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A New Updating Method For The Asymmetric Vibration Systems With No Spill-Over*

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Abstract

In this paper, the model updating problem for quadratic asymmetric vibration systems with no spill-over (MUP-QAV) is considered. The original quadratic asymmetric vibration systems are updated to new systems such that some "troublesome" eigenpairs are replaced by newly measured or given ones, while the remaining eigenpairs are kept unchanged. Firstly, some necessary and sufficient conditions are derived so that the updated system can preserve no spill-over. Then a set of parameter solutions of the MUP-QAV is characterized using only a few eigenpairs to be replaced. A gradient-based optimization algorithm is proposed for the minimum norm solution of the MUP-QAV. Finally, a necessary and sufficient condition is provided so that the sparsity structures of the system matrices and the no spill-over property are preserved simultaneously. The performance of the proposed algorithm is illustrated by some numerical examples.

1 Introduction

Consider an n-degree-of-freedom damped vibration system, which is modeled by the following second-order ordinary differential equations:

$$M\ddot{v}(t) + C\dot{v}(t) + Kv(t) = f(t), \tag{1}$$

where M, C, $K \in \mathbb{R}^{n\times n}$ are mass, damping and stiffness matrices, respectively, $v(t)$ represents the displacement vector and $f(t)$ is the external force control matrix. Additionally, $C = C_s + C_{as}$ and $K = K_s + K_{as}$, where C_s and K_s are symmetric matrices, C_{as} and K_{as} are asymmetric terms which are generated by the non-conservative forces, such as friction and aerodynamic force. Friction induced vibrations [16, 19] and aeroelasticity [20] are some real-world examples of such systems. The vibration of model (1) is governed by eigenvalues and associated eigenvectors of the following quadratic eigenvalue problem (QEP)

$$P(\lambda)x := (\lambda^2 M + \lambda C + K)x = 0, \tag{2}$$

where the scalar λ and the non-zero vector x are eigenvalue and right eigenvectors of the quadratic pencil $P(\lambda)$, respectively. It is known that $P(\lambda)$ has $2n$ finite eigenvalues, provided that the mass matrix M is nonsingular. Generally, the right eigenvector x and left eigenvector y corresponding to the eigenvalue λ of asymmetric pencil $P(\lambda)$ are different. Therefore, we call (λ, x, y) an eigen-triple of $P(\lambda)$. A good survey of mathematical properties and numerical techniques for the QEP can be found in [20].

In practice, the structural dynamic mathematical models, which can be derived analytically by the finite element method, are usually very large and sparse, and there are only a few small number of natural frequencies (eigenvalues) and model shapes (eigenvectors) can be experimentally measured from a realized practical structure. Compare these measured eigenpairs with corresponding ones from the finite element

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model (FEM) of these structures, there may be disagreement between these two sets, i.e., the FEM may be inaccurate. Then the engineers face the problem of updating the existing FEM with minimal changes so that the updated model can predict the dynamic characteristics of the structures accurately. Model updating problem (MUP) concerns correcting the coefficient matrices of FEM so that the updated model will have a behavior closely matching the experimental data. For large scale structures, there are usually only a few eigenpairs which need to be replaced. To ensure the stability of system, the remaining unmeasured natural frequencies and model shapes should be kept unchanged, which means that the updated system should preserve no spill-over, i.e., mathematically, it is required to keep remaining eigenvalues and their associated eigenvectors not to be affected by the model updating.

In recent years, extensive research has been conducted on the MUP of quadratic vibration systems. For damped structured systems, Carvalho [6] considered the eigenvalue embedding problem for the symmetric system, i.e., M, C, K are all symmetric. One can see the model updating methods for the most commonly discussed case when M is positively definite and C, K are symmetric in [12]. Bai et al. [3, 4, 5] proposed three different optimization updating methods such that the updated matrices M and K are symmetric positive semi-definite. However, these results did not consider the spill-over phenomenon. With the mass and stiffness matrices being positive definiteness, Chu [7] gave some sufficient conditions for the solvability conditions of the MUP with no spill-over, and they also pointed out that the necessary condition for the solvability of this problem is that the prescribed eigenvectors should span the same subspaces as those spanned by the original ones. By constructing a parametric symmetric low-rank correction form, Mao [14] provided a new model updating method for the undamped system which can preserve both no spill-over and positive definiteness (semi-definiteness) of the mass and stiffness matrices. With the spectral decomposition of several quadratic pencils, Zhao [21, 22] considered the MUP with no spill-over for the $\star$-palindromic quadratic system and the damped vibroacoustic system. Recently, a necessary and sufficient condition for high-order matrix polynomial to preserve no spill-over is provided in [24]. With some given deflating pairs, analytical expressions of structure preserving no-spillover updating were determined for some specific structured matrix pencils, including symmetric, Hermitian, $\star$-even, $\star$-odd and $\star$-Hamiltonian [1, 8]. Unfortunately, the prescribed eigenvectors in these methods cannot be chosen arbitrarily.

Moreover, all the methods mentioned above can not be used to solve the case that the system has repeated eigenvalues, and can not preserve no spill-over and sparsity structures of system matrices, simultaneously. It is well known that a defective eigenvalue, whose geometric multiplicity is less than its algebraic multiplicity, is generally more sensitive to perturbations than a semi-simple one. Based on a refined Schur method, the robust assignment for the repeated poles was considered in [10]. With the assumption that the prescribed left and right eigenvectors span the same subspace as those spanned by the original ones, respectively, a sufficient condition that the MUP with no spill-over for the quadratic asymmetric vibration system is solvable for the repeated eigenvalues was provided in [23]. Recently, structure preserving updating methods with no spill-over for the undamped systems are provided in [13, 18]. Therefore, MUP with no spill-over for the quadratic asymmetric system remains open in the following three cases: (i) necessary and sufficient condition that the updated system can preserve no spill-over, (ii) MUP for the repeated eigenvalues; and (iii) the sparsity of system matrices are preserved. In this paper, we consider the MUP for quadratic asymmetric vibration systems with no spill-over (MUP-QAV), which can be stated as follows:

MUP-QAV: Given an analytical model $(M,\ C,\ K)$, a set of its eigen-triples $\{\lambda_j,\ x_j,\ y_j\}_{j=1}^{p}$ $(1 \le p < 2n)$, and a set of measured eigenpairs $\{\tilde{\lambda}_j,\ \tilde{x}_j\}_{j=1}^{p}$, where both λ_j and $\tilde{\lambda}_j$ are closed under complex conjugate. Find $\widetilde{M}, \widetilde{C}, \widetilde{K} \in \mathbb{R}^{n\times n}$ such that

- The eigenvalues $\{\lambda_j\}_{j=1}^{p}$ and associated right eigenvectors $\{x_j\}_{j=1}^{p}$ of the original system are replaced by $\{\tilde{\lambda}_i, \tilde{x}_j\}_{j=1}^{p}$ in the updated system $(\widetilde{M},\ \widetilde{C},\ \widetilde{K})$,
- The remaining $2n-p$ unknown eigenvalues and their associated right eigenvectors of the original system are kept unchanged.

The main contributions of this paper are:

(i) Some necessary and sufficient conditions that the updated system can preserve no spill-over is provided.

(ii) Parametric solutions of the MUP-QAV are characterized by using system matrices and a few eigenpairs.

(iii) A gradient-based optimization algorithm is proposed for the minimum norm solution of MUP-QAV, and the prescribed eigenvalues need not to be simple.

(iv) A necessary and sufficient condition for the existence of the structured preserving solution is provided.

Throughout this paper, we denote by $\mathbb{R}^{m\times n}$ the set of all real $m\times n$ matrices, $\mathbb{C}^{m\times n}$ the set of all complex $m\times n$ matrices. Let $\sigma(A)$ be the set of all eigenvalues of A.

