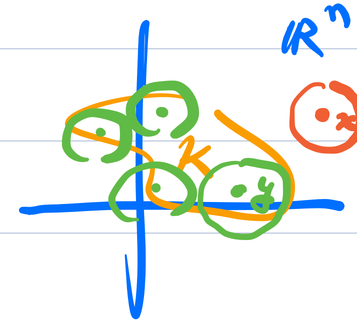
2022/5/12

Theorem 1. Compact sets are closed.



pf. Suppose K is compact.

Given $x \in K^c$.

For any $y \in K$. let $\varepsilon_y = \frac{1}{2}\|x-y\|$.

$\Rightarrow B_{\varepsilon_y}(y) \cap B_{\varepsilon_y}(x) = \phi$.

i.e. $B_{\varepsilon_y}(x) \subset B_{\varepsilon_y}(y)^c$

$\{B_{\varepsilon_y}(y)\}_{y \in K}$ is an open cover for K.

$\Rightarrow \exists$ finite subcover $\{B_{\varepsilon_{y_i}}(y_i)\}_{i=1}^N$. $K \subset \bigcup_{i=1}^N B_{\varepsilon_{y_i}}(y_i)$

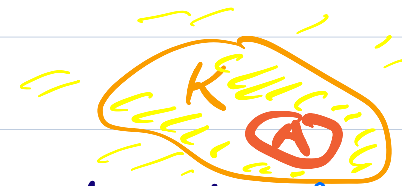
$\Rightarrow K^c \supset \left(\bigcup_{i=1}^N B_{\varepsilon_{y_i}}(y_i)\right)^c = \bigcap_{i=1}^N B_{\varepsilon_{y_i}}(y_i)^c \supset \bigcap_{i=1}^N B_{\varepsilon_{y_i}}(x)$

open nbd. of x

$\therefore K^c$ is open.

i.e. K is closed. QED

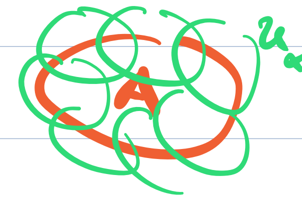
Theorem 2. Closed subsets of a compact set are compact.



pf. Let $A \subset K$. K: compact. A is closed.

Given any open cover $\{U_\alpha\}_{\alpha \in I}$ for A.

$\Rightarrow$ $\{U_\alpha\} \cup \{A^c\}$ is an open cover for K.



$\because$ K is compact.

$\therefore$ $\exists$ finite subcover $\{U_1, U_2, \dots, U_n\} \cup \{A^c\}$.

$\Rightarrow$ $\{U_1, \dots, U_n\}$ is a finite open cover for A.

$\therefore$ Open cover $\{U_\alpha\}$ for A has finite subcover.

$\therefore$ A is compact.

QED.

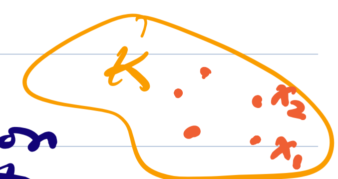
Def. We say $K \subset \mathbb{R}^n$ is <u>sequentially compact</u> if any seq. in K has a subseq. which converges to some pt. in K.

Theorem 3. Compact $\equiv$ Sequentially compact.

pf. "$\Rightarrow$" Assume K is compact.

Suppose K is <u>not</u> seq. compact.

$\Rightarrow \exists$ seq. $S = \{x_n\}$ in K w/o subseq. which conv. to pt. in K.



$\Rightarrow$ S is infinite and has no accumulation pt.

$\Rightarrow$ S is closed $\Rightarrow$ S is compact.

$\forall x_n \in S$. $\exists \varepsilon_n > 0$ s.t. $B_{\varepsilon_n}(x_n) \cap S = \{x_n\}$.

$\Rightarrow \{B_{\varepsilon_n}(x_n)\}$ is an open cover for S w/o finite subcover. $\times$. $\therefore K$ is seq. compact.

"$\Leftarrow$" Assume K is sequentially compact.

Let $\{U_\alpha\}$ be an open cover for K.

First we claim:

(*) $\exists\, \delta > 0$ s.t. $\forall x \in K$, $B_\delta(x) \subset U_\alpha$ for some α

(This δ is called a Lebesgue number)

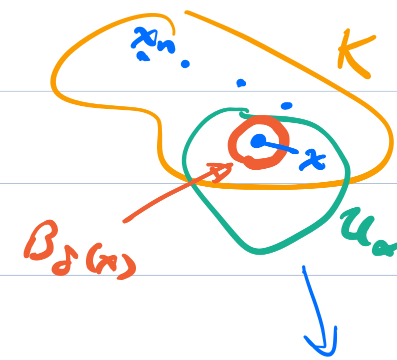
Suppose o.w. Then $\exists$ seq. $\{x_n\} \subset K$ s.t.

$$B_{\frac{1}{n}}(x_n) \not\subset U_\alpha \quad \forall \alpha.$$

$\because$ K is seq. compact.

