2022/6/9

§5. The Stone-Weierstrass theorem

<u>Question</u>: Can we use polynomials (or trignometric polynomials) to approximate a given cont. function on $[a,b]$ (or some compact set in $\mathbb{R}^n$)?

<u>Def</u>. Let $f \in C([0,1],\mathbb{R})$. $n \geq 1$.
The <u>n-th Bernstein polynomial</u> of f is defined by

$$B_n(x;f) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

$(\text{or } B_n(x))$

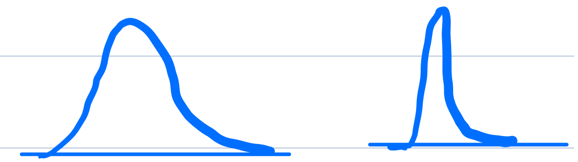
Let $\varphi_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \Rightarrow$ it has max. at $x = \frac{k}{n}$.

Observe: (1) $\displaystyle\sum_{k=0}^{n} \varphi_{n,k} = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$

(2) $B_n(x)$ is weighted sum of $f(0), f(\frac{1}{n}), \ldots, f(\frac{n-1}{n}), f(1)$.

(3) When n is large, $\varphi_{n,k}$ are very "steep".

when $\left|x-\frac{k}{n}\right| \not\approx 0$, $\varphi_{n,k}(x) \approx 0$.



$x(1-x)$ $\qquad$ $x^{10}(1-x)^5$

(4) $B_n(x) - f(x)$

$$= \sum_{k=0}^{n} \left(f\left(\tfrac{k}{n}\right) - f(x)\right) \varphi_{n,k} \approx 0 \quad \text{when } n \text{ is large}$$

(intuitively)

<u>Examples</u>. (a) $f(x) \equiv 1$. $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1 = f(x)$.

(b) $f(x) = x$. $B_n(x) = \sum_{k=0}^{n} \left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$

$$= x \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k}$$

$$= x \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{(n-1)-j} = x = f(x).$$

(c) $f(x) = x^2$. $B_n(x) = \cdots = \frac{n-1}{n} x^2 + \frac{x}{n} \longrightarrow x^2 = f(x)$

as $n \to \infty$.

Lemma. $\displaystyle\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \frac{1}{n}x(1-x).$

pf. l.h.s. $\displaystyle = x^2\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}$

$\displaystyle - 2x\sum_{k=0}^{n}\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$

$\displaystyle + \sum_{k=0}^{n}\left(\frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^{n-k}$

$\displaystyle = x^2 - 2x\cdot x + \left(\frac{n-1}{n}x^2+\frac{x}{n}\right)$

$\displaystyle = \frac{-1}{n}x^2 + \frac{x}{n} = \frac{1}{n}x(1-x).$ QED.

Bernstein Approximation Theorem.

Let $f \in C([0,1], \mathbb{R})$. Then $B_n(x;f)$ conv. uniformly on $[0,1]$ to f as $n \to \infty$.

pf. Write $B_n(x;f)$ as $B_n(x)$.

$$|f(x) - B_n(x)| = \left| \sum_{k=0}^{n} \left(f(x) - f\left(\tfrac{k}{n}\right)\right) \binom{n}{k} x^k (1-x)^{n-k} \right|$$

$$\leq \sum_{k=0}^{n} \left|f(x) - f\left(\tfrac{k}{n}\right)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

Given $\varepsilon > 0$.

$\because$ $[0,1]$ is compact. f is cont.

$\therefore$ $\exists M > 0$ s.t. $|f(x)| \leq M$ $\forall x \in [0,1]$, and f is uniformly cont. on $[0,1]$.

$\therefore$ $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{2}$ whenever $|x-y| < \delta$.

