right) - \beta x y \end{cases}$$

— Lotka–Volterra equation. $(x,y) \in \mathbb{R}_+^2$.

Example 7. Epidemics

S: susceptibles (people who can catch the disease)

I: infectives (people who have the disease and can transmit it)

R : removed class (quarantined, immuned, recovered)

Progress : $S \to I \to R$.

SIR model :

$$\begin{cases} \dfrac{dS}{dt} = -rSI \\ \dfrac{dI}{dt} = rSI - aI \\ \dfrac{dR}{dt} = aI \end{cases}$$

$r > 0$ is infection rate

$a > 0$ is removal rate

Variants : Consider deceased rate, no/low immunity, etc.

Example 8. Enzyme kinetics

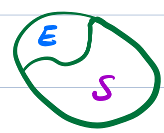
$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} P + E$$

$k_i \geq 0 \ \forall i$.

$k_1, k_2 > 0$.

S : substrate. E : enzyme. C : substrate-enzyme complex

P : product.

C : 

$$\begin{cases} \dot{S} = -k_1 S E + k_{-1} C \\ \dot{E} = -k_1 S E + (k_{-1} + k_2) C - k_{-2} E P \\ \dot{C} = k_1 S E - (k_{-1} + k_2) C + k_{-2} E P. \\ \dot{P} = k_2 C - k_{-2} E P \end{cases}$$

— Michaelis-Menten equation

Usually $k_{-2} = 0$ (irreversible).

Example 9. Evolution of games (economics/social science)

Consider a symmetric game w/ payoff matrix $A = (a_{ij})_{i,j=1}^{n}$,

a_{ij} is the payoff for player to play strategy i against j

Let $x_i \in [0,1]$ be the probability of observing strategy i in a well-mixed population. $\Rightarrow$ $x_1 + \cdots + x_n = 1$.

Expected payoff of a player to play strategy i is

$$\sum_j a_{ij} x_j = (Ax)_i$$

Average payoff of the total population is

$$\sum_i x_i (Ax)_i = \langle x, Ax \rangle = \sum_{i,j} a_{ij} x_i x_j$$

Replicator Equation $\dot{x}_i = [\overbrace{(Ax)_i - \langle x, Ax \rangle}^{\text{relative success/fitness of strategy } i}] x_i$, $i = 1, \cdots, n$.

(Taylor - Jonker, Math. Biosci. 1978, limitation adaptation model).

Study of sol. for this equation would tell us how players (eg. investors) are likely to change their strategies (eg. combination of investments) in the long run.

Chapter 2. Fundamental Theory

§1. Introduction and preliminaries

Consider IVP $\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0. \end{cases}$ $(t, x) \in I \times D \subset \mathbb{R} \times \mathbb{R}^d.$

or IVP w/ parameter(s)

$$\begin{cases} \dot{x} = f_\lambda(t, x). \quad \lambda \in \mathbb{R}^k \\ x(t_0) = x_0. \end{cases}$$

Basic questions:

1. What is the minimum condition on f to ensure local existence of sol. ?
2. When is the sol. unique ?
3. When does the sol. exist globally ?
4. How do sol. vary as initial conditions or parameter(s) change?
   (Do they change continuously ?)

Goal : Answer these basic questions.