2 Necessary and Sufficient Conditions for the No Spill-Over

Suppose that all the distinct eigenvalues of $P(\lambda)$ are λ_1, $\bar{\lambda}_1$, …, λ_l, $\bar{\lambda}_l$, λ_{2l+1}, …, λ_k, where $\lambda_j = \alpha_j + i\beta_j$, $j = 1, \cdots, l$, and $\lambda_j \in \mathbb{R}$, $j = 2l+1, \dots, k$, each of which has algebraic multiplicity n_j, i.e.,

$$2n_1 + \cdots + 2n_l + n_{2l+1} + \cdots + n_k = 2n.$$

Associated with λ_j, let $J(\lambda_j) = \lambda_j I_{n_j} + N_j \in \mathbb{C}^{n_j\times n_j}$ be its Jordan canonical form, where N_j is a nilpotent matrix with at most 1's along its superdiagonal (depending on the geometric multiplicity of λ_j), and let $X_j, Y_j \in \mathbb{C}^{n\times n_j}$ be the corresponding generalized right and left eigenvectors, respectively. Clearly, $J(\lambda_j)$, X_j and Y_j are real, $j = 2l+1, \dots, k$. For $j = 1, \dots, l$, we write $X_j = X_{jR} + iX_{jI}$, $Y_j = Y_{jR} + iY_{jI}$, where X_{jR}, X_{jI}, Y_{jR}, $Y_{jI} \in \mathbb{R}^{n\times n_j}$ and

$$J_R(\lambda_j) = \begin{bmatrix} \alpha_j I_{n_j} + N_j & \beta_j I_{n_j} \\ -\beta_j I_{n_j} & \alpha_j I_{n_j} + N_j \end{bmatrix}. \tag{3}$$

Let

$$X := [X_{1R}, X_{1I}, \dots, X_{lR}, X_{lI}, X_{2l+1}, \dots, X_k] \in \mathbb{R}^{n\times 2n}, \tag{4}$$

$$Y := [Y_{1R}, Y_{1I}, \dots, Y_{lR}, Y_{lI}, Y_{2l+1}, \dots, Y_k] \in \mathbb{R}^{n\times 2n}, \tag{5}$$

$$J := diag\,(J_R(\lambda_1), \dots, J_R(\lambda_l), J(\lambda_{2l+1}), \dots, J(\lambda_k)) \in \mathbb{R}^{2n\times 2n}. \tag{6}$$

Note that (X, J) and (Y, J) are real standard pairs [9] of $P(\lambda)$ if and only if the $2n\times 2n$ matrices $\left[\begin{smallmatrix} Y \\ YJ \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} X \\ XJ \end{smallmatrix}\right]$ are nonsingular and

$$\begin{cases} MXJ^2 + CXJ + KX = 0, \\ (J^T)^2 Y^T M + J^T Y^T C + Y^T K = 0. \end{cases} \tag{7}$$

Partition J, X and Y as

$$J = \mathrm{diag}(J_1, J_2), \quad X = [X_1, X_2], \quad Y = [Y_1, Y_2], \tag{8}$$

where $X_1, Y_1 \in \mathbb{R}^{n\times p}$, and $J_1 \in \mathbb{R}^{p\times p}$ are the real representations of $\{\lambda_j\}_{j=1}^p$ and their associated right and left eigenvectors, respectively, which are to be replaced. Suppose that all the distinct eigenvalues of $\{\tilde{\lambda}_j\}_{j=1}^p$ are $\tilde{\lambda}_1, \bar{\tilde{\lambda}}_1, \cdots, \tilde{\lambda}_{\tilde{s}}, \bar{\tilde{\lambda}}_{\tilde{s}}, \cdots, \tilde{\lambda}_{\tilde{q}}$, where $\tilde{\lambda}_j = \tilde{\omega}_j + i\tilde{\psi}_j$ for $j = 1, \dots, \tilde{s}$, and $\tilde{\lambda}_j \in \mathbb{R}$ for $j = 2\tilde{s}+1, \dots, \tilde{q}$, each of which has algebraic multiplicity $\tilde{r}_j$. Let

$$\tilde{J}_1 := diag(J_R(\tilde{\lambda}_1), \dots, J_R(\tilde{\lambda}_{\tilde{s}}), J(\tilde{\lambda}_{2\tilde{s}+1}), \dots, J(\tilde{\lambda}_{\tilde{q}})), \tag{9}$$

where $J_R(\tilde{\lambda}_j) \in \mathbb{R}^{\tilde{r}_j\times\tilde{r}_j}$ and $J(\tilde{\lambda}_j) \in \mathbb{R}^{\tilde{r}_j\times\tilde{r}_j}$ are the Jordan canonical forms of $\tilde{\lambda}_j$ for $j = 1, \dots, \tilde{s}$ and $j = 2\tilde{s}+1, \dots, \tilde{q}$, respectively. Denoted by Z the matrix of the right eigenvectors corresponding to $\tilde{J}_1$.

Throughout this paper, the following assumptions are made:

(A1) All the eigenvalues of J_1, J_2 and $\tilde{J}_1$ are nonzero.

(A2) $\sigma(J_1) \cap \sigma(J_2) = \emptyset$, $\sigma(J_1) \cap \sigma(\tilde{J}_1) = \emptyset$.

(A3) $\left[\begin{smallmatrix} Y \\ YJ \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} X \\ XJ \end{smallmatrix}\right]$ are nonsingular.

(A4) M and K are nonsingular.

With the notations above, the problem of preserving no spill-over in the MUP-QAV can be mathematically reformulated as follows: Given $M, C, K \in \mathbb{R}^{n\times n}$, find $\Delta M, \Delta C, \Delta K \in \mathbb{R}^{n\times n}$ such that

$$\widetilde{M} X_2 J_2^2 + \widetilde{C} X_2 J_2 + \widetilde{K} X_2 = 0, \tag{10}$$

where $\widetilde{M} = M + \Delta M$, $\widetilde{C} = C + \Delta C$, $\widetilde{K} = K + \Delta K$.

Obviously, we can see from (7) that (10) is equivalent to

$$\Delta M X_2 J_2^2 + \Delta C X_2 J_2 + \Delta K X_2 = 0. \tag{11}$$

Based on the spectral decomposition of $P(\lambda)$, we provide some necessary and sufficient conditions of $(\Delta M,\ \Delta C,\ \Delta K)$ such that the updated system can preserve no spill-over.

Lemma 1 **([23])** *Let $X, Y \in \mathbb{R}^{n\times 2n}$ and $J \in \mathbb{R}^{2n\times 2n}$ be given by (4), (5) and (6), respectively. Then there exist $M, C, K \in \mathbb{R}^{n\times n}$ with M, K being nonsingular such that (7) holds if and only if there exists a nonsingular matrix $\Gamma \in \mathbb{R}^{2n\times 2n}$ satisfying*

$$\Gamma J^T = J\Gamma, \quad X\Gamma Y^T = 0. \tag{12}$$

In this case, the coefficient matrices M, C and K can be given by

$$M = (XJ\Gamma Y^T)^{-1}, \quad C = -MXJ\Gamma J^T Y^T M, \quad K = -(XJ^{-1}\Gamma Y^T)^{-1}.$$

Lemma 2 **([23])** *Under the assumption (A2), the following orthogonal relation*

$$\begin{bmatrix} Y_1^T & J_1^T Y_1^T \end{bmatrix} \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X_2 \\ X_2 J_2 \end{bmatrix} = 0$$

holds.

