Applied Mathematics E-Notes, 26(2026), 184-193 ©
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Regularized Sinc Collocation Method For Solving Fredholm Integral Equations Of The First Kind *

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Received 10 February 2025

Abstract

In this work, we consider a regularized sinc-collocation method for solving Fredholm integral equations of the first kind, which is known to be an ill-posed problem. This numerical method is a combination of the Sinc-Collocation method with a Tikhonov regularization procedure. We show the convergence results of this new technique, and we give some numerical examples to demonstrate the validity and applicability of the proposed method.

1 Introduction

Many inverse problems in mathematical physics and applied mathematics have their most natural mathematical expression in terms of Fredholm integral equations of the first kind (see [4, 9, **?**]). That is, equations of the form

$$\int_a^b k(x,t)u(t)dt = f(x)\ x \in [a,b] \tag{1}$$

where $k(x,t) \in L^2([a,b]\times[a,b])$ is the kernel which is generally not-degenerate, $f(x) \in L^2[a,b]$ is the data and $u(t) \in L^2[a,b]$ is the solution to be determined.

One major difficulty in working with any Fredholm integral equation of the first kind is that the solution does not depend continuously on the given function $f(x)$. Furthermore this instability carries over to the solution of the algebraic system arising from discretization of the integral equation (1).

In mathematical literature, several numerical methods have been developed to solve this category of unstable problems. Among these methods, there are two most used varieties: projections methods which are based on expanding the solution in terms of some basis functions, and the second uses quadrature formulas. In both cases, it has been shown that the convergence rate of these methods is of polynomial order, i.e., of the form $O(1/N^p)$, $p \geq 1$ where N represents the finite dimension of the space on which the projection is made, or the number of points of the quadrature formula.

In [10] it is shown that if we use the Sinc method the convergence rate is of exponential order, i.e., of the form $O(\exp(-c\sqrt{N}))$ with some $c > 0$, although this convergence rate is much faster than that of polynomial order. Subsequently, this result was successfully utilized to numerically approach many problems involving PDEs and integral equations of second-kind. The obtained results showed the effectiveness of the Sinc Collocation method as a numerical approximation tool. Recently, in the work [6], the authors applied this method to a class of integral equations of first-kind, where they showed convergence results and numerically justified the validity of the method.

This work is a continuation in the same direction. In our investigation, we propose the study of the regularized Sinc-Collocation method, i.e., the Sinc-Collocation method combined with a Tikhonov regularization method. Our method consists of reducing the solution of Eq. (1) to a set of algebraic equations by expanding $u(t)$ as Sinc functions with unknown coefficients. The properties of the Sinc function are then utilized to evaluate the unknown coefficients (see [7, 11, 12]). In this study several numerical examples are given to confirm applicability and justify rapid convergence of our method.

*Mathematics Subject Classifications: 45B05, 65R30, 65L60.

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2 Sinc Function and Basic Definition

The goal of this section is to recall notations and definition of the Sinc function that are useful for this paper. These are discussed thoroughly in [5, 10, 11]. The Sinc function is defined on the whole real line by

$$sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{2}$$

2.1 Sinc Approximation on the Real Line

If u is defined on the real line, then for $h > 0$ the series

$$C(u,h)(x) = \sum_{j=-\infty}^{\infty} u(jh)S(j,h)(x) \tag{3}$$

is called the Whittaker cardinal expansion of $u(x)$, whenever this series converges. For purpose of programming, the approximation (3) is rewritten as

$$C_N(u,h)(x) = \sum_{j=-N}^{N} u(jh)S(j,h)(x)$$

where the basis function $S(j,h)(x)$ (the so-called Sinc function) is defined by

$$S(j,h)(x) = sinc\left(\frac{\pi}{h}(x-jh)\right) = \begin{cases} \frac{\sin\left(\frac{\pi}{h}(x-jh)\right)}{\frac{\pi}{h}(x-jh)}, & x \neq jh, \\ 1, & x = jh. \end{cases}$$

To state convergence theorems, we introduce notation and definitions.

