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Inverse Stereographic Projection Of Generalized k-Fibonacci Points*

Baris Ates†, Alp Kustepeli‡

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Abstract

In the present work, using generalized k-Fibonacci numbers, generalized k-Fibonacci points on the plane are constructed. Applying inverse stereographic projection (ISP), the images of these points and their recurrence relations on the sphere are obtained. Rotation of the special generalized k-Fibonacci points located on the positive x-axis is considered, and the relation between the ISPs of those points is derived. As a further application, the generalized k-Fibonacci curve, which passes through the generalized k-Fibonacci points, is considered. Under the ISP, the images of these curves, which form spherical spirals, are obtained. Moreover, two different types of deformations are employed, the first one is the deformation of the curves and the second one is the deformation of the surfaces. In the first case, it is shown that there exist special forms of deformation such that, after the deformation of the generalized k-Binet-Fibonacci spirals, they still pass through the generalized k-Fibonacci points. In the second case, the deformation of the sphere itself is examined. With the help of this new type of deformation, k-Fibonacci points and generalized k-Binet-Fibonacci spirals are transformed onto various new deformed spherical geometries. The mathematical background of all these procedures is explicitly established, and illustrative cases such as spheroidal and pinecone-shaped deformed spherical objects are analyzed in detail.

1 Introduction

Fibonacci numbers, which find a wide range of applications in science and art [3], have many generalizations [11, 14, 15, 21, 22, 23, 28]. Among them, the k-Fibonacci numbers, introduced in [11], and defined by the recurrence relation $F_{n+2} = kF_{n+1} + F_n$ with initial values $F_0 = 0$ and $F_1 = 1$, have many interesting properties [8, 9]. Allowing the initial values to be arbitrary positive integers a and b further generalizes the k-Fibonacci numbers. These generalized numbers obey the following rule [29]

$$F_{n+2}^{a,b} = kF_{n+1}^{a,b} + F_n^{a,b}, \quad n \geq 0 \quad k, a, b \in \mathbf{Z}^+$$

where the upper indices denote the starting values $F_0^{a,b} = a$ and $F_1^{a,b} = b$. The generalized k-Fibonacci numbers can be obtained from a Binet-type expression as

$$F_n^{a,b} = \frac{c\,\alpha^n - d\,\beta^n}{\alpha - \beta}$$

where $c = \alpha a + b - ka$, $d = \beta a + b - ka$, and α, β are the roots of the equation $x^2 - kx - 1 = 0$ and they have following expressions [29]

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2}, \quad \beta = \frac{k - \sqrt{k^2 + 4}}{2}$$

As shown in Table 1, special values of these k-Fibonacci numbers have special names and they lead to special sequences [16]. This means that when a relation, or a property, is found for the generalized k-Fibonacci numbers, it immediately holds for the special cases of the generalized k-Fibonacci numbers. The main motivation of the present work comes from the relation between Fibonacci numbers and quantum mechanics.

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†Department of Mathematics, Izmir Institute of Technology, 35430, Urla, Izmir, Turkey

‡Department of Electrical & Electronics Engineering, Izmir Institute of Technology, 35430, Urla, Izmir, Turkey

k	a	b	Name	First Few Members
1	0	1	Fibonacci Sequence	$0, 1, 1, 2, 3, 5, \ldots$
1	2	1	Lucas Sequence	$2, 1, 3, 4, 7, 11, \ldots$
2	0	1	Pell Sequence	$0, 1, 2, 5, 12, 29, \ldots$
2	2	2	Pell-Lucas Sequence	$2, 2, 6, 14, 34, 82, \ldots$

Table 1: Some special values of Generalized k Fibonacci numbers.

The Fibonacci sequence appears in various contexts within quantum mechanics, particularly in areas such as quasiperiodic systems, quantum walks, and topological quantum computation [7, 18]. In quasiperiodic lattices such as the Fibonacci chain, the arrangement of atomic sites follows the Fibonacci sequence, leading to unique electronic properties and localization phenomena [16, 27]. These systems have been widely studied in the context of quantum transport and spectral analysis [13]. Moreover, in topological quantum computation, hypothetical particles called Fibonacci anyons follow fusion rules based on the Fibonacci sequence and are considered powerful candidates for building fault-tolerant quantum computers [12]. Their braiding operations can implement universal quantum gates, making them highly relevant for quantum information theory [19]. Notably, new models have been developed in which quantum oscillators exhibit energy spectra determined by Fibonacci numbers, introducing concepts such as the "Golden Quantum Oscillator," and exploring supersymmetric extensions that intertwine Fibonacci structures with fermion-boson entanglement. [24, 25].

When studying quantum information theory for qubits, the Bloch sphere is indispensable. The Bloch sphere provides a complete geometric representation of the state space of a single qubit, and any pure qubit state corresponds to a point on its surface [20]. Thus, the aim of the present work is to expand the application area of Fibonacci numbers within quantum information theory by identifying new points on the Bloch sphere associated with Fibonacci numbers.

