Applied Mathematics E-Notes, 26(2026), 298-306 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Strongly Invariant Convergence In Fuzzy Normed Spaces*

Şeyma Yalvaç†, Erdinç Dündar‡

Received 31 December 2025

Abstract

In this study, we define the notions of the invariant convergence and the strongly invariant convergence in fuzzy normed spaces and investigate relations between them. We examine some properties of strongly invariant convergence such as the uniqueness of limits and linearity in fuzzy normed spaces. Also, we prove the equality of strongly invariant convergence and lacunary strongly invariant convergence.

1 Introduction

Banach [4] defined the generalized limit as an application of Hahn-Banach theorem on the set of all bounded real valued sequences. It is also known as Banach limit. Later, Lorentz [22] proposed that if all Banach limits of a given bounded sequence are equal, it is called almost convergent. In further studies [9,30], invariant mean and invariant convergence are given as a more general case of Banach limit and almost convergence. Also, several authors including Schaefer [35], Mursaleen and Edely [26], Mursaleen [27,28], Savaş [31,32] studied on invariant convergent sequences. Lacunary convergence with the relation to strong Cesàro summability was studied by Freedman et al. [14]. Also Das and Patel [8] investigated this issue comprehensively. Savaş [33] proved important theorems about lacunary invariant convergence.

The pioneer of fuzzy logic, which deals with uncertain facts, is Zadeh [41]. With the advent of this important concept, several classical concepts were reconstructed. Fuzzy topological spaces [5, 23], fuzzy metric [15, 19, 21], fuzzy norm [3, 6, 13, 20] are just some of the examples. Felbin's fuzzy norm [13], which is associated with Kaleva and Seikkala [19] type metric space by assigning a non-negative fuzzy real number to each element of a linear space, forms the basis of this study. Das and Das [7] studied fuzzy topology generated by fuzzy norm. Diamond and Kloeden [10] investigated the metric spaces of fuzzy sets-theory and applications. Fang and Huang [12] studied the level convergence of a sequence of fuzzy numbers. Recently, Yalvaç and Dündar [39] introduced the invariant convergence and invariant Cauchy sequence with some properties in fuzzy normed spaces. Also, some other authors [11, 16, 18, 25] studied the notion of fuzzy numbers and fuzzy normed spaces.

Now, we recall the basic notions and some important definitions and notations used in our paper (See [1–3, 6, 9, 11, 13, 16, 17, 20, 22, 24–30, 34–38, 40, 41]). A fuzzy set u is a fuzzy number if it provided that

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$;

(iii) u is upper semi-continuous;

(iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is a compact set.

Let $L(\mathbb{R})$ be the set of all fuzzy numbers. $\mathbb{R}$ can be embedded in $L(\mathbb{R})$ since each $r \in \mathbb{R}$ can be considered as a fuzzy real number $\tilde{r}$ defined by

$$\tilde{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{for } t \neq r. \end{cases}$$

*Mathematics Subject Classifications: 40A30, 40A35, 46S40.

†Faculty of Education, Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey

‡Department of Mathematics, Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey

For $u \in L(\mathbb{R})$, the α-level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1], \\ cl\{x \in \mathbb{R} : u(x) > \alpha\}, & \text{if } \alpha = 0. \end{cases}$$

The α-level set of a fuzzy number denoted by $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ is a non-empty, bounded and closed interval for each $\alpha \in [0,1]$ where $u_\alpha^- = -\infty$ and $u_\alpha^+ = \infty$ are admissible. If $u \in L(\mathbb{R})$ and $u(x) = 0$ for $x < 0$, then u is called a non-negative fuzzy number. Let $L^*(\mathbb{R})$ denote the set of all non-negative fuzzy numbers. It is easy to see that $\tilde{0} \in L^*(\mathbb{R})$. A partial ordering $\preceq$ in $L(\mathbb{R})$ is defined for $u, v \in L(\mathbb{R})$ by

$$u \preceq v \quad \text{iff} \quad u_\alpha^- \leq v_\alpha^- \quad \text{and} \quad u_\alpha^+ \leq v_\alpha^+ \quad \text{for all} \quad \alpha \in [0,1].$$