Definition 1 *Let D_d denote the infinite strip region of width $2d$ $(d > 0)$ in the complex plane:*

$$D_d = \{z \in \mathbb{C} : |Im(z)| < d\}$$

for $0 < \epsilon < 1$, let $D_d(\epsilon)$ be defined by

$$D_d(\epsilon) = \left\{z \in \mathbb{C} : |Im(z)| < d(1-\epsilon),\ |Re(z)| < \frac{1}{\epsilon}\right\}.$$

Let $H^1(D_d)$ be the Hardy space over the region D_d, i.e., the set of functions u analytic in D_d such that

$$\lim_{\epsilon \to 0}\left(\int_{\partial D_d(\epsilon)} |f(z)||dz|\right) < \infty.$$

The following theorem, due to Stenger [10], presents the convergence result on the Sinc approximation.

Theorem 1 (Stenger [10]) *Assume, with positive constants α, β and d, that*

1. *$u \in H^1(D_d)$, and*
2. *u decays exponentially on the real line, that is*

$$|f(x)| \leq \alpha \exp(-\beta|x|), \quad \forall x \in \mathbb{R}.$$

Then we have

$$\sup_{-\infty < x < \infty} |u(x) - C_N(u,h)(x)| \leq CN^{\frac{1}{2}} \exp\left[-(\pi d \beta N)^{\frac{1}{2}}\right]$$

for some C and step size h is taken as

$$h = \left(\frac{\pi d}{\beta N}\right)^{\frac{1}{2}}.$$

2.2 Sinc Approximation on a General Interval $[a, b]$

Let $t = \phi(z)$ denote a conformal map which maps the simply connected domain D with boundary ∂D onto a strip region D_d such that

$$\phi([a,b]) = (-\infty, \infty), \quad \lim_{t\to a} \phi(t) = -\infty \quad \text{and} \quad \lim_{t\to b} \phi(t) = \infty.$$

Now, in order to have the Sinc approximation on a finite interval $[a, b]$ conformal map is employed as follow

$$\phi(x) = \ln\left(\frac{x-a}{b-x}\right).$$

This map carries the eye-shaped complex domain

$$\left\{z = x + iy : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d \leq \frac{\pi}{2}\right\}$$

onto the infinite strip

$$D_d = \left\{\mu = \alpha + \beta i : |\beta| < d < \frac{\pi}{2}\right\}.$$

The basis function on finite interval $[a, b]$ are given by

$$S(j,h) \circ \phi(x) = \frac{\sin\left(\frac{\pi}{h}(\phi(x) - jh)\right)}{\frac{\pi}{h}(\phi(x) - jh)},$$

also, Sinc function for interpolation points $x_k = kh$ is given by

$$S(j,h)(kh) = \delta^{(0)}_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

So, $S(j,h) \circ \phi(x)$ exhibits Kronecker delta behavior on the grid points

$$x_j = \phi^{-1}(jh) = \frac{a + be^{jh}}{1 + e^{jh}}$$

and interpolation and quadrature formulas for $u(x)$ over $[a, b]$ are

$$u(x) \approx u_N(x) = \sum_{j=-N}^{N} u(x_j) S(j,h) \circ \phi(x) \tag{4}$$

and

$$\int_a^b u(x)dx \approx h \sum_{j=-N}^{N} \frac{f(x_j)}{\phi'(x_j)}. \tag{5}$$

Theorem 2 (Stenger [10]) *Assume that, for a variable transformation $z = \phi^{-1}(\xi)$, the transformation function $u(\phi^{-1}(\xi))$ satisfies assumptions (1) and (2) in Theorem (1). with some α, β and d. Then we have*

$$\sup_{a<x<b} |u(x) - u_N(x)| \leq CN^{\frac{1}{2}} \exp[-(\pi d \beta N)^{\frac{1}{2}}]$$

