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The Modification Of Fuzzy Numbers With Its Impact On The Stability Of Fuzzy Number-Valued Functional Equations*

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Abstract

In this paper, we introduce a modified form of fuzzy numbers and investigate its implications for the Hyers-Ulam stability of the following generalized fuzzy number-valued functional equation in Banach spaces:

$$uf(ax+by)+vf(cx-dy)=rf(x)+sf(y),$$

where f denotes a fuzzy number-valued mapping on a Banach space, $a,b,c,d>0$, and $u,v,r,s\in\mathbb{R}$ with $u+v\neq 0$. The obtained results extend and generalize several existing stability results for fuzzy number-valued functional equations.

1 Introduction

The stability theory for functional equations originated from a question on group homomorphisms posed by Ulam [13] in 1940. Hyers [6] provided the first affirmative answer to Ulam's question for the Cauchy functional equation in Banach spaces. Because of Ulam's problem and Hyers' solution, this type of stability is now known as Hyers-Ulam stability of functional equations. Subsequently, numerous mathematicians extended these results to more general settings, as can be seen in the works of Aoki [1], Gajda [4], Gavruta [5], and Rassias [10, 11]. Moreover, the concept of Hyers-Ulam stability has also been generalized to set-valued functional equations. For instance, Cardinali et al. [2] established a stability theorem of Ulam-Hyers type for K-convex set-valued functions, and Chu et al. [3] investigated the stability of a generalized cubic set-valued functional equation. On the other hand, the first investigation of the fuzzy stability of functional equations was conducted by Mirmostafaee et al. [8, 9]. More recently, Wu and Jin [12] attempted to establish a connection between Hyers-Ulam stability and fuzzy number-valued functional equations. They examined the Ulam stability of the following fuzzy number-valued functional equation in Banach spaces by employing a metric defined on the space of fuzzy numbers:

$$f(ax+by)=rf(x)+sf(y), \tag{1}$$

where f is a fuzzy number-valued mapping on a Banach space, $a,b>0$, and $r,s\in\mathbb{R}$. Subsequently, Kunrattanaworawong and Sintunavarat [7] extended certain results of Wu and Jin [12] by investigating the stability of the following functional equation:

$$pf\left(\frac{x+y}{q}\right)=\left(\frac{p}{q}+\frac{q}{p}\right)f(x)+\left(\frac{p}{q}-\frac{q}{p}\right)f(y), \tag{2}$$

where f is a fuzzy number-valued mapping on a Banach space and $p,q\in\mathbb{R}\setminus\{0\}$. In this paper, we focus on the generalized functional equation

$$uf(ax+by)+vf(cx-dy)=rf(x)+sf(y), \tag{3}$$

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where f is a fuzzy number-valued mapping defined on a Banach space, $a, b, c, d > 0$, and $u, v, r, s \in \mathbb{R}$. Equation (3) reduces to equation (1) when $u = 1$ and $v = 0$, and to equation (2) when $a = b = \frac{1}{q}$, $u = p$, $v = 0$, $r = ua + \frac{u}{a}$, and $s = ua - \frac{u}{a}$. This demonstrates that the functional equations considered here constitute a more general form encompassing previously studied cases. The stability of equation (3) is established under the conditions $a, b, c, d > 0$ and $u, v, r, s \in \mathbb{R}$ with $a + b = c - d = r + s \neq |u + v|$. One of the key contributions of this paper lies in the modification of the definition of fuzzy numbers, which serves as a foundation for investigating the stability of fuzzy number-valued functional equations (see Definition 2).

2 Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{I}$ denotes the set of all integers, $\mathbb{R}^+$ denotes the set of all positive real numbers, and $\mathbb{R}$ denotes the set of all real numbers. Unless otherwise specified, X and Y represent Banach spaces, and P_{kc} denotes the family of all nonempty compact convex subsets of X.

