2022/5/5.

Advanced Calculus (I).

§3. Interior, closure, boundary

Recall: (Last time) $A \subset \mathbb{R}^n$.

Interior of $A = A^\circ$ (or $\mathrm{int}(A)$)

$= \bigcup \{ U : U \subset A,\ U \text{ is open} \}$

Closure of $A = \bar{A}$ (or $\mathrm{cl}(A)$)

$= \bigcap \{ V : V \supset A,\ V \text{ is closed} \}$

Boundary of $A = \partial A$ (or $\mathrm{bd}(A)$)

$= \bar{A} \setminus A^\circ$.

Points in A° are called interior points.

" $\bar{A}$ " contact points

" ∂A " boundary points

Remarks

(1) A° is open. ($\because$ it is union of open sets)

(2) $\bar{A}$ is closed. ($\because$ it is intersection of closed sets)

(3) $\partial A = \bar{A} \setminus A^\circ = \underbrace{\bar{A}}_{\text{closed}} \cap \underbrace{(A^\circ)^c}_{\text{closed}}$ is closed.

(4) $\underbrace{A^\circ}_{\text{Largest open subset of } A.} \subset A \subset \bar{A}$: Smallest closed set containing A.

(5) A is open $\iff A = A^\circ$. ($\because A^{\circ\circ} = A^\circ$)

(6) A is closed $\iff A = \bar{A}$. ($\because \bar{\bar{A}} = \bar{A}$)

Theorem (Equivalent Definitions of $A^\circ, \bar{A}, \partial A$.) $A \subset \mathbb{R}^n$

(a) ① $x \in A^\circ$

$\Leftrightarrow$ ② $\exists$ open set U containing x s.t. $U \subset A$.

$\Leftrightarrow$ ③ $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$.

(b) ④ $x \in \bar{A}$

$\Leftrightarrow$ ⑤ $\forall$ open set U containing x, $U \cap A \neq \emptyset$

$\Leftrightarrow$ ⑥ $\forall\, \varepsilon > 0$. $B_\varepsilon(x) \cap A \neq \emptyset$

(c) ⑦ $x \in \partial A$

$\Leftrightarrow$ ⑧ $\forall$ open set U containing x, $U \cap A \neq \emptyset$. $U \cap A^c \neq \emptyset$.

$\Leftrightarrow$ ⑨ $\forall\, \varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$. $B_\varepsilon(x) \cap A^c \neq \emptyset$.

pf. ① $\Rightarrow$ ②. by def. of A°.

② $\Rightarrow$ ③ by def. of open sets.

③ $\Rightarrow$ ① since open balls are open

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④ $\Rightarrow$ ⑤ Suppose o.w. Then $\exists$ open set U containing x s.t. $U \cap A = \phi$.

$\Rightarrow U^c \supset A$. U^c is closed.

$\Rightarrow U^c \supset \bar{A} \ni x \Rightarrow x \notin U$. $\times$

⑤ $\Rightarrow$ ⑥ since open balls are open.

⑥ $\Rightarrow$ ④. Suppose o.w. $x \notin \bar{A}$. i.e. $x \in \bar{A}^c$

$\bar{A}^c$ is open $\Rightarrow \exists B_\varepsilon(x) \subset \bar{A}^c$.

$\Rightarrow B_\varepsilon(x) \cap \bar{A} = \phi$

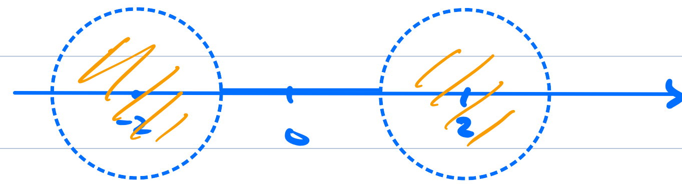
But $B_\varepsilon(x) \cap A \neq \phi$ $\times$

$x \in \partial A = \bar{A} \setminus A^\circ \Leftrightarrow x \in \bar{A},\ x \notin A^\circ$.

$\Leftrightarrow$ $\forall$ open set U containing x. $U \cap A \neq \phi$. $U \cap A^c \neq \phi$.

$\Leftrightarrow \forall \varepsilon > 0$. $B_\varepsilon(x) \cap A \neq \phi$. $B_\varepsilon(x) \cap A^c \neq \phi$. QED.