for some C, where the step size h is taken as

$$h = \left(\frac{\pi d}{\beta N}\right)^{\frac{1}{2}}.$$

3 The Regularization Method

The regularization method was independently established by Tikhonov ([13]) and Phillips [8]. The regularization method consists of transforming first kind integral equations to second kind equations. The regularization method transforms the linear Fredholm integral equation of the first kind

$$\int_a^b k(x,t)u(t)dt = f(x), \tag{6}$$

to the approximation of the Fredholm integral equation

$$\alpha u_\alpha(x) + \int_a^b k(x,t)u_\alpha(t)dt = f(x), \tag{7}$$

where α is a small positive parameter called the regularization parameter.

In this study, we assume that the kernel $k(.,.)$ is a symmetric real function, $k(.,.) \in L^2([a,b]\times[a,b])$, and the associated integral operator ($\mathbf{K}u(x) = \int_a^b k(x,t)u(t)dt$) is positive, i.e.,

$$\begin{cases} k : [a,b]\times[a,b] \longrightarrow \mathbb{R}, \\ \int_a^b \int_a^b k(x,t)^2 dxdt < +\infty, \\ \forall h \in L^2([a,b]),\ \int_a^b \left(\int_a^b k(x,t)h(t)dt\right) h(x)dx \geq 0. \end{cases}$$

Furthermore, the operator equation (6) is uniquely solvable, i.e., $f \in R(\mathbf{K})$ and $\mathbf{K}$ is injective.

It is clear that (7) is an integral equation of the second kind. Moreover, it was proved by Tikhonov [13] and Phillips [8] (see also [15]) that the solution u_α of Eq. (7) converges to the solution u of (6) as $\alpha \to 0$. In other words, it has been shown that

$$\|u - u_\alpha\|_{\mathcal{H}} \longrightarrow 0, \text{ as } \alpha \longrightarrow 0. \tag{8}$$

It is important to note that the Fredholm integral equation of the first kind is ill-posed problem. The solution to an ill-posed problem may not exist, or if a solution exists, it is not unique and may not depend continuously on data f.

As stated before, we apply the regularization method to transform the first kind Fredholm integral equation to the second kind integral equation. The resulting second kind integral equation is then solved by the regularized Sinc-Collocation Method. The proposed method combines the Lavrentiev regularization procedure and spectral approximation based on Sinc functions.

4 Regularized Sinc-Collocation Method

In what follows, we will apply the regularization method combined with Sinc-Collocation methods to solve numerically the integral equation of the first kind denoted by

$$\int_a^b k(x,t)u(t)dt = f(x), \quad a \leq x \leq b. \tag{9}$$

Using the regularization method, Eq. (9) can be transformed to

$$\alpha u_\alpha(x) + \int_a^b k(x,t)u_\alpha(t)dt = f(x) \tag{10}$$

for solving the Eq. (10) with Sinc approximation, we need to chose a method to find unknown coefficients in this expansion. Collocation method is one of the projection methods that is used as follow:

By substituting Sinc approximation (4) expansion of unknown function $u_\alpha(x)$ in the above equation , we have

$$\alpha \sum_{j=-N}^{N} u_{\alpha,N}(\phi^{-1}(jh))S(j,h)\circ\phi(x) + \int_a^b k(x,t)\left(\sum_{j=-N}^{N} u_{\alpha,N}(\phi^{-1}(jh))S(j,h)\circ\phi(t)\right)dt = f(x)$$

which implies

$$\sum_{j=-N}^{N} u_{\alpha,N}(\phi^{-1}(jh))\left[\alpha S(j,h)\circ\phi(x) + \int_a^b k(x,t)S(j,h)\circ\phi(t)dt\right] = f(x).$$

Then we define residual function as follow

$$R_N(x) = \sum_{j=-N}^{N} u_{\alpha,N}(\phi^{-1}(jh))\left[\alpha S(j,h)\circ\phi(x) + \int_a^b k(x,t)S(j,h)\circ\phi(t)dt\right] - f(x).$$