Arithmetic operations (addition $\oplus$, multiplication $\odot$ and scalar multiplication) on $L(\mathbb{R})$ are defined by

(i) $(u \oplus v)(t) = \sup\limits_{s \in \mathbb{R}} \{u(s) \wedge v(t-s)\}, \ t \in \mathbb{R}$,

(ii) $(u \odot v)(t) = \sup\limits_{s \in \mathbb{R} s \neq 0} \{u(s) \wedge v(t/s)\}, \ t \in \mathbb{R}$,

(iii) For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \tilde{0}$, $t \in \mathbb{R}$.

Let $u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$. Arithmetic operations in terms of α-level sets are defined by

(i) $[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$,

(ii) $[u \odot v]_\alpha = [u_\alpha^- v_\alpha^-, u_\alpha^+ v_\alpha^+]$, $u, v \in L^*(\mathbb{R})$,

(ii) $[ku]_\alpha = k[u]_\alpha = \begin{cases} [ku_\alpha^-, ku_\alpha^+] & \text{if } k \geq 0, \\ [ku_\alpha^+, ku_\alpha^-] & \text{if } k < 0. \end{cases}$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined by

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}.$$

One can see that

$$D(u, \tilde{0}) = \sup_{0 \leq \alpha \leq 1} \max\{|u_\alpha^-|, |u_\alpha^+|\} = \max\{|u_0^-|, |u_0^+|\}.$$

Obviously, $D(u, \tilde{0}) = u_0^+$ when $u \in L^*(\mathbb{R})$. A sequence (u_n) in $L(\mathbb{R})$ is called convergent to $u \in L(\mathbb{R})$ denoted by $D^- \lim\limits_{n \to \infty} u_n = u$ if $\lim\limits_{n \to \infty} D(u_n, u) = 0$, i.e., for all given $\varepsilon > 0$ there exists a positive integer $N \in \mathbb{R}$ such that $D(u_n, u) < \varepsilon$, for $n > N$.

Let X be a vector space over $\mathbb{R}$, $\|.\| : X \to L^*(\mathbb{R})$ and the mappings $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. The quadruple $(X, \|.\|, L, R)$ is called a fuzzy normed linear space (FNS) and $\|.\|$ is a fuzzy norm if the following axioms are satisfied

(i) $\|x\| = \tilde{0}$ iff $x = \theta$,

(ii) $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,

(iii) for all $x, y \in X$

  (a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x+y\|_1^-$,

  (b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x+y\|_1^-$.

When $L = \min$ and $R = \max$ are taken in (iii), triangle inequalities become

$$\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^- \text{ and } \|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+,$$

for all $\alpha \in [0,1]$ and $x, y \in X$. Since they fulfil the other conditions of norm, $\|x\|_\alpha^-$ and $\|x\|_\alpha^+$ can be seen as ordinary norms on X.

Example 1 *Let $(X, \|.\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|.\|$ on X can be obtained*

$$\|x\|(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq a\|x\|_C \text{ or } t \geq b\|x\|_C, \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a}, & \text{if } a\|x\|_C \leq t \leq \|x\|_C, \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1}, & \text{if } \|x\|_C \leq t \leq b\|x\|_C, \end{cases}$$

where $\|x\|_C$ is the ordinary norm of $x\ (\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \tilde{0}$. Then $(X, \|.\|)$ is a fuzzy normed linear space.

Throughout this paper, let $(X, \|.\|)$ be a fuzzy normed linear space (FNS). Let us consider the topological structure of an FNS $(X, \|.\|)$. For any $\varepsilon > 0$, $\alpha \in [0,1]$ and $x \in X$, the (ε, α)-neighborhood of x is the set $\mathcal{N}_x(\varepsilon, \alpha) = \{y \in X : \|x-y\|_\alpha^+ < \varepsilon\}$.