In the present work, using generalized k-Fibonacci numbers, so-called generalized k-Fibonacci points are constructed. Using inverse stereographic projection, these points are mapped onto the sphere. This procedure allows generalized k-Fibonacci numbers to be applied to new topological settings. Using the well-known recurrence relations for the generalized k-Fibonacci numbers, the corresponding new relations on the sphere are obtained explicitly. Various generalized k-Fibonacci-related plane structures are mapped onto the sphere. In addition to discrete points, by considering the k-Fibonacci spiral, a continuous set of points is also mapped onto the sphere, and the corresponding spherical spirals are determined. Using the generalized stereographic projection formulated in [6], this spiral and its extensions are mapped onto deformed spheres for the first time. Since the deformation function and deformation parameters are arbitrary, this procedure enables the discovery of new relations on infinitely many distinct geometries.

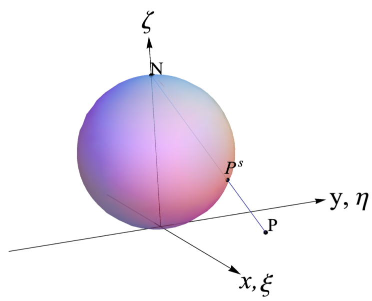
The paper is organized as follows; In section 2, the notion of the generalized k-Fibonacci point is introduced. Their image points under the inverse stereographic projection are found. Rotation of those points and their ISPs are found. Recurrence relations among the image points are given explicitly. Using generalized k-Fibonacci points, the plane are divided into grids and their ISP's on the sphere are given. The well known identities, like Cassini and D'Ocagne identities, for generalized k-Fibonacci points are found. In section 3, generalized k Binet-Fibonacci curve, its rotation and its deformation are considered, and their ISPs are also presented. In section 4, k Binet-Fibonacci spirals and points are mapped onto the deformed spherical surfaces by using the method previously developed in [6].

2 Generalized k-Fibonacci Numbers

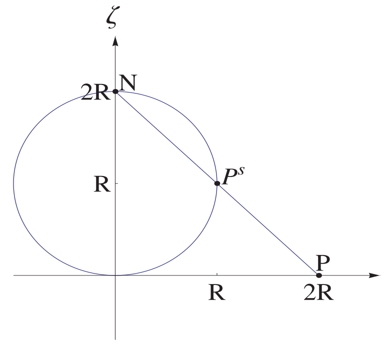
Let us consider a sphere in cartesian coordinates (ξ, η, ζ) whose center is located at $(0, 0, R)$ with radius R. The equation of this sphere is given by the following relation

$$\xi^2 + \eta^2 + (\zeta - R)^2 = R^2. \tag{1}$$

As illustrated in Fig. 1, Stereographic projection enables to establish a $1-1$ and invertible transformation between the points of the sphere and points on the plane [1, 26]. A point $P^s(\xi,\eta,\zeta)$ on the sphere is mapped



(a)



(b)

Figure 1: (a) Stereographic projection uniquely maps points P^s on the sphere to the points P on the plane. (b) Intersection of sphere with plane passing through origin, north pole and point P^s.

to a unique point $P(x,y)$ on the xy plane:

$$P^s(\xi,\eta,\zeta)\rightarrow P(\frac{2R\xi}{2R-\zeta},\frac{2R\eta}{2R-\zeta}). \tag{2}$$

It is clear from the above equation that the stereographic projection of the north pole $N(0,0,2R)$ is not defined [26], as it corresponds to the point at infinity in the complex plane [1]. Inverting equation (2), the ISP of the points $P(x,y)$ is found to be

$$P(x,y)\rightarrow P^s\left(\frac{4R^2x}{4R^2+x^2+y^2},\frac{4R^2y}{4R^2+x^2+y^2},\frac{2R(x^2+y^2)}{4R^2+x^2+y^2}\right). \tag{3}$$

In the rest of the paper, a point in the plane is called "*Generalized k-Fibonacci Point*" if the absolute value of each coordinate is a generalized k-Fibonacci number. Thus, they are in the form of

$$P(x,y)=P(c_1F_n^{a,b},c_2F_m^{a,b}) \tag{4}$$

where c_1,c_2 are equal to $-1,0,1$. These points are the generalization of Fibonacci points introduced in [5]. By considering (2), (3), and the geometry in Fig. (1), one can note that $c_1=c_2=0$ does not result in any problem since the points $P(0,0)\leftrightarrow P^s(0,0,0)$ are interrelated with each other. The ISP of the generalized k-Fibonacci points are the $P^s(\xi_n^{a,b},\eta_n^{a,b},\zeta_n^{a,b})$ where

$$\left(\xi_n^{\mathrm{a,b}},\eta_n^{\mathrm{a,b}},\zeta_n^{\mathrm{a,b}}\right)=\left(\frac{4R^2c_1F_n^{a,b}}{4R^2+(c_1F_n^{a,b})^2+(c_2F_m^{a,b})^2},\frac{4R^2c_2F_m^{a,b}}{4R^2+(c_1F_n^{a,b})^2+(c_2F_m^{a,b})^2},\frac{2R\Big((c_1F_n^{a,b})^2+(c_2F_m^{a,b})^2\Big)}{4R^2+(c_1F_n^{a,b})^2+(c_2F_m^{a,b})^2}\right). \tag{5}$$