So, to find the unknown coefficients $u_{\alpha,N}(\phi^{-1}(jh))$ in Sinc approximation expansion, there are some techniques such as projection methods like Galerkin and collocation. In this study, collocation method which has less computations than Galerkin is applied with some collocation points in interval $[a,b]$ for residual function as follows

$$R_N(x_i) = 0, \;\; i = -N, -N+1, ..., N-1, N$$

in this paper collocation points are

$$x_i = \phi^{-1}(ih) = \frac{a + be^{ih}}{1 + e^{ih}}, \quad i = -N, -N+1, ..., N-1, N. \tag{11}$$

So that

$$\sum_{j=-N}^{N} u_{\alpha,N}(\phi^{-1}(jh))\left[\alpha S(j,h)\circ\phi(x_i) + \int_a^b k(x_i,t)S(j,h)\circ\phi(t)dt\right] = f(x_i).$$

Then integral equation (10) is converted to system of linear algebraic equations

$$A_N X = b_N \tag{12}$$

where

$$A_N = \left[\alpha S(j,h)\circ\phi(x_i) + \int_a^b k(x_i,t)S(j,h)\circ\phi(t)dt\right]_{i=-N}^{N}, \quad j = -N, -N+1, ..., N-1, N,$$

$$X^T = [u_{\alpha,N}(\phi^{-1}(jh))]_{j=-N}^{N},$$

$$b_N = [f(x_i)], \quad i = -N, -N+1, ..., N-1, N.$$

Now, for evaluating matrix elements of algebraic equations we use the quadrature formula (5), then we have

$$\begin{aligned}\int_a^b k(x_i,t)S(k,h)\circ\phi(t)dt &\approx h\sum_{l=-N}^{N} \frac{k(x_i,t_l)S(j,h)\circ\phi(t_l)}{\phi'(t_l)} \\ &= h\sum_{l=-N}^{N} \frac{k(x_i,t_l)\delta_{jl}^{(0)}}{\phi'(t_l)} \\ &= h\frac{k(x_i,t_j)}{\phi'(t_j)}.\end{aligned}$$

We therefore find

$$A_N = \left[\alpha \delta_{ij}^{(0)} + h \frac{k(x_i, t_j)}{\phi'(t_j)} \right]_{i=-N}^{N}, \quad j = -N, -N+1, ..., N-1, N \tag{13}$$

where

$$t_j = \phi^{-1}(jh) = \frac{a + be^{jh}}{1 + e^{jh}}, \quad j = -N, -N+1, ..., N-1, N.$$

By solving the system (12) using the Matlab Linear Algebra Package, we obtain the unknown coefficients $u_{\alpha,N}(\phi^{-1}(jh))$.

Algorithm 1 *Regularized Sinc-Collocation Method.*

1. *Input:* a,b,$\alpha \in (0,1)$, $N \in \mathbb{N}^*$, $k(.,.)$ *and* $f(.)$*;*
2. *Calculates the Sinc-collocation points* $\{x_i\}_{i=-N}^{N}$ *by (11);*
3. *Calculates the coefficients of the matrix* A_N *by (13);*
4. *Calculates the coefficients of the vector* b_N*;*
5. *Compute the coefficients* $[u_{\alpha,N}(\phi^{-1}(jh))]_{j=-N}^{N}$ *by solving the system* $A_N X = b_N$*;*
6. *Determine approximate solution* $u_{\alpha,N}$ *by* $u_{\alpha,N}(x) = \sum_{j=-N}^{N} u_{\alpha,N}(\phi^{-1}(jh)) S(j,h) \circ \phi(x)$.

5 Numerical Examples

In this section, several numerical examples are given to approximate the solution of Fredholm integral equations of the first kind using the numerical method described in this paper. The numerical experiments are implemented in Matlab R2019a software.