A sequence (x_n) in X is convergent to $x \in X$ with respect to the fuzzy norm on X, denoted by $x_n \overset{FN}{\to} x$, provided that $(D) - \lim_{n\to\infty} \|x_n - x\| = \tilde{0}$, i.e., for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n > N$,

$$D(\|x_n - x\|, \tilde{0}) = \sup_{\alpha\in[0,1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon,$$

in terms of neighborhoods, $x_n \in \mathcal{N}_x(\varepsilon, 0)$.

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean if and only if

(i) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ satisfies $x_n \geq 0$ for all n,

(ii) $\phi(e) = 1$, where $e = (1, 1, 1...)$,

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mapping σ is assumed to be one-to-one and satisfy the condition $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Invariant mean, ϕ, is a extension of the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. The sequence is called invariant convergent when its invariant means are equal. In case $\sigma(n) = n+1$, the σ mean is often called a Banach limit and invariant convergent is almost convergent.

A bounded sequence (x_n) is σ-convergent to the number L if and only if $\lim\limits_{m\to\infty} t_{mn} = L$ uniformly in m, where

$$t_{mn} = \frac{x_n + x_{\sigma(n)} + x_{\sigma^2(n)} + \cdots + x_{\sigma^m(n)}}{m+1}.$$

An increasing sequence of non-negative integers $\theta = (k_r)$ with $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ is called a lacunary sequence. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is given by q_r.

2 Main Results

Definition 1 *A sequence $x = (x_n)$ in X is said to be invariant convergent to L with respect to the fuzzy norm and denoted by $x_n \stackrel{\sigma-FN}{\longrightarrow} L$ if*

$$(D) - \lim_{m\to\infty} \|t_{mn} - L\| = \tilde{0},$$

uniformly in n, i.e., for given $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$ and $n \in \mathbb{N}$,

$$D(\|t_{mn} - L\|, \tilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mn} - L\|^+_\alpha = \|t_{mn} - L\|^+_0 < \varepsilon;$$

in terms of neighborhood, for all $m > m_0$ and $n \in \mathbb{N}$, $t_{mn} \in \mathcal{N}_L(\varepsilon, 0)$.

Definition 2 *A sequence (x_n) in X is said to be strongly invariant convergent to L with respect to the fuzzy norm, if*

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} D(\|x_{\sigma^i(n)} - L\|, \tilde{0}) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \|x_{\sigma^i(n)} - L\|^+_0 = 0,$$

i.e., for given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$ and $n \in \mathbb{N}$,

$$\frac{1}{m}\sum_{i=1}^{m} \|x_{\sigma^i(n)} - L\|^+_0 < \varepsilon.$$

In this case, we write $x_n \stackrel{[\sigma-FN]}{\longrightarrow} L$.

Theorem 1 *Let (x_n) be a sequence in X. If (x_n) is strongly invariant convergent, then its limit is unique with respect to the fuzzy norm.*

Proof. Let $x_n \stackrel{[\sigma-FN]}{\longrightarrow} L_1$ and $x_n \stackrel{[\sigma-FN]}{\longrightarrow} L_2$, where $L_1 \neq L_2$. Then, for each $\varepsilon > 0$, there exists $m_1 \in \mathbb{N}$ such that for all $m > m_1$ and $n \in \mathbb{N}$,

$$\frac{1}{m}\sum_{i=1}^{m} \|x_{\sigma^i(n)} - L_1\|^+_0 < \frac{\varepsilon}{2}$$

and also, for given $\varepsilon > 0$, there exists $m_2 \in \mathbb{N}$ such that for all $m > m_2$ and $n \in \mathbb{N}$,

$$\frac{1}{m}\sum_{i=1}^{m} \|x_{\sigma^i(n)} - L_2\|^+_0 < \frac{\varepsilon}{2}.$$