Let us consider special generalized k-Fibonacci points on the x axis whose first coordinates are generalized k-Fibonacci numbers i.e. $P_n^{a,b}(c_1F_n^{a,b},0)$. Since generalized k-Fibonacci numbers and points $P_n^{a,b}(c_1F_n^{a,b},0)$ obey certain recurrence relations, their image points under the inverse stereographic projection

$$P_n^{a,b}(\xi_n^{a,b},\eta_n^{a,b},\zeta_n^{a,b})$$

obey also certain recurrence relation. In order to establish the relation, let us start with the ISP of the points $P_n^{a,b}(c_1F_n^{a,b},0)$ with $c_1\neq 0$,

$$P^s(\xi_n^{a,b},\eta_n^{a,b},\zeta_n^{a,b})=P^s(\frac{4R^2c_1F_n^{a,b}}{4R^2+(c_1F_n^{a,b})^2},0,\frac{2R(c_1F_n^{a,b})^2}{4R^2+(c_1F_n^{a,b})^2}). \tag{6}$$

The equality of the first component of the above relations allows one to write as

$$\xi_n^{a,b}(c_1F_n^{a,b})^2 - 4R^2c_1F_n^{a,b} + 4R^2\xi_n^{a,b} = 0,$$

from which it is found that

$$F_n^{a,b} = \frac{2R^2c_1 + 2R\lambda_n|c_1|\sqrt{R^2 - (\xi_n^{a,b})^2}}{c_1^2\xi_n^{a,b}} \tag{7}$$

where λ_n is simply equal to ± 1 depending on the values of n and R. It should be noted here that, equation (7) is not defined for $\xi_n^{a,b} = 0$. However, in this set up there are only two points which satisfy the condition $\xi = 0$, one of them is $(0,0,0)$ and other one is $(0,0,2R)$ which is the north pole and not covered by the ISP. Thus, there is only one option that the points $P(0,0) \leftrightarrow P^s(0,0,0)$ must be interrelated. One can easily check that, if $R = \frac{F_{N+1}^{a,b}}{2}$ for $N \in \mathbb{N}$

$$\lambda_n(R) = \begin{cases} -c_1, & n < N+1, \\ c_1, & n \geq N+1. \end{cases} \tag{8}$$

Else if $\frac{F_N^{a,b}}{2} < R < \frac{F_{N+1}^{a,b}}{2}$, then

$$\lambda_n(R) = \begin{cases} -c_1, & n \leq N, \\ c_1, & n \geq N+1. \end{cases} \tag{9}$$

In (8), $n = N$ is excluded because for this case the square root in (7) becomes zero thus the effect of the λ vanishes. Considering the equation (6) together with the definition of the $\lambda_n(R)$, one can conclude that coordinates of the ISP of the sequential points $P_n^{a,b}(c_1F_n^{a,b}, 0)$ with $c_1 \neq 0$ satisfy following "*Nonlinear addition*" relations

$$\frac{2R^2c_1 + 2R\lambda_n|c_1|\sqrt{R^2 - (\xi_n^{\mathrm{a,b}})^2}}{\xi_n^{\mathrm{a,b}}} + \frac{2R^2c_1 + 2R\lambda_{n+1}|c_1|\sqrt{R^2 - (\xi_{n+1}^{\mathrm{a,b}})^2}}{\xi_{n+1}^{\mathrm{a,b}}} = \frac{2R^2c_1 + 2R\lambda_{n+2}|c_1|\sqrt{R^2 - (\xi_{n+2}^{\mathrm{a,b}})^2}}{\xi_{n+2}^{\mathrm{a,b}}}, \tag{10}$$

$$\eta_n^{a,b} = \eta_{n+1}^{a,b} = \eta_{n+2}^{a,b} = 0, \tag{11}$$

$$\sqrt{\frac{\zeta_n^{a,b}}{2R - \zeta_n^{a,b}}} + k\sqrt{\frac{\zeta_{n+1}^{a,b}}{2R - \zeta_{n+1}^{a,b}}} = \sqrt{\frac{\zeta_{n+2}^{a,b}}{2R - \zeta_{n+2}^{a,b}}}, \tag{12}$$

where the notation λ_n is used for $\lambda_n(R)$. In addition to the relations (10)–(12), by using the relation (2), one can show that the mixed coordinates of the sequential points on the sphere satisfies following relation

$$\frac{2R\xi_n^{a,b}}{2R - \zeta_n^{a,b}} + k\frac{2R\xi_{n+1}^{a,b}}{2R - \zeta_{n+1}^{a,b}} = \frac{2R\xi_{n+2}^{a,b}}{2R - \zeta_{n+2}^{a,b}}. \tag{13}$$

The above relations (10)–(13) govern the relation between the components of the ISP points of the generalized k-Fibonacci points. It must be noted that, the term $\frac{2R\xi_n^{a,b}}{2R - \zeta_n^{a,b}}$ in Equation (13) is a different representation of the generalized k-Fibonacci numbers $F_n^{a,b}$, with initial values $\frac{2R\xi_0^{a,b}}{2R-\zeta_0^{a,b}} = a$ and $\frac{2R\xi_1^{a,b}}{2R-\zeta_1^{a,b}} = b$. For arbitrary generalized k-Fibonacci numbers from the equation (5) following relation can be found

$$P\left(c_1F_n^{a,b}, c_2F_m^{a,b}\right) = \left(\frac{\xi_{n,m}^{a,b}}{2R - \zeta_{n,m}^{a,b}}, \frac{\eta_{n,m}^{a,b}}{2R - \zeta_{n,m}^{a,b}}\right). \tag{14}$$