- In the following numerical examples, the data are assumed to be exact and free of noise. The regularization parameter α is chosen arbitrarily to ensure good convergence and stability of the method. This choice is made empirically, based on the observed behavior of the numerical solution.
- We introduce the absolute error function by the notation $E_N(t)$ as follows:

$$E_N(t) = |u(t) - u_{\alpha,N}(t)|.$$

Example 1 *Consider the following Fredholm integral equation of the first kind:*

$$\int_0^1 e^{xt} u(t) dt = \frac{e^{1+x} - 1}{1 + x},$$

where the exact solution is $u(t) = e^t$*. The numerical results of example(1) are presented in Figures 1–3.*

Example 2 *As the second example, we consider the following integral equation:*

$$\int_0^{\pi} \cos(x - t) u(t) dt = \frac{\pi}{2} cos(x)$$

with exact solution $u(t) = \cos(t)$*. The numerical results of example(2) are presented in Figures 4–6.*

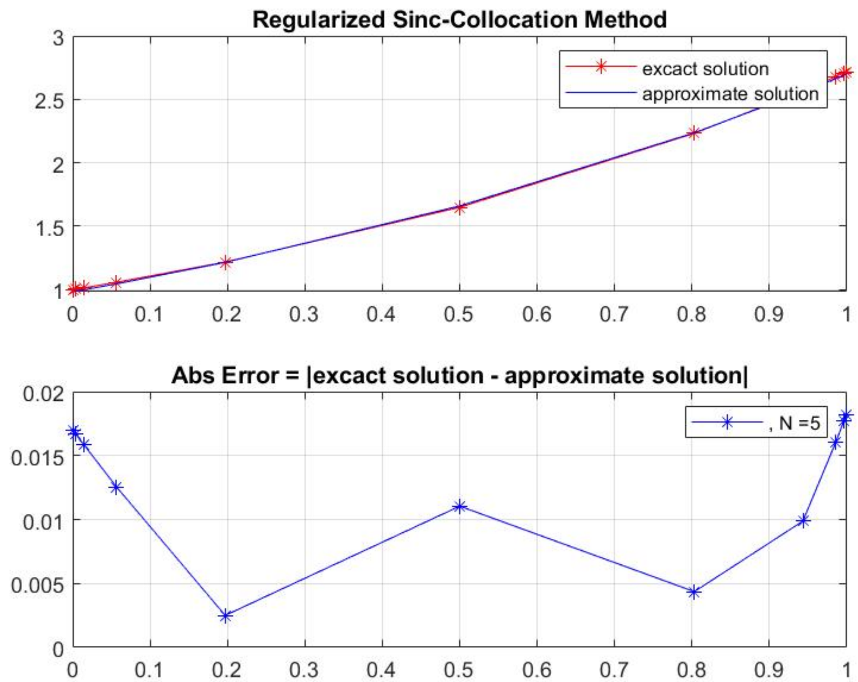

Figure 1: Exact and computed approximate solutions, absolute errors with $N = 5$.

