2022/6/2

Def. Let (X, ρ) be a metric space. $\{x_n\} \subset X$.

We say $\{x_n\}$ converges to $x \in X$ if

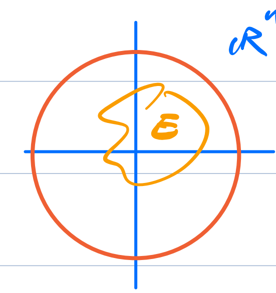
$\forall \varepsilon > 0.\ \exists N \in \mathbb{N}$ s.t. $\rho(x_n, x) < \varepsilon \quad \forall n \geq N.$

In this case, we call x the limit. denoted

$x = \lim_{n\to\infty} x_n$. or $x_n \to x$ as $n \to \infty$

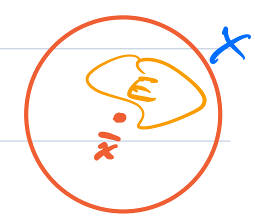
and say $\{x_n\}$ is a convergent sequence.

We say $\{x_n\}$ is divergent if it is not conv.



We say seq. $\{x_n\}$ is bounded if

$\exists \bar{x} \in X$ and $r > 0$ s.t. $\{x_n\} \subset B_r(\bar{x})$



We say $\{x_n\}$ is unbounded if it is not bounded.

We say seq. $\{x_n\}$ is a Cauchy seq. if

$\forall \varepsilon > 0.\ \exists N \in \mathbb{N}$ s.t. $\rho(x_n, x_m) < \varepsilon \quad \forall n \geq m \geq N$

Theorem. (Uniqueness of Limit)

Let (X, ρ) be a metric space. $\{x_n\} \subset X$.

(a) If $\{x_n\}$ conv., then the limit is unique.

(b) If $\{x_n\}$ conv. to $x \in X$, then any subseq. of $\{x_n\}$ conv. to x.

Theorem (Cauchy sequence)

(a) Conv. seq. are Cauchy seq.

(b) Cauchy seq. are bounded.

Remark. Cauchy seq. $\not\Rightarrow$ Conv. seq.

eg. $X = (0,1) \cap \mathbb{Q}^c$ (irrational numbers in $(0,1)$).

The metric is the same as $\mathbb{R}$.

$\left\{\frac{\sqrt{2}}{n}\right\}_{n=2}^{\infty} \subset X$. $\frac{\sqrt{2}}{n} \to 0$ as $n \to \infty$ but $0 \notin X$.

$\therefore$ It is Cauchy but not conv. in X.

Def. We say metric space (X, ρ) is complete if every Cauchy seq. is convergent.

We may consider subspace (E, ρ) of (X, ρ) and call the subspace a complete subspace if (E, ρ) is complete.

eg. $(\mathbb{R}^n, \|\cdot\|)$ is complete.

Open sets containing $x \in X$ are called neighborhoods of x. We say $x \in X$ is an accumulation pt. (or limit pt. or cluster pt.) of $A \subset X$ if $\forall$ neighborhood U of x, U contains infinitely many pts of A. We say $x \in A$ is an isolated pt. if $\exists$ nbd. U of x s.t. $U \cap A = \{x\}$.

Theorem Let (X, ρ) be a metric space.
A set $E \subset X$ is closed if and only if it contains all of its accumulation pts.
(pf. same as $\mathbb{R}^n$).

Theorem. Let (X, ρ) be a complete metric space. $E \subset X$.
Then E is complete if and only if E is closed.

pf. "$\Rightarrow$" Assume E is complete.
Given accumulation pt. x of E. Then $\exists$ seq.
$\{x_n\} \subset E$ s.t. $\rho(x, x_n) < \frac{1}{n}$ $\forall n \in \mathbb{N}$.
$\Rightarrow$ $x_n \to x$ as $n \to \infty$
$\Rightarrow$ $\{x_n\}$ is a Cauchy seq.
By completeness, its limit $x \in E$.
$\therefore$ E contains all of its acc. pts. $\therefore$ E is closed.

"$\Leftarrow$" Assume E is closed.