This equation can be used to determine whether a $3D$ point is a ISP of a generalized k-Fibonacci point or not. More over this equation implies following relations

$$\frac{\xi_{n,m}^{a,b}}{2R - \zeta_{n,m}^{a,b}} + k\frac{\xi_{n+1,m}^{a,b}}{2R - \zeta_{n+1,m}^{a,b}} = \frac{\xi_{n+2,m}^{a,b}}{2R - \zeta_{n+2,m}^{a,b}},$$

$$\frac{\eta^{a,b}_{n,m}}{2R-\zeta^{a,b}_{n,m}} + k\frac{\eta^{a,b}_{n,m+1}}{2R-\zeta^{a,b}_{n,m+1}} = \frac{\eta^{a,b}_{n,m+2}}{2R-\zeta^{a,b}_{n,m+2}}.$$

2.1 Rotated Generalized k-Fibonacci Points

Let us consider special generalized k-Fibonacci points $P_n(F^{a,b}n, 0)$. The rotation of these points about the origin by an angle α can be found using the relations $x' = \cos\alpha,\ x - \sin\alpha, y$ and $y' = \sin\alpha, x + \cos\alpha, y$. The rotated points are then given by $P^{'}n(F^{a,b}n\cos\alpha, F^{a,b}n\sin\alpha)$. Here, it must be noted that the choice of $P_n(F^{a,b}n, 0)$ is sufficient, because in the definition of $Pn(c_1F^{a,b}n, 0)$, if $c_1 = 0$, the point reduces to $(0, 0)$ whose image is $(0, 0, 0)$, and if $c_1 = -1$, it can be obtained by a rotation of angle π. Thus, the choice of $P_n(F^{a,b}n, 0)$ is sufficient. Since the generalized k-Fibonacci points satisfy certain recurrence relations, the same holds for their rotated counterparts. The ISP of the generalized k-Fibonacci points, $P^{'^s}n(\xi^{'}n, \eta^{'}n, \zeta^{'}_n)$, satisfies the following relations:

$$\begin{aligned} &\frac{2R^2\cos\alpha + 2\lambda'_nR\sqrt{R^2\cos^2\alpha - (\xi^{'}_n)^2}}{\xi^{'}_n} + k\frac{2R^2\cos\alpha + 2\lambda'_{n+1}R\sqrt{R^2\cos^2\alpha - (\xi^{'}_{n+1})^2}}{\xi^{'}_{n+1}} \\ = &\frac{2R^2\cos\alpha + 2\lambda'_{n+2}R\sqrt{R^2\cos^2\alpha - (\xi^{'}_{n+2})^2}}{\xi^{'}_{n+2}}, \end{aligned} \tag{15}$$

$$\begin{aligned} &\frac{2R^2\sin\alpha + 2\lambda'_nR\sqrt{R^2\sin^2\alpha - (\eta^{'}_n)^2}}{\eta^{'}_n} + k\frac{2R^2\sin\alpha + 2\lambda'_{n+1}R\sqrt{R^2\sin^2\alpha - (\eta^{'}_{n+1})^2}}{\eta^{'}_{n+1}} \\ = &\frac{2R^2\sin\alpha + 2\lambda'_{n+2}R\sqrt{R^2\sin^2\alpha - (\eta^{'}_{n+2})^2}}{\eta^{'}_{n+2}}, \end{aligned} \tag{16}$$

$$\sqrt{\frac{\zeta^{'}_n}{2R-\zeta^{'}_n}} + k\sqrt{\frac{\zeta^{'}_{n+1}}{2R-\zeta^{'}_{n+1}}} = \sqrt{\frac{\zeta^{'}_{n+2}}{2R-\zeta^{'}_{n+2}}}, \tag{17}$$

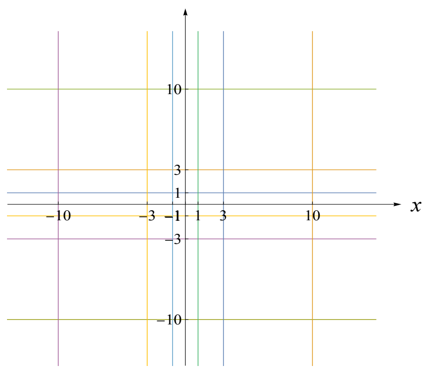
$$\frac{\xi^{'}_n}{2R-\zeta^{'}_n} + k\frac{\xi^{'}_{n+1}}{2R-\zeta^{'}_{n+1}} = \frac{\xi^{'}_{n+2}}{2R-\zeta^{'}_{n+2}}, \tag{18}$$

$$\frac{\eta^{'}_n}{2R-\zeta^{'}_n} + k\frac{\eta^{'}_{n+1}}{2R-\zeta^{'}_{n+1}} = \frac{\eta^{'}_{n+2}}{2R-\zeta^{'}_{n+2}}. \tag{19}$$