Definition 1 **([12])** *A function $u : X \to [0, 1]$ is called a fuzzy number on X if it satisfies the following conditions:*

(i) $[u]^\alpha = \{x \in X : u(x) \geq \alpha\} \in P_{kc}(X)$ *for all* $\alpha \in (0, 1]$;

(ii) $[u]^0 = cl\{x \in X : u(x) > 0\}$ *is a compact set, where cl denotes the closure operation.*

To further elucidate the preceding definition and demonstrate its practical interpretation, we present the following illustrative example of a fuzzy number defined on the Euclidean Banach space $\mathbb{R}$.

Example 1 *Let $X = \mathbb{R}$ be a Euclidean Banach space. Define a function $u : X \to [0, 1]$ by*

$$u(x) = \begin{cases} 2x & \text{if } x \in [0, 0.2], \\ 0.4 & \text{if } x \in (0.2, 0.5], \\ 2x - 0.6 & \text{if } x \in (0.5, 0.8], \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that

$$[u]^\alpha = \begin{cases} \left[\frac{\alpha}{2}, 0.8\right] & \text{if } 0 < \alpha \leq 0.4, \\ \left[\frac{\alpha + 0.6}{2}, 0.8\right] & \text{otherwise.} \end{cases}$$

For each $\alpha \in (0, 1]$, the sets $\left[\frac{\alpha}{2}, 0.8\right]$ and $\left[\frac{\alpha+0.6}{2}, 0.8\right]$ are closed and bounded. According to the Heine-Borel theorem, these sets are compact in X, which implies that $[u]^\alpha \in P_{kc}$ for all $\alpha \in (0, 1]$. Moreover, we have

$$[u]^0 = \overline{(0, 0.8]} = [0, 0.8],$$

which is a compact subset of X. Therefore, u satisfies all the required conditions and is therefore a fuzzy number according to Definition 1.

3 The Modification of Fuzzy Numbers

In the study of functional equations, it is often assumed that the value of an unknown function can be equal to the zero element of its codomain. However, when considering a functional equation for an unknown function f mapping from a given vector space into the set of fuzzy numbers on X, this property can no longer be applied. The reason is that the zero element in the set of fuzzy numbers on X corresponds to the zero function 0, whose α-cut satisfies $[0]^\alpha = \emptyset$ for all $\alpha \in (0, 1]$. Consequently, the zero function 0 does not belong to the set $P_{kc}(X)$, which contradicts condition (i) in Definition 1. This observation motivates a slight modification of the definition of a fuzzy number, as presented below.

Definition 2 *A function $u : X \to [0, 1]$ is called a fuzzy number on X if the following conditions hold:*

(i) $[u]^\alpha := \{x \in X : u(x) \geq \alpha\} \in P_{kc}(X) \cup \{\emptyset\}$ *for all* $\alpha \in (0, 1]$*;*

(ii) $[u]^0 := \overline{\{x \in X : u(x) > 0\}}$ *is a compact set.*

The notion X_F represents the set of all fuzzy numbers on X.

Next, we present an illustrative example of a fuzzy number constructed in accordance with the above definition to provide a clearer understanding of the distinction between Definition 1 and Definition 2.

Example 2 *Let $X = \mathbb{R}$ be a Euclidean Banach space. Define a function $u : X \to [0, 1]$ by*

$$u(x) = \begin{cases} 2x & \text{if } x \in [0, 0.2], \\ 0.4 & \text{if } x \in (0.2, 0.5], \\ 2x - 0.6 & \text{if } x \in (0.5, 0.7], \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that

$$[u]^\alpha = \begin{cases} \left[\frac{\alpha}{2}, 0.7\right] & \text{if } 0 < \alpha \leq 0.4, \\ \left[\frac{\alpha + 0.6}{2}, 0.7\right] & \text{if } 0.4 < \alpha \leq 0.8, \\ \emptyset & \text{if } 0.8 < \alpha \leq 1. \end{cases}$$

Clearly, we have $[u]^\alpha \in P_{kc}(X) \cup \{\emptyset\}$ for all $\alpha \in (0, 1]$. Moreover, $[u]^0 = \overline{(0, 0.7]} = [0, 0.7]$, which is compact. Hence, u is not a fuzzy number in the sense of Definition 1, but it satisfies all the required conditions and is therefore a fuzzy number according to Definition 2.