Now, take $m_0 = \max\{m_1, m_2\}$. For all $m > m_0$ and $n \in \mathbb{N}$, we get

$$\|L_1 - L_2\|^+_0 = \frac{1}{m}\sum_{i=1}^{m} \|L_1 - L_2\|^+_0 \leq \frac{1}{m}\sum_{i=1}^{m} \|x_{\sigma^i(n)} - L_1\|^+_0 + \frac{1}{m}\sum_{i=1}^{m} \|x_{\sigma^i(n)} - L_2\|^+_0 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and so, we have $L_1 = L_2$. ∎

Theorem 2 *Let (x_n) and (y_n) be two sequences in X. If (x_n) and (y_n) are strongly invariant convergent to L_1 and L_2 respectively then the sequence $(x_n + y_n)$ is strongly invariant convergent to $L_1 + L_2$ with respect to the fuzzy norm.*

Proof. Assume that $x_n \stackrel{[\sigma-FN]}{\longrightarrow} L_1$ and $y_n \stackrel{[\sigma-FN]}{\longrightarrow} L_2$. Then, for every $\varepsilon > 0$, there exists $m_1 \in \mathbb{N}$ such that for all $m > m_1$ and $n \in \mathbb{N}$,

$$\frac{1}{m}\sum_{i=1}^{m} \|x_{\sigma^i(n)} - L_1\|^+_0 < \frac{\varepsilon}{2}$$

and also, for given $\varepsilon > 0$, there exists $m_2 \in \mathbb{N}$ such that for all $m > m_2$ and $n \in \mathbb{N}$,

$$\frac{1}{m}\sum_{i=1}^{m}\|y_{\sigma^i(n)} - L_2\|_0^+ < \frac{\varepsilon}{2}.$$

Taking $m_0 = \max\{m_1, m_2\}$ for all $m > m_0$ and $n \in \mathbb{N}$, we obtain that

$$\frac{1}{m}\sum_{i=1}^{m}\left\|\left(x_{\sigma^i(n)} + y_{\sigma^i(n)}\right) - (L_1 + L_2)\right\|_0^+ \leq \frac{1}{m}\sum_{i=1}^{m}\|x_{\sigma^i(n)} - L_1\|_0^+ + \frac{1}{m}\sum_{i=1}^{m}\|y_{\sigma^i(n)} - L_2\|_0^+ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and conclude that $x_n + y_n \overset{[\sigma-FN]}{\longrightarrow} L_1 + L_2$. ■

Theorem 3 *If (x_n) is strongly invariant convergent to L and c is a scalar, then the sequence $(c\,x_n)$ is strongly invariant convergent to cL.*

Proof. Let $x_n \overset{[\sigma-FN]}{\longrightarrow} L$ and c be a scalar. If $c = 0$, then proof is trivial. Assume that $c \neq 0$. Then, for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$ and $n \in \mathbb{N}$,

$$\frac{1}{m}\sum_{i=1}^{m}\|x_{\sigma^i(n)} - L\|_0^+ < \frac{\varepsilon}{|c|}.$$

Since

$$\frac{1}{m}\sum_{i=1}^{m}\|c\,x_{\sigma^i(n)} - c\,L\|_0^+ = |c|\frac{1}{m}\sum_{i=1}^{m}\|x_{\sigma^i(n)} - L\|_0^+ < |c|\frac{\varepsilon}{|c|} = \varepsilon,$$

we get $c\,x_n \overset{[\sigma-FN]}{\longrightarrow} c\,L$. ■

Theorem 4 *Let (x_n) be a sequence in X. If (x_n) is strongly invariant convergent to L, then (x_n) is invariant convergent to L.*