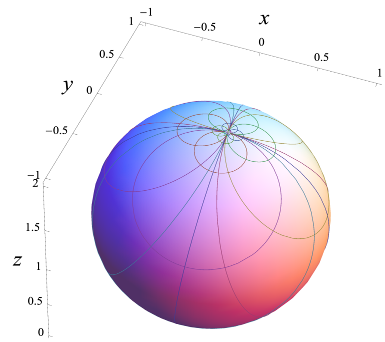
In (15) when $\cos\alpha = 0$ the coordinates $\zeta^{'}_n$, $\zeta^{'}_{n+1}$ and $\zeta^{'}_{n+2}$ are identically zero. For this case, they satisfy trivial identity that $\zeta^{'}_{n+2} = \zeta^{'}_n + k\zeta^{'}_{n+1} = 0$. In a similar way the same reasoning is true for the η' in equation (16) when $\sin\alpha = 0$. Comparing the equations (15) and (16) with the (10), it can be seen that there exist additional $\sin\alpha$, $\cos\alpha$ terms. Thus the value of the λ'_n also depends on these trigonometric functions. For the angle α which makes these functions positive $\lambda'_n = \lambda_n$, for the angle which makes these functions negative $\lambda'_n = -\lambda_n$ where λ_n are defined in the equations (8) and (9).

2.2 Generalized k-Fibonacci Grids

The definition of generalized k-Fibonacci points given in (4) allows to divide two dimensional plane by the grids such that each grid line with $x = c_1F^{a,b}_n$ and $y = c_2F^{a,b}_m$ passes through generalized k-Fibonacci points as illustrated in Fig. 2(a). This new style of grids allows to divide the plane into the rectangular domains whose boundaries and corners can be expressed in terms of generalized k-Fibonacci numbers and k-Fibonacci points, respectively. In Fig. 2(b), the ISP of the grid lines are shown for the unit sphere. The ISP of these rectangular domains are closed regions with curved boundaries on the sphere. Thus, it is resulted in the division of the surface of the sphere into the curved patches that are directly related with the generalized k-Fibonacci numbers. This new type of division of the surface of the sphere could be interesting in geography in order to understand and reinterpret the surface of the earth in terms of generalized k-Fibonacci numbers.



(a)



(b)

Figure 2: (a) Lines that are passing through generalized k Fibonacci points, for $k = 3, a = 0, b = 1$ (b) ISP of that lines.

2.3 New Identities on the Sphere

The well known identities for the generalized k-Fibonacci numbers are also valid for the coordinates of the generalized k-Fibonacci points $P_n^{a,b}(x, y) = (F_n^{a,b}, 0)$ with $c_1 = 1$, $c_2 = 0$. From the relation (7) it can be found that the inverse stereographic projection of these points satisfies recurrence relations. As an illustration, let us consider the Cassini identity which reads as following [17]

$$(F_n^{a,b})^2 - F_{n+1}^{a,b} F_{n-1}^{a,b} = (-1)^{n-1}(b^2 - a^2 - abk)$$

then the coordinates of the image points satisfies following Cassini type relation,

$$\Big(\frac{2R^2 + 2\lambda_n R\sqrt{R^2 - (\xi_n^{a,b})^2}}{\xi_n^{a,b}}\Big)^2 - \frac{2R^2 + 2\lambda_{n+1} R\sqrt{R^2 - (\xi_{n+1}^{a,b})^2}}{\xi_{n+1}^{a,b}}\,\frac{2R^2 + 2\lambda_{n-1} R\sqrt{R^2 - (\xi_{n-1}^{a,b})^2}}{\xi_{n-1}^{a,b}} = (-1)^{n-1}(b^2 - a^2 - abk).$$

Moreover, images of rotated generalized k-Fibonacci numbers also satisfy certain relations, to show that we consider the d'Ocagne's identity

$$F_m^{a,b} F_{n+1}^{a,b} - F_{m+1}^{a,b} F_n^{a,b} = (-1)^n (b F_{m-n}^{a,b} - a F_{m-n+1}^{a,b}).$$

Using equation (7) it is found that coordinates of the images of rotated special generalized k-Fibonacci points satisfy following relations

$$\begin{aligned}
&\frac{2R^2\cos\alpha + 2\lambda'_m R\sqrt{R^2\cos^2\alpha - (\xi'_m)^2}}{\xi'_m}\,\frac{2R^2\cos\alpha + 2\lambda'_{n+1} R\sqrt{R^2\cos^2\alpha - (\xi'_{n+1})^2}}{\xi'_{n+1}}\\
&- \frac{2R^2\cos\alpha + 2\lambda'_{m+1} R\sqrt{R^2\cos^2\alpha - (\xi'_{m+1})^2}}{\xi'_{m+1}}\,\frac{2R^2\cos\alpha + 2\lambda'_n R\sqrt{R^2\cos^2\alpha - (\xi'_n)^2}}{\xi'_n}\\
&= (-1)^n\Big(b\frac{2R^2\cos\alpha + 2\lambda'_{m-n} R\sqrt{R^2\cos^2\alpha - (\xi'_{m-n})^2}}{\xi'_{m-n}} - a\frac{2R^2\cos\alpha + 2\lambda'_{m-n+1} R\sqrt{R^2\cos^2\alpha - (\xi'_{m-n+1})^2}}{\xi'_{m-n+1}}\Big),
\end{aligned}$$