Theorem 1 *The MUP-QAV can avoid spill-over, i.e. (11) is satisfied if and only if ΔM, ΔC, ΔK jointly satisfy*

$$\begin{bmatrix} \Delta M, & \Delta C, & \Delta K \end{bmatrix} \mathcal{H} = 0 \tag{13}$$

where

$$\mathcal{H} = \begin{bmatrix} M^{-1} - X_1 J_1 \Gamma_1 Y_1^T & M^{-1} C M^{-1} + X_1 J_1^2 \Gamma_1 Y_1^T \\ -X_1 \Gamma_1 Y_1^T & -M^{-1} + X_1 J_1 \Gamma_1 Y_1^T \\ -K^{-1} - X_1 J_1^{-1} \Gamma_1 Y_1^T & X_1 \Gamma_1 Y_1^T \end{bmatrix} \in \mathbb{R}^{3n\times 2n}, \tag{14}$$

and

$$\Gamma_1 = \left(\begin{bmatrix} Y_1^T & J_1^T Y_1^T \end{bmatrix} \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_1 J_1 \end{bmatrix} \right)^{-1}. \tag{15}$$

Proof. (Necessity) Since $\left[\begin{smallmatrix} X \\ XJ \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} Y \\ YJ \end{smallmatrix}\right]$ are nonsingular, we can obtain from (7) and Lemma 1 that there exist a nonsingular matrix $\Gamma \in \mathbb{R}^{2n\times 2n}$ such that $J\Gamma = \Gamma J^T$ and

$$X\Gamma Y^T = 0, \quad M = (XJ\Gamma Y^T)^{-1}, \quad C = -MXJ^2\Gamma Y^T M, \; K = -(XJ^{-1}\Gamma Y^T)^{-1}. \tag{16}$$

Since $\sigma(J_1) \cap \sigma(J_2) = \emptyset$, it follows from Lemma 2 that the matrix Γ must be of block diagonal form $\Gamma = diag(\Gamma_1, \Gamma_2)$, which satisfy

$$J_1\Gamma_1 = \Gamma_1 J_1^T, \; J_2\Gamma_2 = \Gamma_2 J_2^T, \tag{17}$$

where Γ_1 is given by (15). Substituting (8) into (16), we can obtain that

$$X_2\Gamma_2Y_2^T = -X_1\Gamma_1Y_1^T, \tag{18}$$
$$X_2J_2\Gamma_2Y_2^T = M^{-1} - X_1J_1\Gamma_1Y_1^T, \tag{19}$$
$$X_2J_2^2\Gamma_2Y_2^T = -M^{-1}CM^{-1} - X_1J_1^2\Gamma_1Y_1^T, \tag{20}$$
$$X_2J_2^{-1}\Gamma_2Y_2^T = -K^{-1} - X_1J_1^{-1}\Gamma_1Y_1^T. \tag{21}$$

Pre-multiplying (18), (19) and (20) by ΔK, ΔC and ΔM, respectively, we can get

$$\left(\Delta MX_2J_2^2 + \Delta CX_2J_2 + \Delta KX_2\right)\Gamma_2Y_2^T := H_1, \tag{22}$$

where

$$H_1 = -\Delta M(M^{-1}CM^{-1} + X_1J_1^T\Gamma_1Y_1^2) + \Delta C(M^{-1} - X_1J_1\Gamma_1Y_1^T) - \Delta KX_1\Gamma_1Y_1^T.$$

Pre-multiplying (18), (19) and (21) by $\Delta C, \Delta M$and ΔK, respectively, we have

$$\left(\Delta MX_2J_2^2 + \Delta CX_2J_2 + \Delta KX_2\right)J_2^{-1}\Gamma_2Y_2^T := H_2, \tag{23}$$

where

$$H_2 = \Delta M(M^{-1} - X_1J_1\Gamma_1Y_1^T) - \Delta CX_1\Gamma_1Y_1^T - \Delta K(K^{-1} + X_1J_1^{-1}\Gamma_1Y_1^T).$$

It is easy to verify that (22) and (23) can be rewritten as

$$AB = \begin{bmatrix} H_1 & H_2 \\ 0 & 0 \end{bmatrix}, \tag{24}$$

where $B = \begin{bmatrix}\Gamma_2Y_2^T & J_2^{-1}\Gamma_2Y_2^T\end{bmatrix} \in \mathbb{R}^{(2n-p)\times 2n}$ and

$$A = \begin{bmatrix} \Delta C & \Delta M \\ \Delta M & 0 \end{bmatrix}\begin{bmatrix} X_2 \\ X_2J_2 \end{bmatrix}J_2 + \begin{bmatrix} \Delta K & 0 \\ 0 & -\Delta M \end{bmatrix}\begin{bmatrix} X_2 \\ X_2J_2 \end{bmatrix} \in \mathbb{R}^{2n\times(2n-p)}.$$

Obviously, $A = 0$ since (11) holds. It follows from (24) that (13) is satisfied.

(Sufficiency) Suppose that (13) holds. We can see from (24) that $AB = 0$, and it follows from the Sylvester's rank inequality that

$$rank(A) + rank(B) \leq 2n - p. \tag{25}$$

Since $\left[\begin{smallmatrix} Y \\ YJ \end{smallmatrix}\right]$ and Γ_2 are nonsingular, it follows from (17) that

$$rank(B^T) = rank\begin{bmatrix} Y_2\Gamma_2^T \\ Y_2\Gamma_2^TJ_2^{-T} \end{bmatrix} = rank\begin{bmatrix} Y_2 \\ Y_2J_2 \end{bmatrix} = 2n - p.$$

Then, we can see from (25) that $A = 0$, i.e., (11) holds. ■

3 Solvability of the MUP-QAV

In this section, we will characterize the solution of MUP-QAV by Theorem 1. Recall that (J_2, X_2) are remaining eigenpair of the original system which should be kept unchanged, and they are generally unknown in practice. Then, it is desirable to characterize the solutions of MUP-QAV without using any information of (J_2, X_2).

The MUP for quadratic asymmetric vibration systems with no spill-over can be stated as follows:

MUP-QAV: Given an analytical model $(M,\ C,\ K)$ and J_1, X_1, $\tilde{J}_1$ defined by (8) and (9), respectively, find ΔM, ΔC and ΔK such that the following equations hold:

(1) Eigenvalue embedding

$$(M + \Delta M)Z\tilde{J}_1^2 + (C + \Delta C)Z\tilde{J}_1 + (K + \Delta K)Z = 0. \tag{26}$$

(2) No spill-over

$$(M + \Delta M)X_2J_2^2 + (C + \Delta C)X_2J_2 + (K + \Delta K)X_2 = 0. \tag{27}$$

Lemma 3 *The rank of the matrix $\mathcal{H}$ is $2n - p$.*

Proof. Substituting (18)–(21) into (14) yields

$$\mathcal{H} = \begin{bmatrix} X_2J_2\Gamma_2Y_2^T & -X_2J_2^2\Gamma_2Y_2^T \\ X_2\Gamma_2Y_2^T & -X_2J_2\Gamma_2Y_2^T \\ X_2J_2^{-1}\Gamma_2Y_2^T & -X_2\Gamma_2Y_2^T \end{bmatrix} = \begin{bmatrix} X_2J_2^2 \\ X_2J_2 \\ X_2 \end{bmatrix} \Gamma_2 \begin{bmatrix} J_2^{-T}Y_2^T & -Y_2^T \end{bmatrix}. \tag{28}$$

Since $\left[\begin{smallmatrix} X \\ XJ \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} Y \\ YJ \end{smallmatrix}\right]$ are nonsingular, it is easy to see that $\left[\begin{smallmatrix} Y \\ YJ^{-1} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -Y \\ YJ \end{smallmatrix}\right]$ are nonsingular, which implies that the matrices

$$\begin{bmatrix} X_2 \\ X_2J_2 \\ X_2J_2^2 \end{bmatrix} \in \mathbb{R}^{3n\times(2n-p)} \quad \text{and} \quad \begin{bmatrix} -Y_2 \\ Y_2J_2^{-1} \end{bmatrix} \in \mathbb{R}^{2n\times(2n-p)}$$

are all of full column rank. Clearly, Γ_2 is nonsingular, it follows that $rank(\mathcal{H}) = 2n - p$. ■

Now, we characterize ΔM, ΔC and ΔK by Theorem 1 such that (26) and (27) are satisfied. Let the QR decomposition of $\mathcal{H}$ be

$$\mathcal{H} = Q \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{29}$$

where $Q := [Q_1, Q_2] \in \mathbb{R}^{3n\times 3n}$ is an orthogonal matrices with $Q_1 \in \mathbb{R}^{3n\times(2n-p)}$, $Q_2 \in \mathbb{R}^{3n\times(n+p)}$, and $R \in \mathbb{R}^{(2n-p)\times 2n}$ is of full row rank.

Theorem 2 *Partition Q_2^T as $Q_2^T = \left[\begin{smallmatrix} Q_{21} \\ Q_{22} \end{smallmatrix}\right]$, where $Q_{21} \in \mathbb{R}^{n\times 3n}$ and $Q_{22} \in \mathbb{R}^{p\times 3n}$. For any nonzero $Z \in \mathbb{R}^{n\times p}$, if*

$$U_Z := Q_{22}Z_L \in \mathbb{R}^{p\times p} \tag{30}$$

is nonsingular, where

$$Z_L = \begin{bmatrix} Z\tilde{J}_1^2 \\ Z\tilde{J}_1 \\ Z \end{bmatrix},$$

then the matrices ΔM, ΔC and ΔK given by

$$[\Delta M,\ \Delta C,\ \Delta K] = -(MZ\tilde{J}_1^2 + CZ\tilde{J}_1 + KZ)U_Z^{-1}Q_{22} \tag{31}$$

form a solution of the MUP-QAV. And in this case, Z is the eigenvector matrix corresponding to $\tilde{J}_1$.