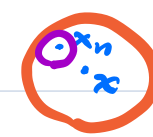
$\therefore$ $\exists$ subseq. w/ limit $x \in K$.

$\Rightarrow$ $x \in U_\alpha$ for some α.



$\Rightarrow$ $\exists\, \delta > 0$ s.t. $B_\delta(x) \subset U_\alpha$.

Choose $n > \frac{2}{\delta}$ large s.t. $\|x_n - x\| < \frac{\delta}{2}$.



$\forall y \in B_{\frac{1}{n}}(x_n)$, $\|y - x\| \le \|y - x_n\| + \|x_n - x\| < \frac{1}{n} + \frac{\delta}{2} < \delta$.

$\therefore$ $B_{\frac{1}{n}}(x_n) \subset B_\delta(x) \subset U_\alpha$. $\times$.

From ⊛, $\exists\, \delta > 0$ s.t. $B_\delta(x) \subset U_{\alpha_x}$ for some α_x.

$\{B_\delta(x) : x \in K\}$ is an open cover for K.

<u>Claim</u>: $\exists$ finite subset $\{x_1, \dots, x_N\}$ s.t.

$\{B_\delta(x_k)\}_{k=1}^{N}$ is an open cover for K.

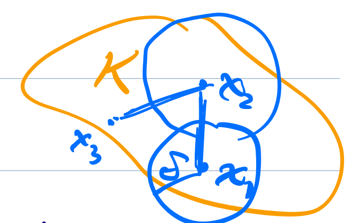
Pick any $x_1 \in K$.

If $B_\delta(x_1) \not\supset K$, then pick $x_2 \in K \setminus B_\delta(x_1)$

If $B_\delta(x_1) \cup B_\delta(x_2) \not\supset K$, then pick $x_3 \in K \setminus (B_\delta(x_1) \cup B_\delta(x_2))$

Continue this process.

If $B_\delta(x_1) \cup \dots \cup B_\delta(x_N) \supset K$, then we are done.



If not, then $\exists$ infinite seq. $\{x_n\}$ in K s.t.

$\|x_i - x_j\| > \delta \quad \forall i \neq j$.

$\Rightarrow \{x_n\}$ has no conv. subseq. $\times$.

$\therefore \{B_\delta(x_i)\}_{i=1}^{N}$ is an open cover for K, for some N.

$\Rightarrow \{U_{\alpha_{x_i}}\}_{i=1}^{N}$ is an open cover for K. $\therefore K$ is cpt. <u>QED</u>

Theorem 4. (Heine-Borel Theorem)

A set $K \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

pf. "$\Rightarrow$" Assume K is compact.

$\Rightarrow K$ is closed. (by Theorem 1).

The open cover $\{B_m(0)\}_{m=1}^{\infty}$ for K has finite cover since K is compact.

$\Rightarrow K \subset B_M(0)$ for some $M > 0$.

$\therefore K$ is bounded.

"$\Leftarrow$" Assume K is closed and bounded.

By Theorem 3, we need show that K is seq. compact.

Given $S = \{x_k\}_{k=1}^{\infty}$ in K.

If S is finite, then $\exists\, x \in S$ s.t. $x_k = x$ for infinitely many k.

$\Rightarrow \exists$ subseq. in S which conv. to $x \in S \subset K$.

If S is infinite, then by Bolzano-Weierstrass theorem, $\exists$ conv. subseq. w/ limit x.

x is an accumulation pt. of S

$\therefore x$ " " " " K

$\therefore x \in K$ since K is closed.

$\therefore K$ is seq. compact.

$\therefore K$ is compact. QED.

§3. Limits of functions.

domain of f

Consider $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$.

$x_0 \in A$ or $\notin A$.

Given accumulation pt. $x_0 \in A$.

We say $f(x)$ converges to $y_0 \in \mathbb{R}^m$ as x approaches x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < \|x - x_0\| < \delta,\ x \in A \quad \text{imply} \quad \|f(x) - y_0\| < \varepsilon.$$

This y_0 is called the limit of $f(x)$ as $x \to x_0$.

Notation: $\lim\limits_{x \to x_0} f(x) = y_0$. or $f(x) \to y_0$ as $x \to x_0$.

Equivalent definition: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$f\big((B_\delta(x_0) \setminus \{x_0\}) \cap A\big) \subset B_\varepsilon(y_0).$$

Theorem! (a) If $\lim_{x\to x_0} f(x)$ exists, then the limit is unique.