$$\frac{2R^2\sin\alpha+2\lambda'_m R\sqrt{R^2\sin^2\alpha-(\eta'_m)^2}}{\eta'_m}\,\frac{2R^2\sin\alpha+2\lambda'_{n+1}R\sqrt{R^2\sin^2\alpha-(\eta'_{n+1})^2}}{\eta'_{n+1}}$$
$$-\frac{2R^2\sin\alpha+2\lambda'_{m+1}R\sqrt{R^2\sin^2\alpha-(\eta'_{m+1})^2}}{\eta'_{m+1}}\,\frac{2R^2\sin\alpha+2\lambda'_n R\sqrt{R^2\sin^2\alpha-(\eta'_n)^2}}{\eta'_n}$$
$$=(-1)^n\Big(b\frac{2R^2\sin\alpha+2\lambda'_{m-n}R\sqrt{R^2\sin^2\alpha-(\eta'_{m-n})^2}}{\eta'_{m-n}}-a\frac{2R^2\sin\alpha+2\lambda'_{m-n+1}R\sqrt{R^2\sin^2\alpha-(\eta'_{m-n+1})^2}}{\eta'_{m-n+1}}\Big),$$

$$\sqrt{\frac{\zeta'_m}{2R-\zeta'_m}}\sqrt{\frac{\zeta'_{n+1}}{2R-\zeta'_{n+1}}}-\sqrt{\frac{\zeta'_{m+1}}{2R-\zeta'_{m+1}}}\sqrt{\frac{\zeta'_n}{2R-\zeta'_n}}=(-1)^n\Big(b\sqrt{\frac{\zeta'_{m-n}}{2R-\zeta'_{m-n}}}-a\sqrt{\frac{\zeta'_{m-n+1}}{2R-\zeta'_{m-n+1}}}\Big).$$

3 Generalized k-Binet-Fibonacci Curve

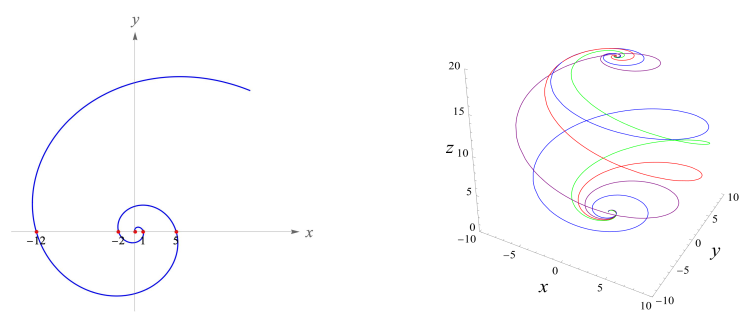
The Binet-Fibonacci spiral for the generalized k-Fibonacci numbers is given with the following relation [10]

$$\gamma^{\pm}(t)=(x(t),y(t))=\Big(\frac{c\,e^{t\ln\alpha}-d\,\cos\pi t\,e^{-t\ln\alpha}}{\alpha-\beta},\pm\frac{-d\,\sin\pi t\,e^{-t\ln\alpha}}{\alpha-\beta}\Big) \tag{20}$$

where $(+)$ sign indicates that it progress in the counter clockwise direction, and $(-)$ sign indicates that it progress in the clockwise direction. The image curve is in the following form

$$\Gamma(\xi(t),\eta(t),\zeta(t))=\Big(\frac{4R^2x(t)}{4R^2+x^2(t)+y^2(t)},\frac{4R^2y(t)}{4R^2+x^2(t)+y^2(t)},\frac{2R(x^2(t)+y^2(t))}{4R^2+x^2(t)+y^2(t)}\Big) \tag{21}$$

where $x(t),y(t)$ are the functions defined in equation (20). A curve from this family is illustrated in the Fig. 3(a) for the values of $k=2$, $a=0$ and $b=1$. Inverse stereographic projection of the spirals for the values of $k=2,3,4,5$ and $a=0$ and $b=1$ are also illustrated in the Fig. 3(b).



(a) (b)

Figure 3: (a) Generalized k Binet-Fibonacci spiral for the values of $k=2$, $a=0$ and $b=1$, the parameter t is in the interval of $-1.5\pi\le t\le 0$. (b) Inverse stereographic projection of the spirals for the values of $k=2,3,4,5$ and $a=0$ and $b=1$, the parameter t is in the interval of $-3\pi\le t\le 0$.