Proof. By Theorem 1, we know that the no spill-over property (27) holds if and only if

$$[\Delta M,\ \Delta C,\ \Delta K]\mathcal{H} = 0.$$

Let

$$[U_1,\ U_2] := [\Delta M,\ \Delta C,\ \Delta K]Q, \tag{32}$$

where $U_1 \in \mathbb{R}^{n\times(2n-p)}$, $U_2 \in \mathbb{R}^{n\times(n+p)}$. It follows from the QR decomposition of $\mathcal{H}$ that

$$[\Delta M,\ \Delta C,\ \Delta K]Q \begin{bmatrix} R \\ 0 \end{bmatrix} = [U_1, U_2] \begin{bmatrix} R \\ 0 \end{bmatrix} = 0, \tag{33}$$

which implies that $U_1 = 0$. Then,

$$\begin{bmatrix}\Delta M, \ \Delta C, \ \Delta K\end{bmatrix} = \begin{bmatrix}U_1, U_2\end{bmatrix} Q^T = U_2 Q_2^T, \tag{34}$$

where U_2 is an arbitrary matrix, which can be further determined to satisfy (26). Obviously, (26) can be rewritten as

$$\begin{bmatrix}\Delta M, \ \Delta C, \ \Delta K\end{bmatrix} Z_L = -(MZ\tilde{J}_1^2 + CZ\tilde{J}_1 + KZ). \tag{35}$$

Substituting (34) into (35) gives

$$U_2 Q_2^T Z_L = -(MZ\tilde{J}_1^2 + CZ\tilde{J}_1 + KZ). \tag{36}$$

Partition $U_2 = [U_{21}, \ U_{22}]$, where $U_{21} \in \mathbb{R}^{n\times n}$ and $U_{22} \in \mathbb{R}^{n\times p}$. We set $U_{21} = 0$. Since U_Z is nonsingular, we can see from (36) that

$$U_{22} = -(MZ\tilde{J}_1^2 + CZ\tilde{J}_1 + KZ)U_Z^{-1}.$$

Thus, the solution of the MUP-QAV can be given by (31). ■

In the Theorem 2, we suppose that the matrix U_Z is nonsingular, which can be guaranteed by choosing appropriate Z. Now, we give some conditions under which U_Z is nonsingular. Let $U_Z := \begin{bmatrix}u_{ij}(Z)\end{bmatrix} \in \mathbb{R}^{p\times p}$ and

$$\tau := \min_{1\le k\le p}\left\{|u_{kk}(Z)| - \frac{1}{2}\left(\sum_{j=1,j\ne k}^{p}|u_{kj}(Z)| + \sum_{j=1,j\ne k}^{p}|u_{jk}(Z)|\right)\right\}.$$

As is known, the singular values of U_Z are the eigenvalues of $(U_Z U_Z^T)^{1/2}$ and are denoted by

$$\sigma_1(U_Z) \ge \sigma_2(U_Z) \ge \cdots \ge \sigma_p(U_Z) \ge 0.$$

If we choose Z such that $\tau > 0$, it follows from Theorem 3 in [11] that $\sigma_p(U_Z) \ge \tau > 0$. Then U_Z is nonsingular. Moreover, since

$$\sigma_1(U_Z) \le \left(\max_{1\le k\le p}\sum_{j=1}^{p}|u_{kj}(Z)|\right)^{\frac{1}{2}}\left(\max_{1\le j\le p}\sum_{k=1}^{p}|u_{jk}(Z)|\right)^{\frac{1}{2}},$$

the upper bound for the condition number of U_Z is

$$cond(U_Z) = \|U_Z\|_2\|U_Z^{-1}\|_2 = \frac{\sigma_1(U_Z)}{\sigma_p(U_Z)} \le \frac{\left(\left(\max_{1\le k\le p}\sum_{j=1}^{p}|u_{kj}(Z)|\right)^{\frac{1}{2}}\left(\max_{1\le j\le p}\sum_{k=1}^{p}|u_{jk}(Z)|\right)^{\frac{1}{2}}\right)}{\tau}.$$

In practice, we prefer the modifications made to the coefficient matrices are as small as possible, which is to solve the following optimization problem:

$$\min_{Z\in\mathbb{R}^{n\times p}} \mathcal{J} := \frac{1}{2}\|\Delta M\|_F^2 + \frac{1}{2}\|\Delta C\|_F^2 + \frac{1}{2}\|\Delta K]\|_F^2. \tag{37}$$

Based on the Theorem 2, the explicit expression for the gradient of $\mathcal{J}$ with respect to the matrix Z can be given by the following theorem.

Theorem 3 *Given any nonzero matrix $Z \in \mathbb{R}^{n\times p}$, suppose that the matrix U_Z defined by (30) is nonsingular and the QR decomposition of H is given by (29). Let $S = -(MZ\tilde{J}_1^2 + CZ\tilde{J}_1 + KZ)U_Z^{-1}$. Partition Q_{22} as $Q_{22} = [W_1, \ W_2, \ W_3]$, where $W_1, W_2, W_3 \in \mathbb{R}^{p\times n}$. Define $V = (V_1 - V_2)^T$, where*

$$V_1 = -(\tilde{J}_1^2 U_Z^{-1} S^T M + \tilde{J}_1 U_Z^{-1} S^T C + U_Z^{-1} S^T K),$$

$$V_2 = \tilde{J}_1^2 U_Z^{-1} S^T S W_1 + \tilde{J}_1 U_Z^{-1} S^T S W_2 + U_Z^{-1} S^T S W_3,$$

Then the gradient $\nabla_Z \mathcal{J}$ of $\mathcal{J}$ with respect to Z is given by

$$\nabla_Z \mathcal{J} = V. \tag{38}$$

Proof. From (31), we can obtain that

$$\big[\Delta M,\ \Delta C,\ \Delta K\big]Q = \big[0 \quad S\big].$$

Recall that Q is an $3n \times 3n$ orthogonal matrix, then

$$\mathcal{J} = \frac{1}{2}\|[\Delta M,\ \Delta C,\ \Delta K]\|_F^2 = \frac{1}{2}\|S\|_F^2 = \frac{1}{2}tr(S^T S).$$

In the following, we establish the gradient $\nabla_Z \mathcal{J}$ of $\mathcal{J}$. The gradient $\nabla_Z \mathcal{J}$ can be deduced from the first order variation:

$$\Delta\mathcal{J} = \frac{1}{2}tr(\Delta S^T S + S^T \Delta S) = \frac{1}{2}tr(\Delta S^T S) + \frac{1}{2}tr(S^T \Delta S). \tag{39}$$

We now express $tr(S^T \Delta S)$ in terms of ΔZ. Note that U_Z is nonsingular, then we have

$$SU_Z = -(MZ\tilde{J}_1^2 + CZ\tilde{J}_1 + KZ). \tag{40}$$

Taking the first order variation of both sides of (40) yields

$$\Delta S U_Z + S\Delta U_Z = -(M\Delta Z\tilde{J}_1^2 + C\Delta Z\tilde{J}_1 + K\Delta Z),$$

which implies that

$$\Delta S = [-(M\Delta Z\tilde{J}_1^2 + C\Delta Z\tilde{J}_1 + K\Delta Z) - S\Delta U_Z]U_Z^{-1}. \tag{41}$$

Substituting (41) into the term $tr(S^T \Delta S)$ gives

$$\begin{aligned} tr(S^T \Delta S) &= tr(S^T[-(M\Delta Z\tilde{J}_1^2 + C\Delta Z\tilde{J}_1 + K\Delta Z) - S\Delta U_Z]U_Z^{-1}) \\ &= tr(U_Z^{-1}S^T[-(M\Delta Z\tilde{J}_1^2 + C\Delta Z\tilde{J}_1 + K\Delta Z) - S\Delta U_Z]) \\ &= tr(V_1\Delta Z) - tr(U_Z^{-1}S^T S\Delta U_Z). \end{aligned} \tag{42}$$