Choose $N \geq \max\left\{ \delta^{-4}, \frac{M^2}{\varepsilon^2} \right\}$, let $n \geq N$, Then $|f(x) - B_n(x)|$

$$\leq \sum_{\substack{k \\ |x-\frac{k}{n}| < \frac{1}{n^{1/4}}}} \left|f(x) - f\left(\tfrac{k}{n}\right)\right| \binom{n}{k} x^k (1-x)^{n-k} + \sum_{\substack{k \\ |x-\frac{k}{n}| \geq \frac{1}{n^{1/4}}}} \left|f(x) - f\left(\tfrac{k}{n}\right)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

$\left(\frac{1}{n^{1/4}} \leq \delta \Leftrightarrow n \geq \delta^{-4}\right)$

$$\leq \frac{\varepsilon}{2} \sum_k \binom{n}{k} x^k (1-x)^{n-k} + 2M \sum_{\substack{k \\ |x-\frac{k}{n}| \geq \frac{1}{n^{1/4}}}} \frac{(x-\frac{k}{n})^2}{(x-\frac{k}{n})^2} \binom{n}{k} x^k (1-x)^{n-k}$$

$\rightarrow \left(\frac{1}{|x-\frac{k}{n}|^2} < \sqrt{n}\right).$

$$\leq \frac{\varepsilon}{2} + 2M\sqrt{n} \sum_{k=0}^{n} \left(x-\frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \frac{\varepsilon}{2} + 2M \frac{\sqrt{n}}{n} \underbrace{x(1-x)}_{\leq \frac{1}{4}} \quad \text{(by Lemma)}$$

$$\leq \frac{\varepsilon}{2} + \frac{M}{2\sqrt{n}} \qquad \left(\frac{M}{\sqrt{n}} \leq \varepsilon \Leftrightarrow \frac{M}{\varepsilon} \leq \sqrt{n} \Leftrightarrow \frac{M^2}{\varepsilon^2} \leq n.\right)$$

$$< \varepsilon \qquad \forall x \in [0,1]. \ \forall n \geq N.$$

$\therefore B_n(x) \to f$ uniformly on $[0,1]$. as $n \to \infty$.

QED.

Weierstrass Approximation Theorem

If $f \in C([a,b], \mathbb{R})$, then f can be uniformly approximated by polynomials.

pf. Let $g(x) = f((b-a)x + a)$, $x \in [0,1]$.

$\Rightarrow$ $g \in C([0,1], \mathbb{R})$, and $g\left(\frac{x-a}{b-a}\right) = f(x)$.

By Bernstein approximation theorem,

$B_n(x;g) \to g$ uniformly on $[0,1]$ as $n \to \infty$.

i.e. $\forall \varepsilon > 0$, $\exists N$ s.t. $|B_n(x;g) - g(x)| < \varepsilon$ $\forall n \geq N$, $\forall x \in [0,1]$.

$$\Rightarrow \left| B_n\left(\frac{x-a}{b-a}; g\right) - \underbrace{g\left(\frac{x-a}{b-a}\right)}_{f(x)} \right| < \varepsilon \quad \forall n \geq N,\ \forall x \in [a,b].$$

$$\therefore B_n\left(\frac{x-a}{b-a}; g\right) \to f \text{ uniformly on } [a,b] \text{ as } n \to \infty.$$

<u>QED</u>.

Recall : $E \subset X$ is dense if $\bar{E} = X$.

Let $P([a,b], \mathbb{R})$ be the set of polynomials defined on $[a,b]$.

$C([a,b], \mathbb{R})$ w/ $\|\cdot\|_\infty$ is a metric space.

f_n conv. uniformly to f as $n \to \infty$.

$\Leftrightarrow$ $\|f_n - f\|_\infty \to 0$ as $n \to \infty$

$\therefore$ Weierstrass Approximation Theorem can be stated as : $P([a,b], \mathbb{R})$ is a dense subset of $C([a,b], \mathbb{R})$.

Next Goal : Extend the theorem to function spaces of the form $C(K, \mathbb{R})$.

$K \subset \mathbb{R}^n$

applicable to Fourier series (next semester).

Prop. Suppose $\mathcal{F} \subset C([a,b], \mathbb{R})$ is dense in $C([a,b], \mathbb{R})$. Then $\forall x_1 \neq x_2$ in $[a,b]$, $\exists f \in \mathcal{F}$ s.t. $f(x_1) \neq f(x_2)$.

pf. Suppose not. i.e. $\exists x_1 \neq x_2$ in $[a,b]$ s.t.

$$f(x_1) = f(x_2) \quad \forall f \in \mathcal{F}.$$

Given $g \in C[a,b]$ s.t. $g(x_1) \neq g(x_2)$.