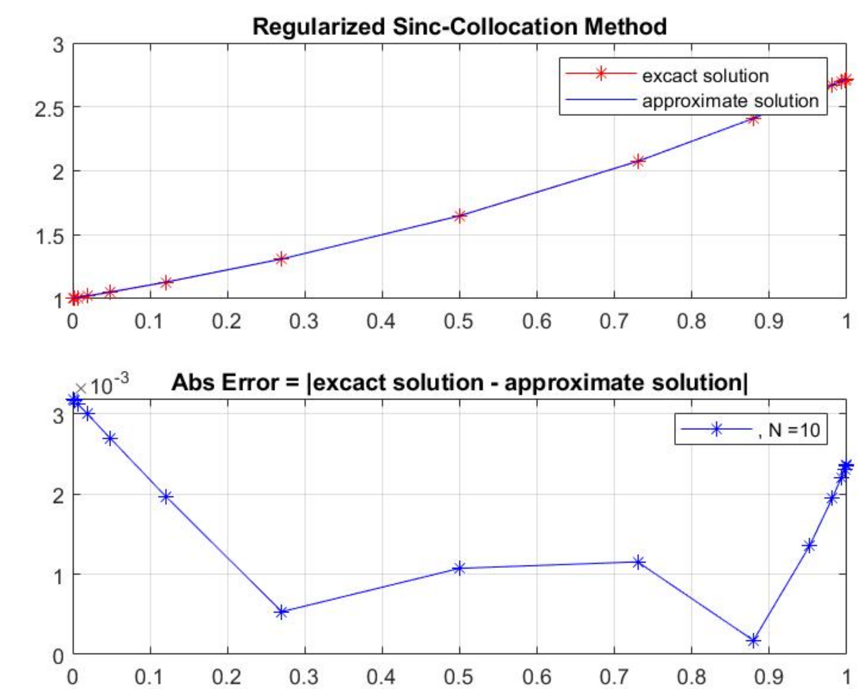

Figure 2: Exact and computed approximate solutions, absolute errors with $N = 10$.

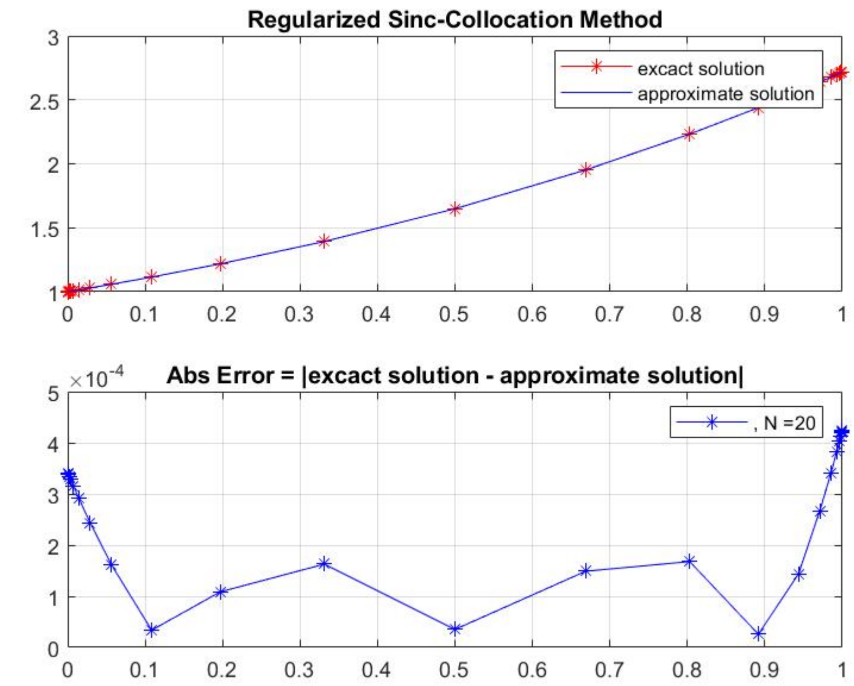

Figure 3: Exact and computed approximate solutions, absolute errors with $N = 20$.