Before proceeding to the next section, which presents the stability results based on the modified fuzzy numbers introduced in Definition 2, we first recall some preliminary concepts that will be useful in the subsequent discussion. Let $u, v \in X_F$. If there exists $w \in X_F$ such that $u = v + w$, then w is called the H-difference of u and v, and it is denoted by $u - v$. Using the Zadeh extension principle, for each $u, v \in X_F$ and $\lambda \in \mathbb{R}$, the operations of addition and scalar multiplication satisfy the following properties:

1. $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$ for all $\alpha \in (0, 1]$;
2. $[\lambda \cdot u]^\alpha = \lambda [u]^\alpha$ for all $\alpha \in (0, 1]$.

Next, define a mapping $D : X_F \times X_F \to [0, \infty)$ by

$$D(u, v) = \sup_{\alpha \in [0,1]} d_H([u]^\alpha, [v]^\alpha),$$

for all $u, v \in X_F$, where d_H denotes the Hausdorff metric. Then (X_F, D) forms a complete metric space, and D satisfies the following properties for all $u, v, w, e \in X_F$ and $\lambda \in \mathbb{R}$:

(P1) $D(\lambda u, \lambda v) = |\lambda| \, D(u, v)$;

(P2) $D(u + w, v + w) = D(u, v)$;

(P3) $D(u + v, w + e) \leq D(u, w) + D(v, e)$.

4 Stability Results Based on Modified Fuzzy Numbers

In this section, we establish the stability result of a fuzzy number-valued functional equation (3). Before going to the main theorem, the following lemma is needed.

Lemma 1 *Let V be a real (or complex) vector space. Suppose that a fuzzy number-valued mapping $f : V \to X_F$ satisfies the following equality*

$$uf(ax+by)+vf(cx-dy)=rf(x)+sf(y) \tag{4}$$

for all $x, y \in V$, where $a, b, c, d > 0$ and $u, v, r, s \in \mathbb{R}$ with $u+v \neq 0$ and $a+b=c-d=r+s \neq |u+v|$. Then

$$f((a+b)^n x)=\left(\frac{a+b}{u+v}\right)^n f(x)$$

for all $x \in V$ and for all $n \in \mathbb{N}$.

Proof. By setting $y = x$ in (4), we obtain

$$uf((a+b)x)+vf((c-d)x)=(r+s)f(x)$$

for all $x \in V$. Since $a+b=c-d=r+s \neq |u+v|$, it follows that

$$uf((a+b)x)+vf((a+b)x)=(a+b)f(x)$$

for all $x \in V$. Hence,

$$f((a+b)x)=\frac{a+b}{u+v}f(x)$$

for all $x \in V$. By induction, we conclude that for each $n \in \mathbb{N}$, we obtain

$$f((a+b)^n x)=\left(\frac{a+b}{u+v}\right)^n f(x),$$

for all $x \in V$. ■

Next, we establish the main stability theorem for equation (3), which utilizes the preceding relation and the framework of the modified fuzzy numbers defined in Definition 2.

Theorem 1 *Let B be a subspace of a Banach space Y. Suppose that a fuzzy number-valued mapping $f : B \to X_F$ satisfies the inequality*

$$D\left(uf(ax+by)+vf(cx-dy), rf(x)+sf(y)\right)<\epsilon \tag{5}$$

for all $x, y \in B$, where $\epsilon, a, b, c, d > 0$ and $u, v, r, s \in \mathbb{R}$ with $u+v \neq 0$ and $a+b=c-d=r+s \neq |u+v|$. Then there exists a unique mapping $T : B \to X_F$ such that

$$D(T(x), f(x)) \leq \frac{\epsilon|u+v|}{(a+b-|u+v|)^2}$$

for all $x \in B$.