$\mathcal{F}$ is dense in $C[a,b]$

$\Rightarrow \exists$ seq. $\{f_n\} \subset \mathcal{F}$ s.t. $f_n \to g$ uniformly on $[a,b]$.

$\Rightarrow g(x_1) = \lim_{n\to\infty} f_n(x_1) = \lim_{n\to\infty} f_n(x_2) = g(x_2)$ $\times$.

<u>QED</u>.

Def. Let $\mathcal{F}$ be a family of functions from a metric space X to $\mathbb{R}$.

We say $\mathcal{F}$ <u>separate points</u> if $\forall x_1 \neq x_2$ in X, $\exists f \in \mathcal{F}$ s.t. $f(x_1) \neq f(x_2)$.

Def. A vector space $\mathcal{A}$ over $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) is called an <u>algebra</u> (over $\mathbb{F}$) if $\mathcal{A}$ has a multiplication $\cdot$ s.t. $\forall a, b, c \in \mathcal{A}$, $\forall \alpha \in \mathbb{F}$.

(1) $a \cdot b \in \mathcal{A}$

(2) $a \cdot (b + c) = a \cdot b + a \cdot c$

(3) $\alpha (a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$

If $\exists e \in \mathcal{A}$ s.t. $e \cdot a = a = a \cdot e$ $\forall a \in \mathcal{A}$, then we say $\mathcal{A}$ is an <u>algebra with identity</u> e.

Remark. Many people add assumption to the def:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Examples. $\mathcal{P}(\mathbb{R})$ — polynomials.

$$\mathcal{P}(\mathbb{R}) \subset C^{\omega}(\mathbb{R}) \subset C^{\infty}(\mathbb{R}) \subset C^{k}(\mathbb{R}) \subset C(\mathbb{R}) \qquad k \in \mathbb{N}$$

Each of them is an algebra. $(f \cdot g)(x) := f(x) \cdot g(x)$.

Let C_T be T-periodic functions in $C(\mathbb{R})$.

$\Rightarrow$ C_T is also an algebra.

$M_{n\times n}$ = the space of $n\times n$ real matrices.

$A, B \in M_{n\times n}$. AB (matrix multiplication) $\in M_{n\times n}$.

$\Rightarrow$ $M_{n\times n}$ is an algebra. (called matrix algebra).

Symm. matrices in $M_{n\times n}$ is a subalgebra of $M_{n\times n}$.

Notation. $f_1 \vee f_2 \vee \cdots \vee f_k = \max\{f_1, \ldots, f_k\}$.

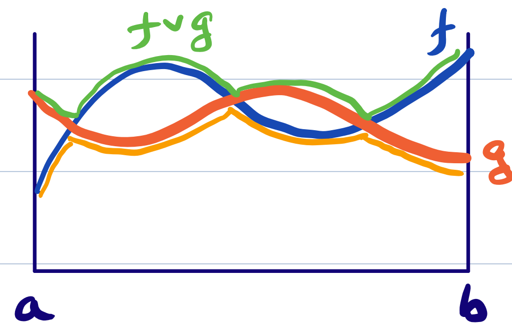
i.e. $(f_1 \vee f_2 \vee \cdots \vee f_k)(x) = \max\{f_1(x), \ldots, f_k(x)\}$.

$f_1 \wedge f_2 \wedge \cdots \wedge f_k = \min\{f_1, \ldots, f_k\}$.

$f, g \in C[a,b]$.

$f \vee g = \max\{f, g\}$

$= \frac{1}{2}\{f + g + |f - g|\}$



$f \wedge g = \min\{f, g\}$

$= \frac{1}{2}\{f + g - |f - g|\}$.

Stone-Weierstrass Theorem

Let K be a compact set of a metric space (X, ρ). If $\mathcal{A}$ is a subalgebra of $C(K, \mathbb{R})$ which contains constant functions and separate points, then $\mathcal{A}$ is dense in $C(K, \mathbb{R})$.

pf. <u>Step 1</u>. $\forall f, g \in \mathcal{A}$. $|f|$. $f \vee g$. $f \wedge g \in \bar{\mathcal{A}}$.

pf: Let $\|f\| = M$. Given $\varepsilon > 0$.