(b) (Sequential characterization of limits).

$$\lim_{x\to x_0} f(x) = y_0 \iff \forall \text{ seq. } \{x_k\} \subset A\setminus\{x_0\} \text{ s.t. } \lim_{k\to\infty} x_k = x_0$$

we have $\lim_{k\to\infty} f(x_k) = y_0$.

(c) (Algebraic properties). $f, g : A \subset \mathbb{R}^n \to \mathbb{R}^m$.

Suppose $\lim_{x\to x_0} f(x)$, $\lim_{x\to x_0} g(x)$ exist. Then

$$\lim_{x\to x_0} (f(x)+g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$$

$$\lim_{x\to x_0} (\alpha f(x)) = \alpha \lim_{x\to x_0} f(x).$$

$$\lim_{x\to x_0} f(x)\cdot g(x) = \left(\lim_{x\to x_0} f(x)\right)\cdot\left(\lim_{x\to x_0} g(x)\right)$$

$$\left\|\lim_{x\to x_0} f(x)\right\| = \lim_{x\to x_0} \|f(x)\|.$$

$m=3$. $\lim_{x\to x_0} f(x)\times g(x) = \left(\lim_{x\to x_0} f(x)\right)\times\left(\lim_{x\to x_0} g(x)\right).$

$m=1$. $\lim_{x\to x_0} g(x) \neq 0. \Rightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} f(x) \Big/ \lim_{x\to x_0} g(x).$

Denote $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ by $f = (f_1, f_2, \dots, f_m)$.

Theorem 2. $\lim_{x \to \bar{x}} f(x) = \bar{y} = (\bar{y}_1, \dots, \bar{y}_m)$.

$\Leftrightarrow \lim_{x \to \bar{x}} f_k(x) = \bar{y}_k \quad \forall k$.

pf. Observe: $\underset{\max\limits_{k=1,\dots,N} |z_k|}{\| z \|_\infty} \le \underset{\underline{\sqrt{z_1^2 + \dots + z_N^2}}}{\| z \|} \le \sqrt{n} \| z \|_\infty \le \sqrt{n} \max\limits_{k=1,\dots,N} |z_k|$. $\forall z \in \mathbb{R}^N$

$\| f(x) - \bar{y} \| \to 0$ as $x \to x_0$

$\Leftrightarrow \| f(x) - \bar{y} \|_\infty \to 0$ " "

$\Leftrightarrow \max\limits_k | f_k(x) - \bar{y}_k | \to 0$ " "

$\Leftrightarrow | f_k(x) - \bar{y}_k | \to 0$ as $x \to x_0 \quad \forall k$.

QED

Example! Define $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ by

$$f(x,y) = \frac{2xy}{x^2+y^2}.$$

Question : $\lim_{(x,y)\to(0,0)} f(x,y)$ exists or not ?

Take $(x_n, y_n) = (\frac{1}{n}, 0)$. $f(x_n, y_n) = 0$

$$\therefore \lim_{n\to\infty} f(x_n,y_n) = 0.$$

Take $(\tilde{x}_n, \tilde{y}_n) = (\frac{1}{n}, \frac{1}{n})$. $f(\tilde{x}_n, \tilde{y}_n) = \dfrac{2(\frac{1}{n})(\frac{1}{n})}{\frac{1}{n^2}+\frac{1}{n^2}} = 1.$

$$\therefore \lim_{n\to\infty} f(\tilde{x}_n,\tilde{y}_n) = 1.$$

(x_n, y_n), $(\tilde{x}_n, \tilde{y}_n) \to (0,0)$ as $n \to \infty$.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does <u>not</u> exist.

__Example 2__. Define $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ by

$$f(x,y) = \frac{xy^2}{x^2+y^4}.$$

Question : $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exists or not?

Let $(x_n, y_n) = (\frac{1}{n}, 0)$ be as in Example 1.

$f(x_n, y_n) = 0 \quad \forall n. \Rightarrow \displaystyle\lim_{n\to\infty} f(x_n,y_n) = 0.$

Let $(\tilde{x}_n, \tilde{y}_n) = (\frac{1}{n^2}, \frac{1}{n})$. $f(x_n,y_n) = \dfrac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^4}+\frac{1}{n^4}} = \frac{1}{2} \quad \forall n.$

$\Rightarrow \displaystyle\lim_{n\to\infty} f(\tilde{x}_n, \tilde{y}_n) = \frac{1}{2}.$

$\because (x_n, y_n), (\tilde{x}_n, \tilde{y}_n) \to (0,0)$ as $n \to \infty$.

$\therefore \displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does <u>not</u> exist.