Proof. Replacing y by x in (5), we obtain

$$D\big(uf((a+b)x)+vf((c-d)x),\,(r+s)f(x)\big)<\epsilon \tag{6}$$

for all $x \in B$. Next, we divide the proof into two cases as follows:

Case 1 Assume that $a+b > |u+v|$. Replacing x by $(a+b)^n x$, where $n \in \mathbb{N}$, in (6), we obtain

$$D\big(uf((a+b)^{n+1}x) + vf((a+b)^{n+1}x),\ (a+b)f((a+b)^n x)\big) < \epsilon \tag{7}$$

for all $x \in B$. This implies that

$$D\left(\left(\frac{u+v}{a+b}\right)^{n+1} f((a+b)^{n+1}x),\ \left(\frac{u+v}{a+b}\right)^{n} f((a+b)^{n}x)\right) < \frac{\epsilon|u+v|^n}{(a+b)^{n+1}} \tag{8}$$

for all $x \in B$. For a given $x \in B$, we will define a sequence $\{f_n(x)\}$ in X_F by

$$f_n(x) = \left(\frac{u+v}{a+b}\right)^{n} f((a+b)^{n}x)$$

for all $n \in \mathbb{N}$. We will now show that, for each $x \in B$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in X_F. Let $m, n \in \mathbb{N}$ and assume without loss of generality that $m < n$. Then

$$\begin{aligned} D(f_m(x), f_n(x)) &\le D(f_m(x), f_{m+1}(x)) + D(f_{m+1}(x), f_{m+2}(x)) + \cdots + D(f_{n-1}(x), f_n(x)) \\ &< \frac{\epsilon|u+v|^m}{(a+b)^{m+1}} + \frac{\epsilon|u+v|^{m+1}}{(a+b)^{m+2}} + \cdots + \frac{\epsilon|u+v|^{n-1}}{(a+b)^{n}} \\ &= \frac{\epsilon}{(a+b-|u+v|)}\left(\frac{|u+v|}{a+b}\right)^{m} \end{aligned} \tag{9}$$

for all $x \in B$. Letting $m, n \to \infty$ in (9), we deduce that $\{f_n(x)\}$ is a Cauchy sequence in X_F. Since X_F is complete, we can define a mapping $T : B \to X_F$ by

$$T(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in B$. Now, set $f_0 := f$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} D(f_n(x), f(x)) &\le \sum_{k=1}^{\infty} D(f_k(x), f_{k-1}(x)) \\ &\le \sum_{k=1}^{\infty} \frac{\epsilon}{(a+b-|u+v|)}\left(\frac{|u+v|}{a+b}\right)^{k} \\ &= \frac{\epsilon|u+v|}{(a+b-|u+v|)^2} \end{aligned}$$

for all $x \in B$. Taking the limit as $n \to \infty$ in (9), it follows that

$$D(T(x), f(x)) \le \frac{\epsilon|u+v|}{(a+b-|u+v|)^2} \tag{10}$$

for all $x \in B$. Next, we will show that T satisfies (3). For each $n \in \mathbb{N}$, we have

$$\begin{aligned} &D\left(uf_n(ax+by) + vf_n(cx-dy),\ rf_n(x) + sf_n(y)\right) \\ &= \left(\frac{|u+v|}{a+b}\right)^{n} D\left(uf((a+b)^n(ax+by)) + vf((a+b)^n(cx-dy)),\ rf((a+b)^n x) + sf((a+b)^n y)\right) \\ &< \epsilon\left(\frac{|u+v|}{a+b}\right)^{n} \end{aligned}$$

