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Location Of Zeros Of Complex Polynomials*

Mayanglambam Singhajit Singh[†], Barchand Chanam[‡]

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Abstract

In this paper, we discuss the necessary and sufficient conditions for a complex polynomial $p(z)$ to have all its zeros inside or outside an open disc under certain conditions involving the derivative of the reciprocal polynomial of $p(z)$ and the reciprocal of the derivative of $p(z)$. Among others, as some consequences of one of our results, we also obtain some interesting generalizations of the well-known classical Theorem of Laguerre.

1 Introduction

There have been significant applications of zero bounds in scientific disciplines such as stability theory, mathematical biology, communication theory and computer engineering. So, it has become interesting to identify problems concerning the suitable regions containing the zeros of polynomials. The first to obtain a classical solution to such a kind of problem was Cauchy [1], and several related results followed in the literature. One such problem is to study the circumstances under which a polynomial has some, all or none of its zeros in a disc. For detailed information, we refer to the monographs by Marden [7] and Milovanović et al. [8]. With a greater emphasis on determining the number of zeros in a given domain, the location of zeros has been of great interest for numerous authors. But we could not come across any exclusive work on a necessary and sufficient condition for a polynomial to have all its zeros in the unit disc except the well-known Schur-Cohn algorithm [2, 10] which is often used to determine whether a given polynomial has no zero in the closed unit disc. The Schur-Cohn algorithm with some other hypotheses can also be applied to verify the number of zeros in a circular region. Most results in this direction have included coefficients as the main parameters. The Eneström-Kakeya Theorem [3] and its generalizations like the one given by Govil and Rahman [4] are classic and significant examples of this kind. In this paper, we establish some elegant results concerning the conditions for a polynomial to have none or all of its zeros in an open disc in terms of a simple inequality involving the derivative of the reciprocal of the polynomial and the reciprocal of the derivative of the polynomial.

2 Main Results

We first present the following result, which discusses the necessary condition for a polynomial $p(z)$ of degree n to have all its zeros in $|z| < k$, $k \leq 1$.

Theorem 1 *If $p(z) = a_0 + a_1 z + ... + a_n z^n$ is a polynomial of degree n having all its zeros in $|z| < k$, $k \leq 1$, then for each z on $|z| = 1$ for which $p(z) \neq 0$,*

$$\left|\frac{T(z)}{S(z)}\right| < \frac{k+1}{2},$$

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[†]Department of Mathematics, National Institute of Technology Manipur, Imphal 795004, India

[‡]Department of Mathematics, National Institute of Technology Manipur, Imphal 795004, India

where

$$T(z) = nz^{n-1}p\left(\frac{1}{z}\right) - \frac{k+1}{2}z^{n-2}p'\left(\frac{1}{z}\right) + k\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)z^{n-1}p\left(\frac{1}{z}\right) \text{ and } S(z) = z^{n-1}p'\left(\frac{1}{z}\right).$$

Setting $k = 1$, Theorem 1 reduces to the following interesting result recently proved by Kumar and Dhankhar [5].

Corollary 1 *Let $p(z) = a_0 + a_1z + ... + a_nz^n$ be a polynomial of degree n. If $p(z)$ has all its zeros in $|z| < 1$, then on $|z| = 1$,*

$$|T_0(z)| < |S(z)|,$$

where

$$T_0(z) = \left\{z^np\left(\frac{1}{z}\right)\right\}' + \left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)z^{n-1}p\left(\frac{1}{z}\right) \text{ and } S(z) = z^{n-1}p'\left(\frac{1}{z}\right).$$

If $p(z)$ is a polynomial of degree n having no zero in $|z| \le k$, $k \ge 1$, then $q(z) = z^np\left(\frac{1}{z}\right)$ has all its zeros in $|z| < \frac{1}{k}$, $\frac{1}{k} \le 1$. Therefore applying Theorem 1 to $q(z)$, we will also obtain a necessary condition for a polynomial to have no zero in the closed disc, which is given below.

Corollary 2 *If $p(z)$ is a polynomial of degree n having no zero in $|z| \le k$, $k \ge 1$, then on $|z| = 1$,*

$$\left|\frac{T^*(z)}{S^*(z)}\right| < \frac{k+1}{2k},$$

where

$$T^*(z) = nz^{n-1}q\left(\frac{1}{z}\right) - \frac{k+1}{2k}z^{n-2}q'\left(\frac{1}{z}\right) + \frac{1}{k}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|}\right)z^{n-1}q\left(\frac{1}{z}\right),$$

$$S^*(z) = z^{n-1}q'\left(\frac{1}{z}\right) \text{ and } q(z) = z^np\left(\frac{1}{z}\right).$$

Further we can also establish the sufficiency analogue of Theorem 1 as given below.