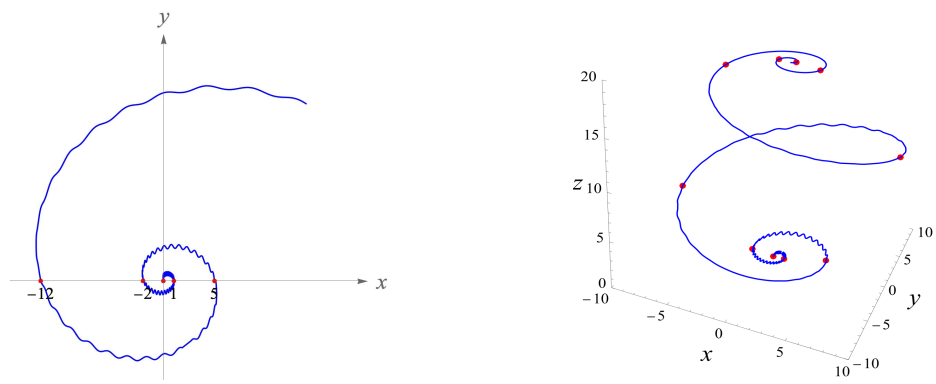
The rotation of the positive and negative winding curves (20) are given with the following relations:

$$\gamma^{\pm}(\alpha,t)=\Big(x(t)\cos(\alpha)\mp y(t)\sin(\alpha),x(t)\sin(\alpha)\pm y(t)\cos(\alpha)\Big).$$

In addition to the rotated form of the generalized k Binet-Fibonacci spiral, by applying perturbation method to the curve in (20), one can construct new spirals $\gamma_d(t)$ passing through the generalized k-Fibonacci points but their graphs have been perturbed.

$$\gamma_d(t) = (x_d(t), y_d(t)) = \left(\frac{c\, e^{t \ln \alpha} - d\, \cos \pi t\, e^{-t \ln \alpha}}{\alpha - \beta},\ \frac{-d\, \sin \pi t\, e^{-t \ln \alpha}}{\alpha - \beta} + \beta g(\sin(m\pi t)) \right)$$

where g arbitrary smooth function, β is smallness parameter and m is arbitrary integer. This deformed spiral again passes through the generalized k-Fibonacci points but its graph has been deformed. Application of the method is illustrated in Fig. 4(a) for the choices of the parameters $\beta = 0.4$, $g(x) = x^2$, $m = 15$, $k = 2$, $a = 0$, $b = 1$ together with $-1.5\pi \leq t \leq 0$. Inverse stereographic projection of this curve is illustrated in Fig. 4(b) for $-3\pi \leq t \leq 0$. As depicted from the figures, effect of the deformation getting bigger and bigger as spiral approaching to the origin.



(a) (b)

Figure 4: (a) Deformation of the generalized k Binet-Fibonacci spiral for the choices $\beta = 0.4$, $g(x) = x^2$, $m = 15$, $k = 2$, $a = 0$, $b = 1$ together with $-1.5\pi \leq t \leq 0$. (b) ISP of the deformed spiral for $-3\pi \leq t \leq 0$.

4 Inverse Stereographic Projection onto the Deformed Spheres

Inverse stereographic projection is a 1-1 mapping between the points on the plane with the points on the sphere [1]. This correspondence can be generalized to the deformed spheres [6]. Position vector of a point on a sphere is given by $\vec{r}_{sph} = R\,\hat{r}$. Deformation of this sphere in the normal direction can be expressed with

$$\vec{r}_d = R(1 + \beta h(\theta, \varphi))\,\hat{r} \tag{22}$$

such that β is the deformation parameter and $h(\theta, \varphi)$ is the single valued deformation function [2]. Dependence of the function $h(\theta, \varphi)$ to the spherical angles θ and φ is crucial here because it clearly shows that the amount of the deviation depend on the values of the these angles, in other words amount of the deviation depends on the position of the point. If a parametric plane curve $\vec{\gamma} = (x(t), y(t))$ is given, its image curve $\vec{\Gamma}^1(t) = (\xi(t), \eta(t), \zeta(t))$ can be found from the definition (3). Shifting center of the sphere to the origin results in a new curve $\vec{\Gamma}^2(t) = (\xi'(t), \eta'(t), \zeta'(t)) = (\xi(t), \eta(t), \zeta(t) - R)$. Since position of the points on the plane curve are t dependent and hence position of the image curves are also t dependent. Thus, t dependent spherical angles can be written as

$$\theta(t) = \arccos\left(\frac{\xi'(t)}{R}\right),\ \ \varphi(t) = \arctan\left(\frac{y(t)}{x(t)}\right).$$

Deformation of the surface of the sphere in the normal direction obeying the rule (22) enables to obtain final curve on the deformed surface as $\vec{\Gamma}^3(t) = (\xi''(t), \eta''(t), \zeta''(t))$ where

$$\xi''(t) = \xi'(t) + \beta R\, h\big(\theta(t), \varphi(t)\big) \sin\big(\theta(t)\big) \cos\big(\varphi(t)\big), \tag{23}$$

$$\eta''(t) = \eta'(t) + \beta R\, h\big(\theta(t), \varphi(t)\big) \sin\big(\theta(t)\big) \sin\big(\varphi(t)\big), \tag{24}$$

$$\zeta''(t) = \zeta'(t) + \beta R\, h\big(\theta(t), \varphi(t)\big) \cos\big(\theta(t)\big). \tag{25}$$

The uniqueness of the above procedure is discussed in the [6]. In Fig. (5) a plane curve obtained with the choice of the parameters $k = 2$, $a = 0$, $b = 1$ and $-5\pi \le t \le 0$ is mapped onto the θ and φ dependent surface obtained from the deformation of sphere with deformation function $h(\theta, \varphi) = Re\left(Y_8^4(\theta, \varphi)\right)$ and deformation parameter $\beta = 0.2$.