for all $x, y \in B$. Hence,

$$\lim_{n\to\infty} D(uf_n(ax+by) + vf_n(cx-dy),\ rf_n(x) + sf_n(y)) = 0$$

for all $x, y \in B$. By the continuity of D, we obtain

$$D(uT(ax+by)+vT(cx-dy),\, rT(x)+sT(y))=0,$$

which implies that

$$uT(ax+by)+vT(cx-dy)=rT(x)+sT(y)$$

for all $x, y \in B$. To prove the uniqueness of T, let $T_1, T_2 : B \to X_F$ be mappings satisfying (3) and

$$D(T_i(x), f(x)) \leq \frac{\epsilon|u+v|}{(a+b-|u+v|)^2}$$

for all $x \in B$ and for all $i = 1, 2$. By Lemma 1, we have

$$\left(\frac{a+b}{u+v}\right)^n T_i(x) = T_i((a+b)^n x)$$

for all $x \in B$ and for all $i = 1, 2$. Therefore,

$$\begin{aligned} D(T_1(x), T_2(x)) &= \left(\frac{|u+v|}{a+b}\right)^n D\left(T_1((a+b)^n x), T_2((a+b)^n x)\right) \\ &\leq \left(\frac{|u+v|}{a+b}\right)^n \Big[D(T_1((a+b)^n x), f((a+b)^n x)) + D(T_2((a+b)^n x), f((a+b)^n x))\Big] \\ &\leq \frac{2\epsilon|u+v|}{(a+b-|u+v|)^2}\left(\frac{|u+v|}{a+b}\right)^n \end{aligned} \tag{11}$$

for all $x \in B$ and for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (11), we conclude that $T_1(x) = T_2(x)$ for all $x \in B$. Hence, the mapping T is unique.

Case 2 Assume that $0 < a+b < |u+v|$. Replacing x by $\frac{x}{(a+b)^{n+1}}$, where $n \in \mathbb{N}$, in (6), we obtain

$$D\left((u+v)f\left(\frac{x}{(a+b)^n}\right),\;\; (a+b)f\left(\frac{x}{(a+b)^{n+1}}\right)\right) < \epsilon \tag{12}$$

for all $x \in B$. Hence,

$$D\left(\left(\frac{a+b}{u+v}\right)^n f\left(\frac{x}{(a+b)^n}\right),\;\; \left(\frac{a+b}{u+v}\right)^{n+1} f\left(\frac{x}{(a+b)^{n+1}}\right)\right) < \frac{\epsilon(a+b)^n}{|u+v|^{n+1}} \tag{13}$$

for all $x \in B$. For a given $x \in B$, we will define a sequence $\{f_n(x)\}$ in X_F by

$$f_n(x) = \left(\frac{a+b}{u+v}\right)^n f\left(\frac{x}{(a+b)^n}\right)$$

for all $n \in \mathbb{N}$. Then we have

$$D(f_n(x), f_{n+1}(x)) < \frac{\epsilon(a+b)^n}{|u+v|^{n+1}}$$

for all $n \in \mathbb{N}$, which implies that the sequence $\{f_n(x)\}$ is a Cauchy sequence in X_F. Since X_F is complete, we can define a mapping $T : B \to X_F$ by

$$T(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in B$. By arguments analogous to those in Case 1, we conclude that T satisfies (3) and

$$D(T(x), f(x)) \leq \frac{\epsilon(a+b)}{(|u+v|-(a+b))^2} \tag{14}$$

for all $x \in B$. Moreover, by employing a reasoning similar to that in Case 1, we obtain the uniqueness of T.

■

In [12] and [7], the authors investigated the stability of the following functional equations:

$$f(ax+by) = rf(x) + sf(y)$$

and

$$pf\left(\frac{x+y}{q}\right) = \left(\frac{p}{q}+\frac{q}{p}\right)f(x) + \left(\frac{p}{q}-\frac{q}{p}\right)f(y),$$

which represent special cases of the general functional equation (3), depending on the parameters a, b, p, q, r and s. Hence, the following corollaries can be directly derived from Theorem 1.