Proof. Assume that the sequence (x_n) is strongly invariant convergent to L. Then, for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$ and $n \in \mathbb{N}$, we have

$$\frac{1}{m}\sum_{i=1}^{m}\|x_{\sigma^i(n)} - L\|_0^+ < \varepsilon.$$

From the inequality for all $m > m_0$ and $n \in \mathbb{N}$,

$$\|t_{mn} - L\|_0^+ = \|\frac{1}{m}\sum_{i=1}^{m}x_{\sigma^i(n)} - L\|_0^+ \leq \frac{1}{m}\sum_{i=1}^{m}\|x_{\sigma^i(n)} - L\|_0^+ < \varepsilon,$$

for all $m > m_0$ and $n \in \mathbb{N}$, clearly the sequence (x_n) is invariant convergent to L. ■

Lemma 5 *Let (x_n) be a sequence in X. Assume that for any given $\varepsilon > 0$, there exist $m_0, n_0 \in \mathbb{N}$ such that*

$$\frac{1}{m}\sum_{i=1}^{m}\left\|x_{\sigma^i(n)} - L\right\|_0^+ < \varepsilon$$

for all $m \geq m_0$ and $n \geq n_0$. Then, (x_n) is strongly invariant convergent to L.

Proof. Firstly, for any $\varepsilon > 0$, let's choose $m_1, n_0 \in \mathbb{N}$ such that

$$\frac{1}{m}\sum_{i=1}^{m} \left\| x_{\sigma^i(n)} - L \right\|_0^+ < \frac{\varepsilon}{2}, \tag{1}$$

for all $m \geq m_1$ and $n \geq n_0$. We need to prove that there exists $m_2 \in \mathbb{N}$ such that for all $m \geq m_2$ and $0 \leq n \leq n_0$,

$$\frac{1}{m}\sum_{i=1}^{m} \left\| x_{\sigma^i(n)} - L \right\|_0^+ < \varepsilon. \tag{2}$$

Since, taking $m_0 = \max\{m_1, m_2\}$, for all $m \geq m_0$ and $n \in \mathbb{N}$,

$$\frac{1}{m}\sum_{i=1}^{m} \left\| x_{\sigma^i(n)} - L \right\|_0^+ < \varepsilon$$

is satisfied, which is the definition of strongly invariant convergence.

Once n_0 has been choosen, n_0 is fixed. So, there exists M such that

$$\sum_{i=1}^{n_0} \| x_i - L \|_0^+ = M. \tag{3}$$

Now, let's take $0 \leq n \leq n_0$ and $m \geq m_0$. Also for fixed n_0, there exists m' such that $\sigma^{m'}(n) \leq n_0$. So we have

$$\begin{aligned}
\frac{1}{m}\sum_{i=1}^{m} \left\| x_{\sigma^i(n)} - L \right\|_0^+ &= \frac{1}{m}\sum_{i=1}^{m'} \left\| x_{\sigma^i(n)} - L \right\|_0^+ + \frac{1}{m}\sum_{i=m'+1}^{m} \left\| x_{\sigma^i(n_0)} - L \right\|_0^+ \\
&\leq \frac{1}{m}\sum_{i=1}^{n_0} \| x_i - L \|_0^+ + \frac{1}{m}\sum_{i=1}^{m} \left\| x_{\sigma^i(n_0)} - L \right\|_0^+ \\
&\leq \frac{M}{m} + \frac{1}{m}\sum_{i=1}^{m} \left\| x_{\sigma^i(n_0)} - L \right\|_0^+ \quad \text{(from 3)} \\
&\leq \frac{M}{m} + \frac{\varepsilon}{2} \quad \text{(from 1)}.
\end{aligned}$$

If m is sufficiently large, we obtain $\frac{M}{m} + \frac{\varepsilon}{2} < \varepsilon$, which gives (2) and the proof is completed. ■

Definition 3 *For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary strongly invariant convergent to L with respect to the fuzzy norm and it is denoted by $x_n \overset{[\sigma-FN]_\theta}{\longrightarrow} L$ if*

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} D(\|x_{\sigma^i(n)} - L\|, \tilde{0}) = \lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L\|_0^+ = 0,$$

uniformly in n; i.e., for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r} D(\|x_{\sigma^i(n)} - L\|, \tilde{0}) = \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$.