Substituting the partition of Q_{22} into (30) yields

$$U_Z = W_1 Z\tilde{J}_1^2 + W_2 Z\tilde{J}_1 + W_3 Z.$$

It follows that

$$\Delta U_Z = W_1\Delta Z\tilde{J}_1^2 + W_2\Delta Z\tilde{J}_1 + W_3\Delta Z,$$

which implies that

$$tr(U_Z^{-1}S^T S\Delta U_Z) = tr(U_Z^{-1}S^T S(W_1\Delta Z\tilde{J}_1^2 + W_2\Delta Z\tilde{J}_1 + W_3\Delta Z)) = tr(V_2\Delta Z). \tag{43}$$

Then, we can obtain from (42) and (43) that

$$tr(S^T\Delta S) = tr((V_1 - V_2)\Delta Z) = tr(V^T\Delta Z). \tag{44}$$

Similar to the proof of (44), we can get the term $tr(S\Delta S^T)$ in (39) by ΔZ as follows:

$$tr(S\Delta S^T) = tr(V\Delta Z^T). \tag{45}$$

Finally, substituting (44) and (45) into (39) gives

$$\Delta\mathcal{J} = \frac{1}{2}tr(V^T\Delta Z) + \frac{1}{2}tr(V\Delta Z^T), \tag{46}$$

which implies that the gradient of $\mathcal{J}$ can be given by (38). ∎

We summarize the discussion above as the following Algorithm 1 for minimum solution of the MUP-QAV.

Algorithm 1 Find a minimum solution to the MUP-QAV.

Input:
1. The matrices M, C, $K \in \mathbb{R}^{n\times n}$.
2. A set of targeted or newly measured eigenvalues $\{\tilde{\lambda}_j\}_{j=1}^{p}$.
3. ϵ = Termination tolerance and N = Maximum number of iteration.

Output: The matrices ΔM, ΔC and ΔK such that the objective function $\mathcal{J}$ in (37) is minimized.
Step 1. Form the matrices J_1, X_1, Y_1, $\tilde{J}_1$ by (8) and (9).
Step 2. Compute the QR decomposition of $\mathcal{H}$ by (28). (This requires $O(n^3)$ operations.) **Set** $k = 1$.
Step 3. Randomly choose Z.
Step 4. Compute U_Z by (30) and compute cond(U_Z). If cond(U_Z) is large, return to **Step 3**.
Step 5. Form the matrices S, V_1, V_2 and compute Grad := $\nabla_Z\mathcal{J}$ by Theorm 3. (This requires $O(p^3 + np^2)$ operations.) If $||\nabla_Z\mathcal{J}|| < \epsilon$ or if the number of iterations exceeds N, go to **Step 7**; else go to **Step 6.**
Step 6. Compute a new Z by gradient-based optimization method (we use the BFGS method [15]) . **Set** $k = k + 1$ and return to **Step 4**.
Step 7. Compute the matrices ΔM, ΔC and ΔK defined in Theorem 2 by Z which minimizes the value of $\mathcal{J}$. **Stop.**

Next, we will characterize the solutions of MUP-QVA that the sparsity structures of M, C, K are preserved. Let $\mathfrak{M}, \mathfrak{C}, \mathfrak{K} \in \mathbb{R}^{n\times n}$ be the subspaces of the structured matrices, which have the same sparsity structures with M, C, K, respectively. Without loss of generality, we can assume that $\dim(\mathfrak{M}) = r_1$, $\dim(\mathfrak{C}) = r_2$ and $\dim(\mathfrak{K}) = r_3$, and the basis for $\mathfrak{M}$ is $\{S_{j1}, j = 1, 2, \ldots, r_1\}$, for $\mathfrak{C}$ is $\{S_{j2}, j = 1, 2, \ldots, r_2\}$, and for $\mathfrak{K}$ is $\{S_{j3}, j = 1, 2, \ldots, r_3\}$. Then the matrices M, C and K can be expressed as

$$M = \sum_{j=1}^{r_1} m_j S_{j1}, \quad C = \sum_{j=1}^{r_2} c_j S_{j2}, \quad K = \sum_{j=1}^{r_3} k_j S_{j3},$$

where m_j, c_j, k_j are the structure parameters of the matrices M, C, K. Define

$$\mathbf{vec}_s(M) = \begin{bmatrix} m_1 & m_2 & \cdots & m_{r_1}\end{bmatrix}^T \in \mathbb{R}^{r_1\times 1}, \tag{47}$$
$$\mathbf{vec}_s(C) = \begin{bmatrix} c_1 & c_2 & \cdots & c_{r_2}\end{bmatrix}^T \in \mathbb{R}^{r_2\times 1}, \tag{48}$$
$$\mathbf{vec}_s(K) = \begin{bmatrix} k_1 & k_2 & \cdots & k_{r_3}\end{bmatrix}^T \in \mathbb{R}^{r_3\times 1}. \tag{49}$$

Let

$$\begin{aligned} B_m &= \begin{bmatrix}\mathbf{vec}(S_{11}) & \mathbf{vec}(S_{21}) & \cdots & \mathbf{vec}(S_{r_1 1})\end{bmatrix} \in \mathbb{R}^{n^2\times r_1}, \\ B_c &= \begin{bmatrix}\mathbf{vec}(S_{12}) & \mathbf{vec}(S_{22}) & \cdots & \mathbf{vec}(S_{r_2 2})\end{bmatrix} \in \mathbb{R}^{n^2\times r_2}, \\ B_k &= \begin{bmatrix}\mathbf{vec}(S_{13}) & \mathbf{vec}(S_{23}) & \cdots & \mathbf{vec}(S_{r_3 3})\end{bmatrix} \in \mathbb{R}^{n^2\times r_3}, \\ D &= \mathrm{diag}(B_m, B_c, B_k), \end{aligned}$$

where **vec** is matrix operator which stack all columns of a matrix and transform it into a vector. Thus, it is easy to see that

$$\mathbf{vec}(M) = B_m\mathbf{vec}_s(M), \ \mathbf{vec}(C) = B_c\mathbf{vec}_s(C), \ \mathbf{vec}(K) = B_k\mathbf{vec}_s(K).$$

Theorem 4 *The MUP-QVA has solution $(\Delta M, \Delta C, \Delta K) \in \mathfrak{M}\times\mathfrak{C}\times\mathfrak{K}$ if and only if $PP^\dagger b = b$, where*

$$P = \begin{bmatrix} Z_L^T \otimes I_n \\ H^T \otimes I_n \end{bmatrix} D \in \mathbb{R}^{(2n^2+np)\times r}, \tag{50}$$

$$b = \begin{bmatrix} -\mathbf{vec}(MZ\tilde{J}_1^2 + CZ\tilde{J} + KZ) \\ 0 \end{bmatrix} \in \mathbb{R}^{(2n^2+np)\times 1}. \tag{51}$$

And in this case, the solutions are given by

$$\mathbf{vec}(\Delta M) = B_m \begin{bmatrix} I_{r_1} & 0_{r_1\times(r_2+r_3)} \end{bmatrix} \left(P^\dagger b + (I_r - P^\dagger P)y\right), \tag{52}$$
$$\mathbf{vec}(\Delta C) = B_c \begin{bmatrix} 0_{r_2\times r_1} & I_{r_2} & 0_{r_2\times r_3} \end{bmatrix} \left(P^\dagger b + (I_r - P^\dagger P)y\right), \tag{53}$$
$$\mathbf{vec}(\Delta K) = B_k \begin{bmatrix} 0_{r_3\times(r_1+r_2)} & I_{r_3} \end{bmatrix} \left(P^\dagger b + (I_r - P^\dagger P)y\right), \tag{54}$$

where $r = r_1+r_2+r_3$, $y \in \mathbb{R}^{r\times 1}$ *is arbitrary. Moreover, MUP-QVA has an unique solution* $(\Delta M, \Delta C, \Delta K) \in \mathfrak{M}\times\mathfrak{C}\times\mathfrak{K}$ *if and only if* $PP^\dagger b = b$ *and* $P^\dagger P = I_r$*. The solution is given by*

$$\mathbf{vec}(\Delta M) = B_m \begin{bmatrix} I_{r_1} & 0_{r_1\times(r_2+r_3)} \end{bmatrix} P^\dagger b, \tag{55}$$
$$\mathbf{vec}(\Delta C) = B_c \begin{bmatrix} 0_{r_2\times r_1} & I_{r_2} & 0_{r_2\times r_3} \end{bmatrix} P^\dagger b, \tag{56}$$
$$\mathbf{vec}(\Delta K) = B_k \begin{bmatrix} 0_{r_3\times(r_1+r_2)} & I_{r_3} \end{bmatrix} P^\dagger b, \tag{57}$$