Example. $B_1(-2,0) \cup B_1(2,0) \cup \{(x,0) : -1 \le x \le 1\} = A$.



$$A^\circ = B_1(-2,0) \cup B_1(2,0).$$

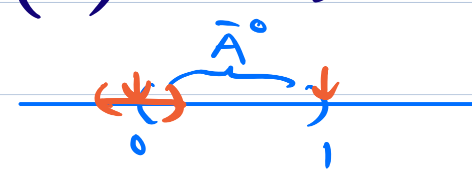
$$\bar{A} = \overline{B_1(-2,0)} \cup \overline{B_1(2,0)} \cup \{(x,0) : -1 \le x \le 1\}.$$

$$\partial A = \partial B_1(-2,0) \cup \partial B_1(2,0) \cup \{(x,0) : -1 \le x \le 1\}.$$

Example. $A = [0,1] \cap \mathbb{Q}$.

$A^\circ = \emptyset$. $\bar{A} = [0,1] = \partial A$.

$\bar{A}^\circ = (0,1)$. $\overline{(\bar{A}^\circ)} = [0,1]$. $\partial(\bar{A}^\circ) = \{0,1\}$.



Remark. $\partial A = \bar{A} \setminus A^\circ = \bar{A} \cap (A^\circ)^c$

$A \subset \mathbb{R}$. "cl. int. complement". Q: How many different sets can you get?

Theorem. Given $A, B \subset \mathbb{R}^n$.

(a) $(A\cup B)^\circ \supset A^\circ \cup B^\circ$. $(A\cap B)^\circ = A^\circ \cap B^\circ$.

(b) $\overline{A\cup B} = \bar{A}\cup \bar{B}$. $\overline{A\cap B} \subset \bar{A}\cap\bar{B}$.

(c) $\partial(A\cup B) \subset \partial A \cup \partial B$. $\partial(A\cap B) \subset \partial A\cup\partial B$.

pf. (a) $A^\circ \subset A \subset A\cup B \Rightarrow A^\circ \subset (A\cup B)^\circ$

Similarly, $B^\circ \subset (A\cup B)^\circ$. $\therefore A^\circ \cup B^\circ \subset (A\cup B)^\circ$.

$(A\cap B)^\circ$ is open subset of A, & B.

$\therefore (A\cap B)^\circ \subset A^\circ \cap B^\circ$.

On the other hand, $A^\circ \cap B^\circ \subset A\cap B$

$A^\circ \cap B^\circ$ is open $\Rightarrow A^\circ \cap B^\circ \subset (A\cap B)^\circ$

$\therefore (A\cap B)^\circ = A^\circ \cap B^\circ$

(b) $A\subset \bar{A}$, $B\subset\bar{B}$. $\Rightarrow A\cup B \subset \bar{A}\cup\bar{B}$. — closed.

$\Rightarrow \overline{A\cup B} \subset \bar{A}\cup\bar{B}$.

On the other hand, $\bar{A}\subset\overline{A\cup B}$, $\bar{B}\subset\overline{A\cup B}$.

$\therefore \bar{A}\cup\bar{B}\subset\overline{A\cup B}$. $\therefore \overline{A\cup B} = \bar{A}\cup\bar{B}$. (exercise — rest of pf.).

Remark. $(A \cup B)^\circ \supset A^\circ \cup B^\circ$. "$\supset$" can't be replaced by "$=$".

e.g. $A = (0,1)$. $B = [1,2]$.

$(A \cup B)^\circ = (0,2) \neq (0,1) \cup (1,2) = A^\circ \cup B^\circ$.

$\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. "$\subset$" can't be replaced by "$=$"

e.g. $A = [0,1] \cap \mathbb{Q}$. $B = [0,1] \setminus \mathbb{Q}$.

$\overline{A \cap B} = \emptyset \neq [0,1] = \bar{A} \cap \bar{B}$.

Def. We say $A \subset B$ is <u>dense</u> in B if $\bar{A} = B$.

Chap 9. Convergence in $\mathbb{R}^n$.

§1. Limits of Sequences.

<u>Def</u>. Given seq. $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$.