Theorem 6 *Let (x_n) be a sequence in X. Then, $x_n \overset{[\sigma-FN]_\theta}{\longrightarrow} L \Leftrightarrow x_n \overset{[\sigma-FN]}{\longrightarrow} L$.*

Proof. Let $x_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} L$. Given $\varepsilon > 0$, there exists r_0 such that

$$\frac{1}{h_r}\sum_{i=1}^{h_r} \|x_{\sigma^i(n)} - L\|_0^+ < \varepsilon$$

for $r \geq r_0$ and $n = \sigma^{k_{r-1}}(s)$, $s > 0$. Take $m \geq h_r$ and write

$$m = t\,h_r + \theta, \quad \text{where } \ t \ \text{ is an integer and } \ 0 \leq \theta \leq h_r. \tag{4}$$

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^{m} \|x_{\sigma^i(n)} - L\|_0^+ &\leq \frac{1}{m}\sum_{i=1}^{(t+1)h_r} \|x_{\sigma^i(n)} - L\|_0^+ \\ &= \frac{1}{m}\sum_{\mu=0}^{t}\sum_{i=\mu h_r}^{(\mu+1)h_r} \|x_{\sigma^i(\sigma^{k_{r-1}}(n))} - L\|_0^+ \leq \frac{t+1}{m}\, h_r\, \varepsilon \leq \frac{2th_r\varepsilon}{m} \qquad (t \geq 1).\end{aligned}$$

From (4), we have $\frac{th_r}{m} \leq 1$. So

$$\frac{1}{m}\sum_{i=1}^{m} \|x_{\sigma^i(n)} - L\|_0^+ \leq 2\varepsilon.$$

Hence from lemma 5, $x_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} L \ \Rightarrow \ x_n \stackrel{[\sigma-FN]}{\longrightarrow} L$. It is trivial that

$$x_n \stackrel{[\sigma-FN]}{\longrightarrow} L \ \Rightarrow \ x_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} L.$$

As a result, our proof is completed. ■

References

[1] M. Ayman-Mursaleen, Approximation and convergence analysis of blending-type q-Baskakov operators using wavelet transformations, Filomat, 39(2025), 11117–11132.

[2] M. Ayman-Mursaleen, On q-statistical approximation of wavelets aided Kantorovich q-Baskakov operators, Filomat, 38(2024), 3261–3274.

[3] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed spaces, Ann. Fuzzy Math. Inform., 22(2009), 1700–1704.

[4] S. Banach, Théorie des Opérations Linéaires, Warszawa, 1932.

[5] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., 24(1968), 191–201.

[6] S. C. Cheng and J. N. Mordeson, Fuzzy linear operator and fuzzy normed linear spaces, Bull. Calcutta Math. Soc., 86(1994), 429–436.

[7] N. R. Das and P. Das, Fuzzy topology generated by fuzzy norm, Fuzzy Sets Syst., 107(1999), 349–354.

[8] G. Das and B. K. Patel, Lacunary distribution of sequences, Indian J. Pure Appl. Math., 20(1989), 64–74.

[9] D. Dean and R. A. Raimi, Permutations with comparable sets of invariant means, Duke Math J., 27(1960), 467–479.

[10] P. Diamond and P. Kloeden, Metric Spaces of Fuzzy Sets-Theory and Applications, World Scientific, Singapore, 1994.

[11] J. X. Fang, A note on the completions of fuzzy metric spaces and fuzzy normed space, Fuzzy Sets Syst., 131(2002), 399–407.

[12] J. X. Fang and H. Huang, On the level convergence of a sequence of fuzzy numbers, Fuzzy Sets Syst., 147(2004), 417–435.

[13] C. Felbin, Finite dimensional fuzzy normed linear space, Fuzzy Sets Syst., 48(1992), 239–248.

[14] A. R. Freedman, J. J. Sember and M. Raphael, Some Cesàro-type summability spaces, Proc. London Math. Soc., 37(1978), 508–520.