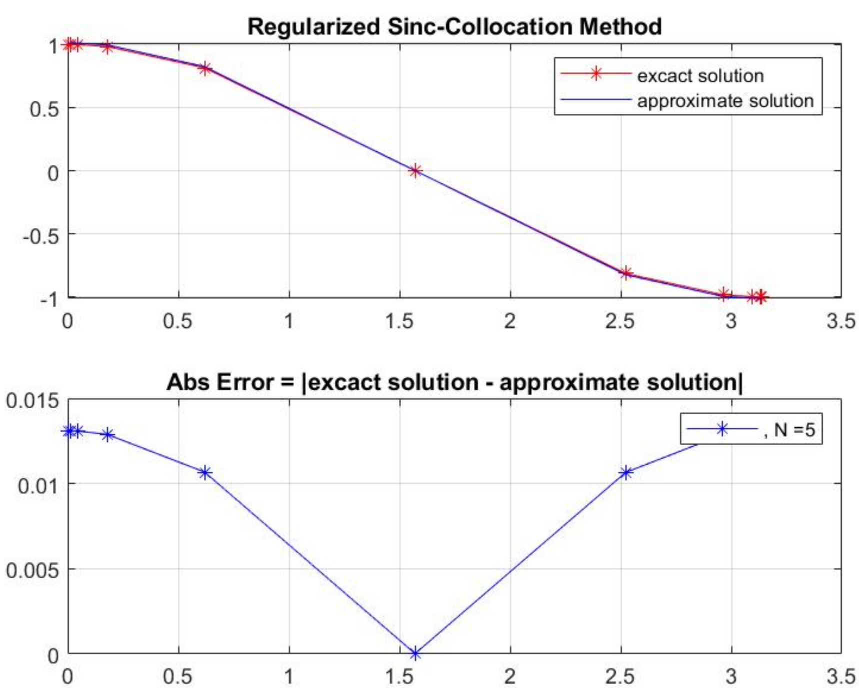

Figure 4: Exact and computed approximate solutions, absolute errors with $N = 5$.

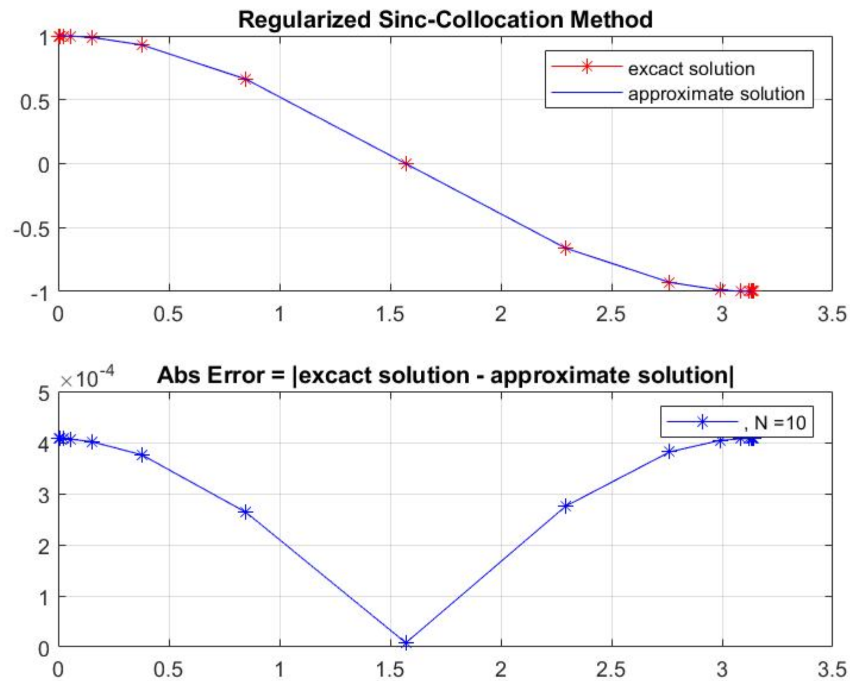

Figure 5: Exact and computed approximate solutions, absolute errors with $N = 10$.