We say $\{x_k\}$ conv. to $x \in \mathbb{R}^n$ if

$$\forall \varepsilon > 0. \ \exists N \in \mathbb{N} \ \text{s.t.} \ \|x_k - x\| < \varepsilon \quad \forall k \geq N.$$

In this case, we say x is the limit of $\{x_k\}$.

Denoted by $\lim\limits_{k\to\infty} x_k = x$ or $x_k \to x$ as $k \to \infty$.

If $\{x_k\}$ converges to some x, then we say $\{x_k\}$ is convergent, o.w., we say it is divergent.

We say $\{x_k\}$ is bounded if $\exists M > 0$ s.t.

$$\|x_k\| \leq M \quad \forall k.$$

We say $\{x_k\}$ is unbounded if it is not bounded.

We say $\{x_k\}$ is a Cauchy seq. if
$\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\|x_k - x_m\| < \varepsilon$ $\forall k > m \geq N$.

Theorem 1. (Uniqueness of Limit).
(a) If $\{x_k\}$ conv., then its limit is unique.
(b) If $x_k \to x$ as $k \to \infty$, then any subseq. of $\{x_k\}$ also conv. to x.

Theorem 2. (Algebraic properties of Limit).
Suppose $\lim_{k\to\infty} x_k = x$. $\lim_{k\to\infty} y_k = y$. Then
(a) $\lim_{k\to\infty} (x_k + y_k) = x + y$.
(b) $\lim_{k\to\infty} (\alpha x_k) = \alpha \lim_{k\to\infty} x_k$ $\forall \alpha \in \mathbb{R}$.
(c) $\lim_{k\to\infty} x_k \cdot y_k = x \cdot y$
(d) $n=3$. $\lim_{k\to\infty} x_k \times y_k = x \times y$.

Theorem 3 (Criteria for Convergence).

(a) Conv. seq. are Cauchy.

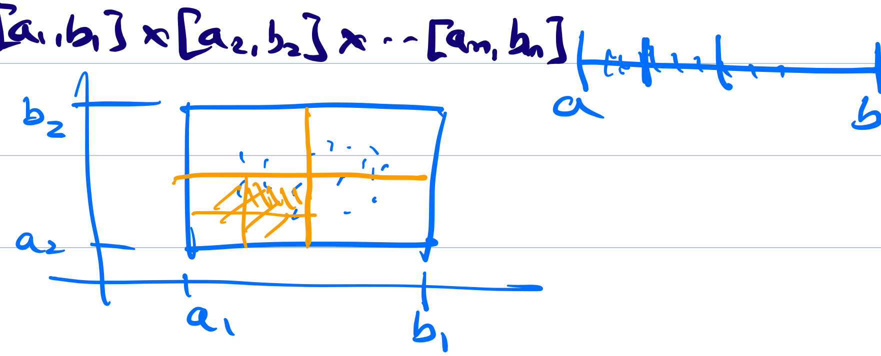
(b) Cauchy seq are bounded.

(c) Bounded seq. has conv. subseq.
(Bolzano - Weierstrass theorem)

(d) Cauchy seq. are conv.
((a) + (d) is the Cauchy criterion)

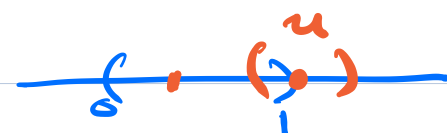
pf. Same as the case $n=1$.
Just replace absolute value $|\cdot|$ by norm $\|\cdot\|$.
and intervals by rectangles.

$[a,b]$. $[a_1,b_1]\times[a_2,b_2]\times\cdots[a_n,b_n]$



Def. Given $A \subset \mathbb{R}^n$. We say $x \in \mathbb{R}^n$ is an accumulation point (or limit points, or cluster points) of A if $\forall$ nbd. $\mathcal{U}$ of x, $\mathcal{U}$ contains infinitely many points of A.

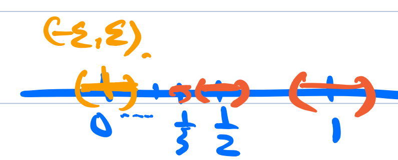
Example. $A = (0,1)$. $x = 1$ is an accumulation pt. of A.



$A = [0,1] \cap \mathbb{Q}$. any $x \in [0,1]$ is an acc. pt. of A.

We say $x \in A$ is an isolated point of
$\exists$ nbd. $\mathcal{U}$ of x s.t. $\mathcal{U} \cap A = \{x\}$.