By Weierstrass approx. theorem. $\exists\, a_0, a_1, \dots, a_m \in \mathbb{R}$ s.t.

$$\left| \sum_{k=0}^{m} a_k y^k - |y| \right| < \varepsilon \quad \forall y \in [-M, M].$$

$$\Rightarrow \left| \sum_{k=0}^{m} a_k (f(x))^k - |f(x)| \right| < \varepsilon \quad \forall x \in K.$$

$$\Rightarrow \left\| \sum_{k=0}^{m} a_k f^k - |f| \right\| < \varepsilon$$ (note: $\neq \varepsilon$ since K is cpt.)

$\because \sum_{k=0}^{m} a_k f^k \in \mathcal{A}$ (since $\mathcal{A}$ is an algebra containing const. functions)

$\therefore |f| \in \bar{\mathcal{A}}$ since $\varepsilon > 0$ is arbitrary.

$\therefore |f+g|$. $|f-g| \in \bar{\mathcal{A}}$ since $f+g$. $f-g \in \mathcal{A}$.

$$\Rightarrow f \vee g = \tfrac{1}{2}(f+g) + \tfrac{1}{2}|f-g| \in \bar{\mathcal{A}}$$

$$f \wedge g = \tfrac{1}{2}(f+g) - \tfrac{1}{2}|f-g| \in \bar{\mathcal{A}}.$$

This proves Step 1

Step 2. $\forall x_1 \neq x_2$ in K. $\forall y_1, y_2 \in \mathbb{R}$. $\exists f \in \mathcal{A}$ s.t.

$$f(x_1) = y_1,\ f(x_2) = y_2.$$

pf: Choose $g \in \mathcal{A}$ s.t. $g(x_1) \neq g(x_2)$. ($\because \mathcal{A}$ separate pts).

Let $f(x) = y_1 + (y_2 - y_1) \dfrac{g(x) - g(x_1)}{g(x_2) - g(x_1)}$

$\Rightarrow f \in A$ since A is algebra containing const. functions.

$f(x_1) = y_1$. $f(x_2) = y_2$.

Step 3. Given $\varepsilon > 0$. $\forall f \in C(K, \mathbb{R})$. $\forall z \in K$ $\exists h_z \in \bar{\mathcal{A}}$ s.t. $h_z(z) = f(z)$ and $h_z(x) > f(x) - \varepsilon$ $\forall x \in K$.

pf: $\forall x \in K$, by Step 2 we may choose $h_{xz} \in \mathcal{A}$ s.t.

$$h_{xz}(x) = f(x),\ h_{xz}(z) = f(z).$$

By continuity, $U_x = \{ t \in K : h_{xz}(t) > f(t) - \varepsilon \}$ is open. Clearly $x, z \in U_x$.

By compactness. $\exists$ finite set $x_1, \dots, x_m$ s.t. $U_{x_1} \cup \cdots \cup U_{x_m} \supset K$.

Let $h_z = h_{x_1 z} \vee h_{x_2 z} \vee \cdots \vee h_{x_m z}$

$\Rightarrow h_z(x) > f(x) - \varepsilon \quad \forall x \in K$, and $h_z \in \bar{A}$ (by Step 1).

Moreover, $h_z(z) = \max\{h_{x_1 z}(z), \ldots, h_{x_m z}(z)\} = f(z)$.

Step 4. Given $\varepsilon > 0$. $\forall f \in C(K, \mathbb{R})$, $\exists h \in \bar{A}$ s.t.

$$|f(x) - h(x)| < \varepsilon \quad \forall x \in K.$$

pf: Given $z \in K$. Let $h_z \in \bar{A}$ be as in Step 3.

By continuity, $V_z = \{t \in K : h_z(t) < f(t) + \varepsilon\}$ is open, and $z \in V_z$.

By compactness, $\exists$ finite set $z_1, \ldots, z_n \in K$ s.t.

$$V_{z_1} \cup V_{z_2} \cup \cdots \cup V_{z_n} \supset K.$$

Let $h = h_{z_1} \wedge h_{z_2} \wedge \cdots \wedge h_{z_n}$

$$\Rightarrow f(x) - \varepsilon < h(x) < f(x) + \varepsilon \quad \forall x \in K.$$

i.e. $\|f - h\| < \varepsilon$.

By Step 4. $\bar{A} = C(K, \mathbb{R})$. QED.