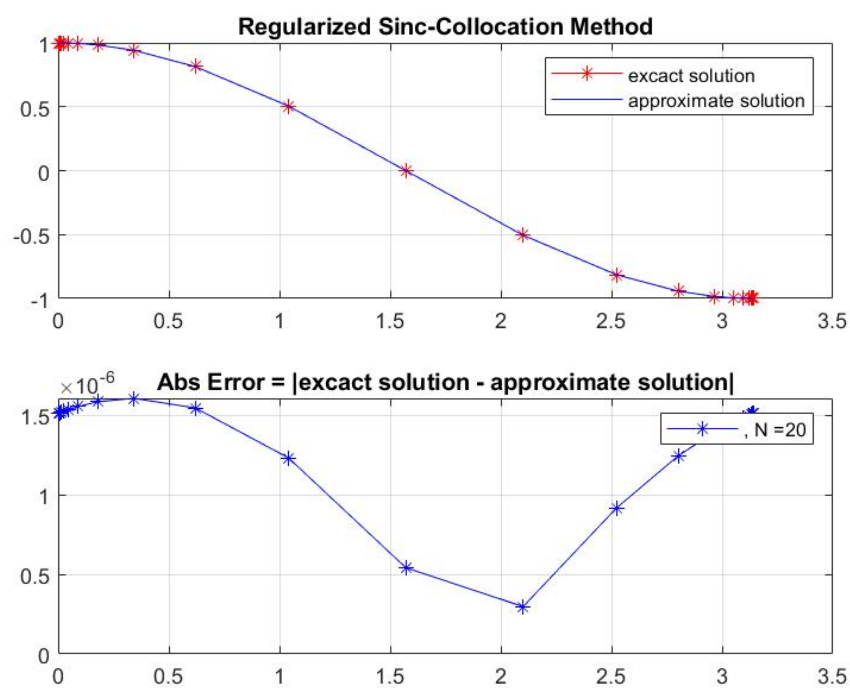

Figure 6: Exact and computed approximate solutions, absolute errors with $N = 20$.

6 Conclusion

In this work, we have employed an efficient method to solve Fredholm integral equations of the first kind. This method is a combination of Sinc collocation method and the Tikhonov regularization method. The numerical experiments have demonstrated the validity and the applicability of the suggested method. Our method is shown to be of good convergence, easy to program. So, we expect our method can be extended to the Volterra integral equation of first kind. This is left to our next paper.

References

[1] T. Bechouat, A collocation method for Fredholm integral equations of the first kind via iterative regularization scheme, Math. Model. Anal., 28(2023), 237–254.

[2] T. Bechouat and N. Boussetila, On the numerical solution of weakly singular integral equations of the first kind using a regularized projection method, Int. J. Comput. Math., 101(2024), 561–576.

[3] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, Spectral Methods: Fundamentals in Single Domains, Springer, Berlin, 2007.

[4] C. W. Groetsch and A. Neubauer, Regularization of ill-posed problems: optimal parameter choice in finite dimensions, J. Approx. Theory, 58(1989), 184–200.

[5] J. Lund and K. Bowers, Sinc Methods for Quadrature and Differential Equations, SIAM, Philadelphia, PA, 1992.

[6] K. Maleknejad, R. Mollapourasl and M. Alizadeh, Convergence analysis for numerical solution of Fredholm integral equation by Sinc approximation, Commun. Nonlinear Sci. Numer. Simul., 16(2011), 2478–2485.

[7] M. Muhammad, A. Nurmuhammad, M. Mori and M. Sugihara, Numerical solution of integral equations by means of the Sinc collocation method based on the double exponential transformation, J. Comput. Appl. Math., 177(2005), 269–286.

[8] D. L. Phillips, A technique for the numerical solution of certain integral equations of the first kind, J. Assoc. Comput. Mach., 9(1962), 84–97.

[9] P. C. Sabatier, Inverse Problems: An Interdisciplinary Study, Proceedings of the meeting held in Montpellier, December 1–5, 1986.

[10] F. Stenger, Handbook of Sinc Numerical Methods, Numerical Analysis and Scientific Computing. CRC Press, Boca Raton, FL, 2011.

[11] M. Sugihara and T. Matsuo, Recent developments of the Sinc numerical methods, J. Comput. Appl. Math., 164/165(2004), 673–689.

[12] M. Sugihara, Near optimality of the Sinc approximation, Math. Comp., 72(2003), 767–786.

[13] A. N. Tikhonov, On the solution of incorrectly posed problem and the method of regularization, Soviet Math., 4(1963), 1035–1038.

[14] A. N. Tikhonov, Regularization of incorrectly posed problems, Soviet Math. Dokl. 4(1963), 1624–1627.

[15] A. M. Wazwaz, The regularization method for Fredholm integral equations of the first kind, Comput. Math. Appl., 61(2011), 2981–2986.