Given Cauchy seq. $\{x_n\} \subset E$.

It is also a Cauchy seq. in X

(note: X and E have the metric).

$\because$ X is complete.

$\therefore$ $\{x_n\}$ is a convergent seq. in X.

Let x be its limit.

If $x_n \neq x$ for finitely many n.

then $x \in E$.

If $x_n \neq x$ for infinitely many n.

then x is an acc. pt. of E.

$\Rightarrow x \in E$ since E is closed.

$\therefore$ E is complete. $\underline{\text{QED}}$.

$x_n \neq x$ $\forall$ ∞-ly n.

$x_n \neq x$ for finitely many n.

§2. Limits of functions.

Given metric spaces (X, ρ), (Y, d).
Consider $f : A \subset X \to Y$. Let x_0 be acc. pt. of A.
We say $f(x)$ converges to $y_0 \in Y$ as x approaches x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < \rho(x, x_0) < \delta, \ x \in A \ \text{ imply } \ d(f(x), y_0) < \varepsilon.$$

This y_0 is called the limit of $f(x)$ as $x \to x_0$.
Denote it by $\lim_{x \to x_0} f(x) = y_0$, or $f(x) \to y_0$ as $x \to x_0$.

<u>Theorem</u>. (a) If $\lim_{x \to x_0} f(x)$ exists then the limit is unique.
(b) $\lim_{x \to x_0} f(x) = y_0 \iff \forall$ seq. $\{x_n\} \subset A \setminus \{x_0\}$ s.t. $x_n \to x_0$ as $n \to \infty$, we have $\lim_{n \to \infty} f(x_n) = y_0$.

(c) If Y is $\mathbb{R}^m$ (or a normed vector space), then algebraic properties of limits hold:

" $\lim_{x \to x_0} f(x)$, $\lim_{x \to x_0} g(x)$ exist " imply

$$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

$$\lim_{x \to x_0} (\alpha f(x)) = \alpha \lim_{x \to x_0} f(x).$$

$$\lim_{x \to x_0} (f(x) \cdot g(x)) = \left(\lim_{x \to x_0} f(x)\right) \cdot \left(\lim_{x \to x_0} g(x)\right)$$

$$\left\| \lim_{x \to x_0} f(x) \right\| = \lim_{x \to x_0} \| f(x) \|.$$

<u>Def</u>. We say $f : A \subset X \to Y$ is <u>continuous at</u> <u>x_0</u> $\in A$ if $\forall \varepsilon > 0$. $\exists \delta > 0$ s.t.

$$\rho(x, x_0) < \delta.\ x \in A \text{ imply } d(f(x), f(x_0)) < \varepsilon.$$

i.e. $f(x) \to f(x_0)$ as $x \to x_0$.

We say f is <u>continuous</u> (on A) if f is continuous at every $x_0 \in A$.

<u>Theorem</u>. The followings are equivalent.

(a) f is continuous (on A).

(b) $f^{-1}(U)$ is open $\forall$ open set $U \subset Y$.

(c) $f^{-1}(V)$ is closed $\forall$ closed set $V \subset Y$.

(d) $f(\bar{W}) \subset \overline{f(W)}$ $\forall W \subset A$.

(e) $\forall x_0 \in A$. $\forall$ seq. $\{x_n\}$ in A conv. to x_0, we have $\lim_{n \to \infty} f(x_n) = f(x_0)$.

(pf. Same as $\mathbb{R}^n$).

§3. Interior, closure, and boundary.

Def. $A \subset X$. (X, ρ) a metric space.

The interior of A is

$$A^\circ = \bigcup \{ u : u \subset A,\ u \text{ is open} \}.$$

The closure of A is

$$\bar{A} = \bigcap \{ V : V \supset A,\ V \text{ is closed} \}$$

The boundary of A is $\partial A = \bar{A} \setminus A^\circ$.

Pts in A° are called interior pts.

" $\bar{A}$ " " contact pts.

" ∂A " " boundary pts.

Theorem (a) $x \in A^\circ \Leftrightarrow \exists$ open set $U \ni x$ s.t. $U \subset A$.