Corollary 1 *Let B be a subspace of a Banach space Y. Suppose that a fuzzy number-valued mapping $f : B \to X_F$ satisfies the inequality*

$$D\left(f\left(ax+by\right), rf(x) + sf(y)\right) < \epsilon$$

for all $x, y \in B$, where $\epsilon, a, b > 0$ and $r, s \in \mathbb{R}$ and $a + b = r + s \neq 1$. Then there exists a unique mapping $T : B \to X_F$ such that

$$D(T(x), f(x)) \leq \frac{\epsilon}{(a+b-1)^2}$$

for all $x \in B$.

Proof. Letting $u = 1$ and $v = 0$ in Theorem 1, we obtain this result. ■

Corollary 2 *Let B be a subspace of a Banach space Y. Suppose that a fuzzy number-valued mapping $f : B \to X_F$ satisfies the inequality*

$$D\left(pf\left(\frac{x+y}{q}\right), \left(\frac{p}{q}+\frac{q}{p}\right)f(x) + \left(\frac{p}{q}-\frac{q}{p}\right)f(y)\right) < \epsilon$$

for all $x, y \in B$, where $\epsilon, q > 0$ and $p \in \mathbb{R} - \{0\}$. Then there exists a unique mapping $T : B \to X_F$ such that

$$D(T(x), f(x)) \leq \frac{\epsilon|p|}{\left(\frac{2}{q} - |p|\right)^2}$$

for all $x \in B$.

Proof. Letting $a = b = \frac{1}{q}$, $u = p$, $v = 0$, $r = ua + \frac{u}{a}$ and $s = ua - \frac{u}{a}$ in Theorem 1, we obtain this result. ■

5 Conclusions

The modifications proposed in this study refine the classical definition of fuzzy numbers and uncover new perspectives on their applicability, particularly within the framework of Hyers-Ulam stability. By extending the stability analysis to a generalized fuzzy-number-valued functional equation in Banach spaces, this work establishes a unified framework encompassing a wider class of functional equations. Such a generalization not only broadens the theoretical landscape of fuzzy analysis but also opens new avenues for further research on the stability of fuzzy systems, particularly at the intersection of fuzzy logic and nonlinear analysis. The results presented herein contribute to a fundamental understanding of fuzzy number-valued mappings and provide valuable insights for both theoretical advances and practical applications in mathematics and the applied sciences.

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References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2(1950), 64–66.

[2] T. Cardinali, K. Nikodem and F. Papalini, Some results on stability and characterization of K-convexity of set-valued functions, Ann. Polon. Math., 58(1993), 185–192.

[3] H. Y. Chu, A. Kim and S. K. Yoo, On the stability of the generalized cubic set-valued functional equation, Appl. Math. Lett., 37(2014), 7–14.

[4] Z. Gajda, On stability of additive mappings, Int. J. Math. Math. Sci., 14(1991), 431–434.

[5] P. Găvruţa, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184(1994), 431–436.

[6] D. H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA, 27(1941), 222–224.

[7] J. Kunrattanaworawong and W. Sintunavarat, On the Ulam stability of two fuzzy number-valued functional equations via the metric defined on a fuzzy number space, Thai J. Math., 20(2022), 103–112.

[8] A. K. Mirmostafaee, M. Mirzavaziri and M. S. Moslehian, Fuzzy stability of the Jensen functional equation, Fuzzy Sets Syst., 159(2008), 730–738.

[9] A. K. Mirmostafaee and M. S. Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, Fuzzy Sets Syst., 159(2008), 720–729.

[10] T. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72(1978), 297–300.

[11] T. M. Rassias, On a modified Hyers-Ulam sequence, J. Math. Anal. Appl., 158(1991), 106–113.

[12] J. R. Wu and Z. Y. Jin, A note on Ulam stability of some fuzzy number-valued functional equations, Fuzzy Sets Systems, 375(2019), 191–195.

[13] S. M. Ulam, A Collection of Mathematical Problems, Interscience, New York, 1960.