Proof. From Theorem 1, we know that the sufficient and necessary conditions for the existence of a solution of MUP-QVA without the structure constraints are given by (13) and (26). In order to get the solution $(\Delta M, \Delta C, \Delta K) \in \mathfrak{M}\times\mathfrak{C}\times\mathfrak{K}$, we need to characterize the structured solutions of (13) and (26). Let

$$z = \begin{bmatrix} \mathbf{vec}_s(\Delta M) \\ \mathbf{vec}_s(\Delta C) \\ \mathbf{vec}_s(\Delta K) \end{bmatrix} \in \mathbb{R}^{r\times 1}.$$

Then, it is easy to verify that $\mathbf{vec}[\Delta M \;\; \Delta C \;\; \Delta K] = Dz$. From (13), we have

$$\begin{aligned} & [\Delta M \;\; \Delta C \;\; \Delta K]H = 0, \\ \Rightarrow \quad & (H^T \otimes I_{3n})\mathbf{vec}[\Delta M \;\; \Delta C \;\; \Delta K] = 0, \\ \Rightarrow \quad & (H^T \otimes I_{3n})Dz = 0. \end{aligned} \tag{58}$$

From (26), we get

$$\begin{aligned} & \Delta M Z \tilde{J}_1^2 + \Delta C Z \tilde{J}_1 + \Delta K Z = -(MZ\tilde{J}^2 + CZ\tilde{J}_1 + KZ), \\ \Rightarrow \quad & \mathbf{vec}\left([\Delta M \;\; \Delta C \;\; \Delta K]Z_L\right) = -\mathbf{vec}\left(MZ\tilde{J}^2 + CZ\tilde{J}_1 + KZ\right), \\ \Rightarrow \quad & (Z_L^T \otimes I_{3n})Dz = -\mathbf{vec}\left(MZ\tilde{J}^2 + CZ\tilde{J}_1 + KZ\right). \end{aligned} \tag{59}$$

Combining (58) and (59) yields

$$Pz = b, \tag{60}$$

where P and b are given by (50) and (51), respectively. It is well known that the system of linear equations (60) is solvable if and only if $PP^\dagger b = b$, and the solutions can be given by

$$z = P^\dagger b + (I_r - P^\dagger P)y,$$

where $y \in \mathbb{R}^{r\times 1}$ is arbitrary. Therefore, the MUP-QVA has solution $(\Delta M, \Delta C, \Delta K) \in \mathfrak{M}\times\mathfrak{C}\times\mathfrak{K}$ if and only if $PP^\dagger b = b$, and it is easy to verify that the solutions $\Delta M, \Delta C, \Delta K$ can be characterized by (52), (53) and (54), respectively. Similarly, we can prove the case of $PP^\dagger b = b$ and $P^\dagger P = I_r$. This completes the proof. ■

Clearly, the freedoms in (52), (53) and (54) can be used to get the minimum norm solution. Based on the Theorem 4, the problem of finding the minimum norm structured solution of MUP-QVA can be formulated as the following constrained linear optimization problem:

$$\begin{aligned} \min_{z\in\mathbb{R}^{r\times 1}} \quad & ||\Delta M||_F^2 + ||\Delta C||_F^2 + ||\Delta K]||_F^2 \\ s.t. \quad & Pz - b = 0, \end{aligned} \tag{61}$$

where P and b are given by (50) and (51), respectively. This is a simple convex optimization problem. In our numerical example, we have used MATLAB function **fmincon** to find the minimum of (61). If the optimal solution z^* is obtained, we can reconstruct ΔM, ΔC and ΔK by (52), (53) and (54), respectively.

4 Numerical Examples

In this section, some numerical examples are provided to illustrate the performance of Algorithm 1. All calculations are performed using MATLAB R2016a. We compute the relative residuals of the updated system ($Res.U$) and the original system ($Res.O$) as

$$Res.U_j = \frac{\|\widetilde{M}Z_j\tilde{J}_j^2 + \widetilde{C}Z_j\tilde{J}_j + \widetilde{K}Z_j\|_2}{(\|\widetilde{M}\|_2\|\tilde{J}_j^2\|_2 + \|\widetilde{C}\|_2\|\tilde{J}_j\|_2 + \|\widetilde{K}\|_2)\|Z_j\|_2},$$
$$Res.O_j = \frac{\|MX_jJ_j^2 + CX_jJ_j + KX_j\|_2}{(\|M\|_2\|J_j^2\|_2 + \|C\|_2\|J_j\|_2 + \|K\|_2)\|X_j\|_2}, \quad j = 1, 2.$$

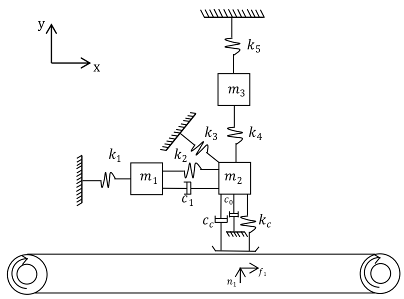

Figure 1: Four degree of freedom model in Example 1.

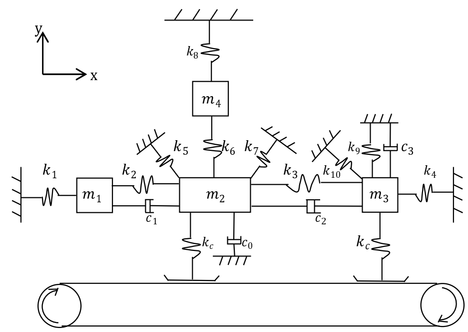

Figure 2: Six degree of freedom model in Example 2.

Example 1 ([17]) *In this example, we consider a friction-induced vibration system with four degrees of freedoms, which is shown in Figure 1, and the coefficient matrices are* $M = diag(m_1, m_3, m_2, m_2)$ *and*

$$C = \begin{bmatrix} c_1 & 0 & -c_1 & 0 \\ 0 & 0 & 0 & 0 \\ -c_1 & 0 & c_1 & 0 \\ 0 & 0 & 0 & c_0 \end{bmatrix}, \quad K = \begin{bmatrix} k_1+k_2 & 0 & -k_2 & 0 \\ 0 & k_4+k_5 & 0 & -k_4 \\ -k_2 & 0 & k_2+0.5k_3 & -0.5k_3+\mu k_c \\ 0 & -k_4 & -0.5k_3 & k_4+0.5k_3+k_c \end{bmatrix}.$$

Where $m_i = 1kg(i = 1, 2, 3)$, $c_i = 0.5Ns/m(i = 0, 1)$, $k_i = 100N/m(i = 1, 2, 3, 4, 5)$ *and* $k_c = 2k_1$. *Moreover, the friction coefficient* μ *is set to be 0.3868.*

Suppose that the eigenvalues $\pm 8.7334i$ are replaced by $-1 \pm 15i$, and the other eigenvalues and associated eigenvectors are kept unchanged. By Algorithm 1, we randomly choose Z and obtain that

$$\widetilde{M} = \begin{bmatrix} 0.7737 & 0.0161 & -0.4561 & 0.0343 \\ 0.0324 & 0.9999 & 0.0580 & -0.0018 \\ -0.3839 & 0.0306 & 0.2151 & 0.0628 \\ 0.0496 & 0.0018 & 0.0822 & 1.0000 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} 1.0152 & -0.0571 & -0.1189 & -0.0535 \\ -0.7805 & 0.1088 & -0.9368 & 0.1337 \\ -0.6874 & 0.0542 & -0.1786 & 0.0985 \\ -1.8120 & 0.2545 & -2.2048 & 0.8148 \end{bmatrix}$$

and

$$\widetilde{K} = \begin{bmatrix} 199.4916 & -0.1094 & -148.4304 & -2.0043 \\ 0.8557 & 200.1819 & 5.4772 & -98.9403 \\ -99.6865 & 0.0642 & 65.6441 & 25.1201 \\ 1.9938 & -99.5763 & -42.8972 & 352.2968 \end{bmatrix}.$$

The relative residues of the updated system and the original system are

$$Res.U_1 = 1.1924e-17, \quad Res.U_2 = 3.6514e-15,$$
$$Res.O_1 = 3.7000e-15, \quad Res.O_2 = 3.3551e-15.$$

Numerical results show that the prescribed eigenvalues are embedded perfectly into the updated system, and the remaining eigenvalues are kept unchanged.