$\Leftrightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$.

(b) $x \in \bar{A} \Leftrightarrow \forall$ open set $U \ni x$, $U \cap A \neq \phi$.

$\Leftrightarrow \forall \varepsilon > 0$. $B_\varepsilon(x) \cap A \neq \phi$.

(c) $x \in \partial A \Leftrightarrow \forall$ open set $U \ni x$, $U \cap A \neq \phi$. $U \cap A^c \neq \phi$.

$\Leftrightarrow \forall \varepsilon > 0$. $B_\varepsilon(x) \cap A \neq \phi$. $B_\varepsilon(x) \cap A^c \neq \phi$.

Theorem. Given $A, B \subset X$.

(a) $(A \cup B)^\circ \supset A^\circ \cup B^\circ$. $(A \cap B)^\circ = A^\circ \cap B^\circ$

(b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$. $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

(c) $\partial(A \cup B) \subset \partial A \cup \partial B$. $\partial(A \cap B) \subset \partial A \cup \partial B$.

(pf. same as $\mathbb{R}^n$).

Example 1. $A = [0,1] \cap \mathbb{Q}$.

$(A, |\cdot|)$ is a metric subspace of $(\mathbb{R}, |\cdot|)$.

As a subset of $\mathbb{R}$, $A^\circ = \phi$. ("open sets" are open in $\mathbb{R}$)

As a metric space, $A^\circ = A$. ("open sets" are rel. open sets in A).

As a subset of $\mathbb{R}$, $\bar{A} = [0,1]$. $\partial A = [0,1]$.

As a metric space, $\bar{A} = A$. $\partial A = \phi$.

Example 2. $(\mathbb{R}, \rho)$. ρ = discrete metric. $A \subset \mathbb{R}$.

Any $\frac{1}{2}$-nbd. $B_{\frac{1}{2}}(x)$ of x consists of only one pt. x.

$\therefore$ Singletons are open.

"A" is union of singletons. $\therefore$ A is open.

$\therefore$ $A^\circ = A$. Similarly, A^c is open. $\therefore$ A is closed.

$\therefore$ $\bar{A} = A$. $\partial A = \phi$.

§4. Compact sets.

Def. Given $E \subset X$. We say $\{U_i\}_{i\in I}$ is an open cover of E if each U_i is open and $E \subset \bigcup_{i\in I} U_i$.
We say $K \subset X$ is compact if every open cover has a finite subcover.
We say $K \subset X$ is sequentially compact if every seq. in K has a subseq. which conv. to some pt. in K.

Theorem (a) Compact sets are closed.
(b) Closed subsets of compact sets are compact.
(c) Compact $\equiv$ sequentially compact.
(d) $f: K \to Y$ cont. K cpt. $\Rightarrow$ $f(K)$ is cpt.
(e) $f: K \to Y$ cont. 1-1. onto. $\Rightarrow$ f^{-1} is cont.

(f) (Extreme value theorem).

$f: K \to \mathbb{R}$ is cont. K is cpt.

$\Rightarrow$ f attains its infimum & supremum on K.

(g) $f: K \to Y$ is cont. K is cpt.

$\Rightarrow$ f is uniformly continuous.

(pf. same as $\mathbb{R}^n$).

What is missing?

We don't have Heine-Borel Theorem in metric spaces!

Heine-Borel Theorem is false in general metric spaces.

Example. $X = C[0,1]$. $(X, \|\cdot\|_\infty)$ is a normed vector space $\Rightarrow$ it is a metric space.

Cauchy criterion for uniform convergence implies $(X, \|\cdot\|_\infty)$ is complete.

$f_n(x) = x^n$. $\Rightarrow$ $f_n \in X$. $\|f_n\|_\infty = 1$ $\forall n$.

$\therefore \{f_n\} \subset \overline{B_1(0)}$ (unit closed ball in X).

$$f_n \to \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1). \end{cases} \notin X.$$

$\{f_n\}$ is bounded but w/o conv. subseq.

$\therefore$ $\overline{B_1(0)}$ is bdd. closed, but not compact.