[15] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst., 64(1994), 395–399.

[16] R. Goetschel and W. Voxman, Elementary fuzzy calculus, Fuzzy Sets Syst., 18(1986), 31–43.

[17] E. Gülle and U. Ulusu, Wijsman deferred invariant statistical and strong p-deferred invariant equivalence of order α, Fundam. J. Math. Appl., 6(2023), 211–217.

[18] M. Itoh and M. Cho, Fuzzy bounded operators, Fuzzy Sets Syst., 93(1998), 353–362.

[19] O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets Syst., 12(1984), 215–229.

[20] A. K. Katsaras, Fuzzy topological vector spaces II, Fuzzy Sets Syst., 12(1984), 143–154.

[21] I. Kramosil and J. Michalek, Fuzzy metrics and statistical metric spaces, Kybernetika, 5(1975), 336–344.

[22] G. Lorentz, A contribution to the theory of divergent sequences, Acta Math. 80(1948), 167–190.

[23] J. Michalek, Fuzzy topologies, Kybernetika, 11(1975), 345–354.

[24] H. I. Miller and C. Orhan, On almost convergent and statistically convergent subsequences, Acta Math. Hungar., 93(2001), 135–151.

[25] M. Mizumoto and K. Tanaka, Some properties of fuzzy numbers, in: M. M. Gupta et al. (Eds.), Adv. Fuzzy Set Theory Appl., North-Holland, Amsterdam, (1979), 153–164.

[26] M. Mursaleen and O. H. H. Edely, On the invariant mean and statistical convergence, Appl. Math. Lett., 22(2009), 1700–1704.

[27] M. Mursaleen, On some new invariant matrix methods of summability, Quart. J. Math., Oxford, 34(1983), 77–86.

[28] M. Mursaleen, Matrix transformations between some new sequence spaces, Houston J. Math., 9(1983), 505–509.

[29] M. Nasiruzzaman, Wavelets associated approximation enhanced by Szász-type operators initiated by q-properties, J. Inequal. Appl., 2026(2026), Paper No. 1, 19 pp.

[30] R. A. Raimi, Invariant means and invariant matrix methods of summability, Duke Math. J., 30(1963), 81–94.

[31] E. Savaş, Some sequence spaces involving invariant means, Indian J. Math., 31(1989), 1–8.

[32] E. Savaş, Strong σ-convergent sequences, Bull. Calcutta Math. Soc., 81(1989), 295–300.

[33] E. Savaş, On lacunary strong $\sigma-$convergence, Indian J. Pure Appl. Math., 21(1990), 359–365.

[34] R. Savaş, Multidimensional strongly deferred invariant convergence, Numer. Funct. Anal. Optim., 42(2021), 1323–1333.

[35] P. Schaefer, Infinite matrices and invariant means, Proc. Amer. Math. Soc., 36(1972), 104–110.

[36] C. Şençimen and S. Pehlivan, Statistical convergence in fuzzy normed linear spaces, Fuzzy Sets Syst., 159(2008), 361–370.

[37] U. Ulusu, F. Nuray and E. Dündar, $\mathfrak{I}$-limit and $\mathfrak{I}$-cluster points for functions defined on amenable semigroups, Fundam. J. Math. Appl., 4(2021), 45–48.

[38] J. Xiao and X. Zhu, On linearly topological structure and property of fuzzy normed linear space, Fuzzy Sets Syst., 125(2002), 153–161.

[39] Ş. Yalvaç and E. Dündar, Invariant convergence in fuzzy normed spaces, Honam Math. J., 43(2021), 433–440.

[40] Ş. Yalvaç and E. Dündar, Lacunary strongly invariant convergence in fuzzy normed spaces, Math. Sci. Appl. E-Notes, 11(2023), 89–96.

[41] L. A. Zadeh, Fuzzy sets, Inform. and Control, 8(1965), 338–353.