Example 2 *In this example, we consider a mass-spring-damper system on a conveyor belt in [2] (see Figure 2). The coefficient matrices of this model are* $M = diag(m_1, m_3, m_3, m_4, m_2, m_1)$ *and*

$$C = \begin{bmatrix} c_1 & 0 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 0 & c_1+c_2 & -c_2 & 0 & 0 \\ 0 & 0 & -c_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_{11} & 0 & k_{13} & 0 & 0 & 0 \\ 0 & k_{22} & 0 & 0 & k_{25} & 0 \\ k_{31} & 0 & k_{33} & k_{34} & k_{35}+\mu k_c & 0 \\ 0 & 0 & k_{43} & k_{44} & 0 & k_{46} \\ 0 & k_{52} & k_{53} & 0 & k_{55} & 0 \\ 0 & 0 & 0 & k_{64}+\mu k_c & 0 & k_{66} \end{bmatrix}.$$

where $k_{11} = k_1 + k_2$, $k_{13} = k_{31} = -k_2$, $k_{22} = k_6 + k_8$, $k_{25} = k_{52} = -k_6$, $k_{33} = k_2 + k_3 + 0.5(k_5 + k_7)$, $k_{34} = k_{43} = -k_3$, $k_{35} = k_{53} = 0.5(k_7 - k_5)$, $k_{44} = k_3 + k_4 + 0.5k_{10}$, $k_{46} = k_{64} = -0.5k_{10}$, $k_{55} = k_c + k_6 + 0.5(k_5 + k_7)$, $k_{66} = k_c + k_9 + 0.5k_{10}$. *We take the mass* $m_i = 1kg\,(i = 1, 2, 3, 4)$, *damping* $c_i = 0.5Ns/m\,(i = 0, 1, 2, 3)$, *stiffness* $k_i = 100N/m$ $(i = 1, 2, 3, 4, 5, 6, 8, 9, 10)$, $k_7 = 50N/m$, *contact stiffness* $k_c = 110N/m$. *Besides, the friction coefficient* μ *is set to be 0.5.*

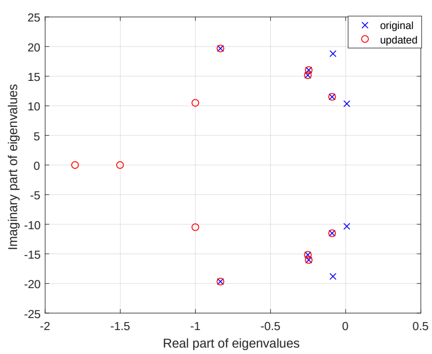

Figure 3: Eigenvalues of updated and original systems in Example 2: Case i.

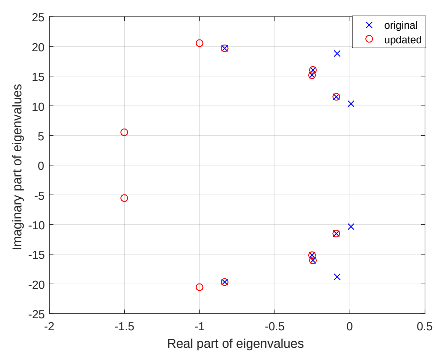

Figure 4: Eigenvalues of updated and original systems in Example 2: Case ii.

The eigenvalues of the original system are $\{0.0069 \pm 10.3843i$, $-0.0838 \pm 18.8646i$, $-0.0903 \pm 11.4497i$, $-0.2465 \pm 15.9791i$, $-0.2517 \pm 15.2078i$, $-0.8346 \pm 19.6958i.\}$ Suppose that we update $0.0069 \pm 10.3843i$ and $-0.0838 \pm 18.8646i$ to $\tilde\lambda_1, \tilde\lambda_2, \tilde\lambda_3, \tilde\lambda_4$ as the following four cases:

(i) $\tilde\lambda_1 = -1.5$, $\tilde\lambda_2 = -1.8$ and $\tilde\lambda_{3,4} = -1 \pm 10.5i$;

(ii) $\tilde\lambda_{1,2} = -1.5 \pm 5.5i$ and $\tilde\lambda_{3,4} = -1 \pm 20.5$;

(iii) $\tilde\lambda_1 = -2$, $\tilde\lambda_2 = -2.5$, $\tilde\lambda_3 = -1$ and $\tilde\lambda_4 = -1.5$.

(iv) $\tilde\lambda_{1,2} = -2 \pm 15.5i$ with algebraic multiplicity two, and the Jordan canonical form of $\tilde\lambda$ is

$$J_R(\tilde\lambda) = \begin{bmatrix} -2I_2 + N_2 & 15.5I_2 \\ -15.5I_2 & -2I_2 + N_2 \end{bmatrix}.$$

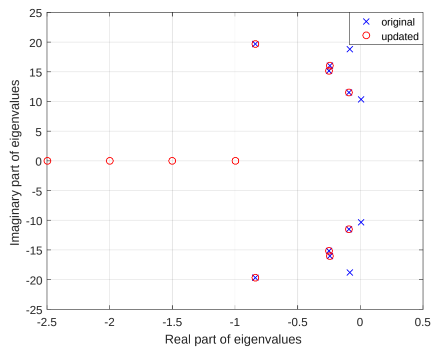

Figure 5: Eigenvalues of updated and original systems in Example 2: Case iii.

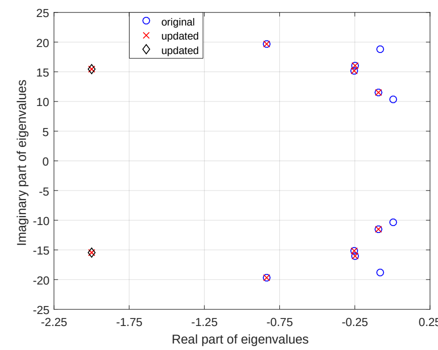

Figure 6: Eigenvalues of updated and original systems in Example 2: Case iv.

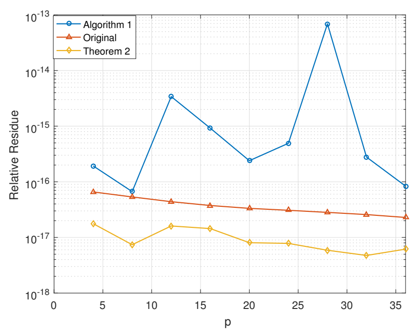

Figure 7: The relative residue $Res.U1$ of the updated and original systems in Example 3

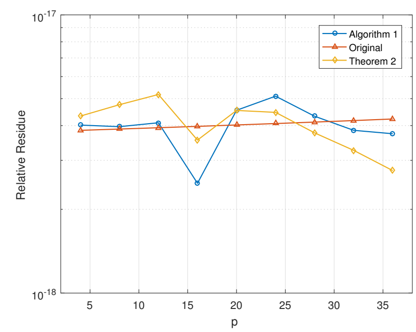

Figure 8: The relative residue $Res.U2$ of the updated and original systems in Example 3

For Cases (i)-(iii), the coefficient matrices $\widetilde{M}$, $\widetilde{C}$, and $\widetilde{K}$ are computed by Algorithm 1, and the eigenvalues of the original and updated systems are shown in Figures 3, 4 and 5, respectively. For Case (iv), Figure 6 demonstrates that the Algorithm 1 is also effective for solving the MUP-QAV problem when the prescribed eigenvalues are not simple.