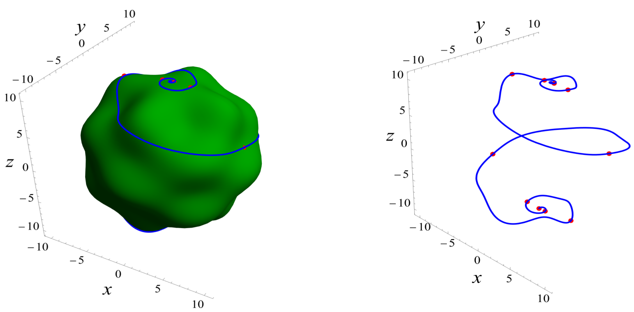
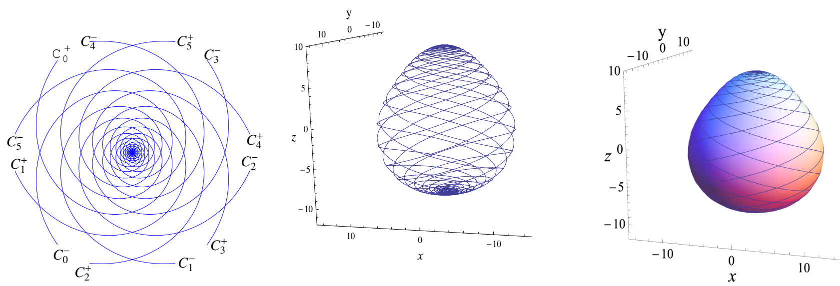


Figure 5: (a) ISP of the generalized k Binet-Fibonacci curve which is obtained with the choice of the parameters $k = 2$, $a = 0$, $b = 1$ and $-5\pi \le t \le 0$ and generalized k Fibonacci points $P_n^{0,1} = (0, F_n^{0,1})$, $n = 0, \ldots, 9$, on to the θ and φ dependent surface obtained from the deformation of sphere with deformation function $h(\theta, \varphi) = Re\left(Y_8^4(\theta, \varphi)\right)$ and deformation parameter $\beta = 0.2$. (b) Image of generalized k Binet-Fibonacci spiral and generalized k Fibonacci points.

In the Final figure, Fig. 6, generalized k Binet-Fibonacci curves and their rotations are considered. In the Fig. 6(a), the curves $\vec{\gamma}^{\,+}(\alpha, t)$ and $\vec{\gamma}^{\,-}(\alpha, t)$ are shown for the values of $\alpha = \frac{m\pi}{6}$, $m = 0, \ldots, 5$, $k = 2$, $a = 0$, $b = 1$ and $-2\pi \le t \le 0$. In Fig. 6(b), ISP of that curves are found on the deformed pinecone shaped surface obtained by the deformation function $H_3(\cos\theta)$ and parameter $\beta = 0.03$ where H function is the Hermite function [4]. In the Fig. 6(c) the curves are shown with the deformed surface.

5 Conclusion

In the present work, using generalized k-Fibonacci numbers, a new class of points called generalized k-Fibonacci points is introduced. Corresponding recurrence relations are derived for these generalized k-Fibonacci points. Explicit forms of the Cassini and D'Ocagne identities are also given for both the generalized k-Fibonacci points and their rotated forms. In addition, the generalized k-Fibonacci points are used to divide the plane with special lines, yielding a new type of grid on the plane. Their ISPs are obtained, which divide the surface of the sphere into curved regions associated with the generalized k-Fibonacci points. The work is further extended to include the generalized k Binet-Fibonacci spirals and their rotated and deformed forms. All cases are analytically and explicitly discussed, and their inverse stereographic projections are obtained. Finally, using the method introduced in [6], the generalized k Binet-Fibonacci spirals and points are mapped onto arbitrarily shaped deformed spheres such as spheroidal and pinecone-shaped geometries. The present work can be applied in many different research areas. In quantum information theory, the results can be interpreted by considering the sphere as the Bloch sphere. Since any pure qubit state corresponds to a point on the surface of the Bloch sphere [20], the ISP of the Fibonacci points corresponds to a pure qubit



(a) (b) (c)

Figure 6: (a) Positive and negative oriented generalized k Binet-Fibonacci spirals for the values of $\alpha = \frac{m\pi}{6}$, $m = 0, \ldots, 5$, $k = 2$, $a = 0$, $b = 1$ and $-2\pi \leq t \leq 0$. (b) ISP of that curves for $-5\pi \leq t \leq 0$ are found on the deformed pinecone shaped surface obtained by the deformation function $H_3(\cos\theta)$ and parameter $\beta = 0.03$. (c) Curves are shown with the deformed surface.

state. In biology, by employing different types of deformations, various realistic shapes can be modeled, such as cancer cells, blood cells (erythrocytes), pine cones, and many others. For future studies, the obtained results may be generalized to higher-dimensional hyperspherical structures by employing continuous maps from homotopy theory.

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