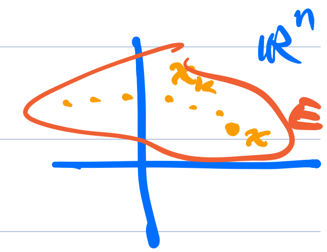
Example. $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.



$\frac{1}{n}$ is isolated pt. $\forall n$.

0 is accumulation pt. of A.

Remark. Suppose $S=\{x_k\}\subset E\subset \mathbb{R}^n$. and $x_k\to x$ as $k\to\infty$.



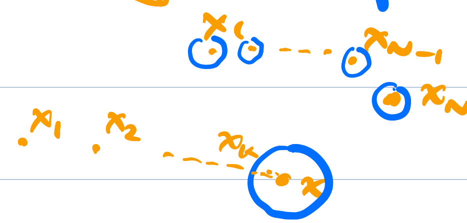
Two cases:

① $x_k = x$ $\forall k\geq N$. for some $N\in\mathbb{N}$.

Every pt. in S is isolated in S.

② $\exists$ ∞-ly many $x_k \neq x$.

x is an acc. pt. of S.



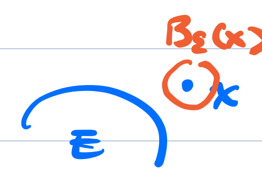
Theorem 4. A set $E\subset\mathbb{R}^n$ is closed if and only if it contains all of its accumulation pts.

pf. E is closed $\Leftrightarrow$ E^c is open

$\Leftrightarrow$ $\forall x\in E^c$, $\exists\, \varepsilon>0$ s.t. $B_\varepsilon(x)\subset E^c$ (i.e. $B_\varepsilon(x)\cap E=\emptyset$)

$\Leftrightarrow$ $\forall x\in E^c$, x is not an acc. of E.

$\Leftrightarrow$ E contains all of its acc. pts.



QED.

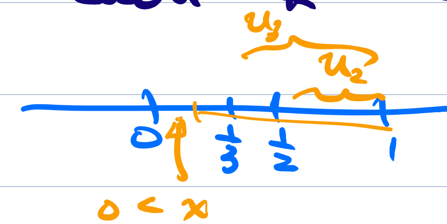
§2. Compact sets.

Def. Given $E \subset \mathbb{R}^n$. A cover (or covering) of E is a collection $\{U_i\}_{i \in I}$ of sets s.t. $\bigcup_{i \in I} U_i \supset E$.
In this case we say $\{U_i\}_{i \in I}$ covers E.
If each U_i is open, then we say $\{U_i\}_{i \in I}$ is an open cover (or open covering) of E.

We say $E \subset \mathbb{R}^n$ is compact if every open cover of E has a finite subcover.
i.e. $\forall$ open cover $\{U_i\}_{i \in I}$ of E. $\exists$ finite collection $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$ s.t. $E \subset \bigcup_{j=1}^{m} U_{i_j}$

Example. (1) $E = (0,1)$. $U_k = (\frac{1}{k}, 1)$. — each U_k is open.

$$\bigcup_{k=1}^{\infty} U_k = E$$



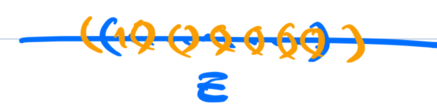
∴ $\{U_k\}$ is an open cover of E w/o finite subcover.

∴ $(0,1)$ is not compact.

(2) $E = (0,1) \cap \mathbb{Q}$. $\forall x \in E$. Take $\varepsilon > 0$.

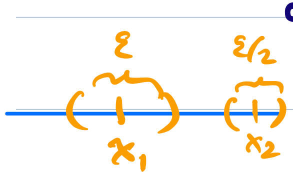
$\{B_\varepsilon(x) : x \in E\}$ is an open cover of E.

it has finite subcover.



We can find an open cover w/o finite subcover:

Let $E = \{x_k\}_{k=1}^{\infty}$. Fix $0 < \varepsilon < \frac{1}{2}$.



$\{B_{\varepsilon/2^k}(x_k)\}_{k=1}^{\infty}$ is an open cover for E.

Their total length $\leq \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \cdots\cdots$

$= 2\varepsilon < 1$

This open cover has no finite subcover.