Example 3 *In this example, we consider an 100 order quadratic asymmetric vibration system, with the coefficient matrices M, C and K being randomly generated. Suppose that we update p eigenvalues λ_j of $P(\lambda)$ (varying from 4 to 36 in increments of 4) to randomly generated real numbers $\tilde{\lambda}_j$.*

The relative residues $Res.U1$ and $Res.U2$ of the updated and original system are illustrated in Figure 7 and 8, respectively. The prescribed eigenvalues $\tilde{\lambda}$ are perfectly embedded into the updated system, and the remaining eigenvalues and their associated eigenvectors are kept unchanged.

For all p, the results are very close, or at least comparable. In the correction of quadratic vibration systems, it is essential to keep disturbances as small as possible because smaller disturbances help maintain the stability and performance of the system. As shown in Table 1, the norms of ΔM, ΔC and ΔK obtained by Algorithm 1 are much smaller than those of Theorem 3.2. Therefore, it can be concluded that Algorithm 1 is efficient for the minimum norm solution of the MUP-QAV.

Example 4 *Consider a quadratic vibration system with the system matrices as follows:*

$$M = \begin{bmatrix} 0.7684 & 3.6679 & 3.7304 & 0.4366 & 1.5003 & 2.6117 \\ 0.1457 & 1.5055 & 4.0492 & 4.1436 & 2.0566 & 1.4809 \\ 0.0459 & 2.4781 & 3.7264 & 3.4299 & 1.1830 & 2.3141 \\ 2.9825 & 1.2914 & 1.6859 & 1.3369 & 0.9747 & 4.6263 \\ 3.0455 & 3.6645 & 2.9219 & 4.8475 & 3.5270 & 1.0797 \\ 4.5947 & 0.5837 & 2.3449 & 0.9200 & 0.9032 & 0.0057 \end{bmatrix}, \quad C = \begin{bmatrix} 4.5332 & 3.4003 & 0 & 0 & 0 & 2.5458 \\ 3.4003 & 2.5750 & 2.6106 & 1.2344 & 0 & 0.2273 \\ 0 & 2.6106 & 0.5153 & 4.9843 & 0 & 4.2088 \\ 0.2415 & 0 & 4.9843 & 1.7948 & 3.1265 & 1.5820 \\ 3.9172 & 4.8621 & 2.9324 & 3.1265 & 1.9675 & 0.0386 \\ 0 & 0 & 3.8901 & 3.6385 & 0.0386 & 2.7263 \end{bmatrix},$$

Table 1: Numerical results of Example 3

p	Alg	$\|\Delta M\|_F$	$\|\Delta C\|_F$	$\|\Delta K\|_F$
4	Theorem 2	16.0605	44.9849	15.0213
	Algorithm 1	2.0230	4.5811	1.9658
8	Theorem 2	110.188	333.353	91.2415
	Algorithm 1	5.6202	13.3973	5.7456
12	Theorem 2	75.944	158.701	85.3559
	Algorithm 1	10.3064	23.0454	10.3488
16	Theorem 2	50.1678	97.6717	51.3775
	Algorithm 1	13.5087	33.2102	14.7694
20	Theorem 2	46.4630	89.5189	47.8009
	Algorithm 1	20.2596	44.7294	22.4738
24	Theorem 2	84.6463	181.016	102.895
	Algorithm 1	26.2396	61.2226	31.3358
28	Theorem 2	401.638	869.697	473.843
	Algorithm 1	26.6405	65.199	36.5226
32	Theorem 2	109.495	205.977	121.763
	Algorithm 1	30.8353	65.0218	37.4174
36	Theorem 2	342.898	779.189	375.449
	Algorithm 1	33.6178	73.535	37.3525

$$K = \begin{bmatrix} 3.2553 & 3.3229 & 0 & 0 & 1.9910 & 3.7566 \\ 3.3229 & 4.6940 & 2.6757 & 0 & 0 & 2.6117 \\ 0 & 2.6757 & 1.9924 & 3.3523 & 0 & 0 \\ 2.4526 & 0 & 3.3523 & 2.2031 & 0.6641 & 0 \\ 0 & 0 & 0 & 0.6641 & 2.1965 & 2.7384 \\ 1.2540 & 0 & 0.4433 & 0 & 2.7384 & 1.9758 \end{bmatrix}.$$

The eigenvalues of original system are $\{-1.1555 \pm 2.2186i,\ 1.2693 \pm 1.1299i,\ -0.4712 \pm 0.6932i,\ -0.3789 \pm 0.3867i,\ -1.4932,\ 0.6184,\ 0.1765,\ -0.7192\}$. *Suppose that the eigenvalues* $\{-1.1555 \pm 2.2186i\}$ *are replaced by* $\{-10 \pm 5i\}$.

There are 84 nonzero entries in M, C, K. We choose

$$Z^T = \begin{bmatrix} 2.2379 & 3.1899 & 3.5473 & 4.9632 & 4.6611 & 0.4616 \\ 4.7679 & 0.8148 & 4.8523 & 2.9852 & 1.2019 & 0.3522 \end{bmatrix},$$

and compute P and b by (50) and (51), respectively. It is easy to verify that $PP^{\dagger}b = b$ and $rank(P^{\dagger}P) = 72 < 84$, which means that the solution set of problem (61) is nonempty. Applying the function **fmincon** in MATLAB, we get

$$\widetilde{M} = \begin{bmatrix} -1.3059 & 1.5905 & 1.2535 & -0.4073 & -1.4638 & 4.7856 \\ -0.9665 & -0.2678 & 0.5427 & 0.8431 & -0.8177 & 1.4792 \\ -1.1650 & 0.5732 & 0.8565 & 0.4204 & -0.9676 & 0.4887 \\ -0.1720 & 1.3465 & 0.3831 & -0.7971 & -0.5424 & 3.6218 \\ 0.6073 & 0.3929 & -0.0676 & -1.3458 & 0.8488 & 0.8256 \\ 0.0121 & -0.1312 & -0.0380 & 0.0889 & 0.0416 & -0.2479 \end{bmatrix},$$

$$\widetilde{C} = \begin{bmatrix} 0.2374 & -0.4478 & 0 & 0 & 0 & 1.6411 \\ -0.4478 & -0.7134 & 0.4974 & 0.1358 & 0 & 0.2025 \\ 0 & 0.4974 & -1.7393 & 0.2640 & 0 & 2.0728 \\ 0.1717 & 0 & 0.2640 & -0.3539 & 1.4540 & 1.4770 \\ -0.9350 & -0.3343 & 0.7516 & 1.4540 & -1.1472 & 0.0274 \\ 0 & 0 & 0.0125 & -0.0515 & 0.0274 & -0.0928 \end{bmatrix},$$

$$\widetilde{K} = \begin{bmatrix} 3.4842 & 1.4642 & 0 & 0 & 1.8128 & 2.1431 \\ 1.4642 & 1.0869 & 0.8583 & 0 & 0 & 0.3662 \\ 0 & 0.8583 & -0.0957 & 0.7959 & 0 & 0 \\ 1.4480 & 0 & 0.7959 & 1.3120 & 0.4004 & 0 \\ 0 & 0 & 0 & 0.4004 & 0.2116 & -0.1523 \\ -0.1284 & 0 & 0.0800 & 0 & -0.1523 & -0.1274 \end{bmatrix},$$

and $||\Delta M||_F = 13.2730$, $||\Delta C||_F = 12.2500$ and $||\Delta K||_F = 6.6537$. The residuals of the updated system are

$$ResU_1 = 1.5440e - 11, \quad ResU_2 = 1.6089e - 11,$$

which mean that the prescribed eigenvalues are embedded into the updated system, and the sparsity of C, K and the no spill-over property are preserved, simultaneously.

5 Conclusion

The model updating problem for quadratic asymmetric vibration systems with no spill-over is considered. A necessary and sufficient condition that the updated system can preserve no spill-over is provided. The methods proposed in this paper, (i) work directly in the second-order model, (ii) can be implemented with the knowledge of only a small number of eigenpairs which are to be replaced, (iii) can preserve the sparsity and no spill-over property, simultaneously. Numerical results show that the proposed algorithm constitute a practical and reliable way to replace the "troublesome" eigenpairs, and the spill-over can be prevented efficiently.

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