Theorem 2 *Let $p(z)$ be a polynomial of degree n. If $U(z)$ and $S(z)$ satisfy*

$$\left|\frac{U(z)}{S(z)}\right| < \frac{k+1}{2k^2}$$

on $|z| \le \frac{1}{k}$, $\frac{1}{k} \ge 1$, where $U(z) = \frac{k+1}{2k}\left\{z^np\left(\frac{1}{z}\right)\right\}'$ and $S(z) = z^{n-1}p'\left(\frac{1}{z}\right)$, then $p(\frac{z}{k^2})$ has all its zeros in $|z| < k$, $k \le 1$.

Setting $k = 1$, Theorem 2 reduces to the following result established by Kumar and Dhankhar [5].

Corollary 3 *Let $p(z) = a_0 + a_1z + ... + a_nz^n$ be a polynomial of degree n. If $U_0(z)$ and $S(z)$ satisfy*

$$|U_0(z)| < |S(z)|$$

on $|z| \le 1$, where $U_0(z) = \left\{z^np\left(\frac{1}{z}\right)\right\}'$ and $S(z) = z^{n-1}p'\left(\frac{1}{z}\right)$, then $p(z)$ has all its zeros in $|z| < 1$.

The well-known Theorem of Laguerre [6] states that if $p(z)$ is a polynomial of degree n having no zero in the disc $|z| < 1$, then the polynomial

$$D_\beta p(z) = np(z) + (\beta - z)p'(z)$$

has no zero in $|z| < 1$ for any complex number β with $|\beta| < 1$. There arises a question: what happens if any positive real number c replaces n, which in particular is the degree of $p(z)$ in the expression for $D_\beta p(z)$? As a result, in view of the above question, Kumar and Dhankhar [5] established the following theorem.

Theorem 3 *Let $p(z)=\prod_{j=1}^{n}(z-z_j)$ be a polynomial of degree n having no zero in the disc $|z|<1$. Then*

$$cp(z)+(\alpha-z)\sum_{j=1}^{n}\alpha_j p_j(z) \tag{1}$$

has no zero in the disc $|z|<1$ for all α with $|\alpha|<1$ and $c\geq 1$ where $p_j(z)=\prod_{i=1,i\neq j}^{n}(z-z_j)$, $1\leq j\leq n$ and $\alpha_j>0$ with $\sum_{j=1}^{n}\alpha_j=1$.

Naturally, there may arise a question "What would be the analogous result of Theorem 3 for any open disc?", and to give an answer to this question, we establish the following interesting extension and generalization of Theorem 3 by considering the more general class of polynomials having no zero in open discs. More precisely, we prove the following theorem.

Theorem 4 *Let $p(z)=\prod_{j=1}^{n}(z-z_j)$ be a polynomial of degree n having no zero in the disc $|z|<k$, $k>0$. Then*

$$cp(z)+(\alpha-z)k\sum_{j=1}^{n}\alpha_j p_j(z) \tag{2}$$

has no zero in the disc $|z|<k$ for any complex number α with $|\alpha|<k$ and $c\geq k$ where $p_j(z)=\prod_{i=1,i\neq j}^{n}(z-z_i)$, $1\leq j\leq n$ and $\alpha_j>0$ with $\sum_{j=1}^{n}\alpha_j=1$.

Putting $k=1$, (2) yields (1) and on setting $\alpha_j=\frac{1}{n}$, $1\leq j\leq n$, Theorem 4 reduces to the following interesting result that extends and generalizes the Theorem of Laguerre for any open disc.

Corollary 4 *Let $p(z)=\prod_{j=1}^{n}(z-z_j)$ be a polynomial of degree n having no zero in the disc $|z|<k$, $k>0$. Then*

$$cp(z)+(\alpha-z)kp'(z)$$

has no zero in the disc $|z|<k$ for any complex number α with $|\alpha|<k$ and $c\geq nk$.

It may be remarked here that for $k=1$, $c=n$ and $|\alpha|<1$, Corollary 4 reduces to the well-known Theorem of Laguerre.

3 Lemmas

We need the following lemmas to prove the theorems. The first lemma is due to Rather et al. [9].

Lemma 1 *If $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then for each point z on $|z|=1$ for which $p(z)\neq 0$,*

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right)\geq\frac{n}{1+k}\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}.$$

Lemma 2 *Let $p(z)=\prod_{j=1}^{n}(z-z_j)$ be a polynomial of degree n having no zero in $|z|<k$, $k>0$. Then the polynomial*

$$cp(z)+(\gamma-k)z\sum_{j=1}^{n}\alpha_j p_j(z) \tag{3}$$

has no zero in $|z|<k$, $k>0$ for any complex number γ with $|\gamma|\leq k$ where $p_j(z)=\prod_{i=1,i\neq j}^{n}(z-z_i)$, $c\geq k$ and $\alpha_j>0$, $1\leq j\leq n$, with $\sum_{j=1}^{n}\alpha_j=1$.

Proof. If $\gamma = k$, then the result follows directly. So we assume that $\gamma \neq k$ and take $w_j = \frac{1}{z_j}$, $1 \leq j \leq n$. Since $c \geq k$, we can write $c = k + a$, for some $a \geq 0$. For each point z in $|z| < k$, $k > 0$ and using the fact that $\sum_{j=1}^{n} \alpha_j = 1$, we have

$$\begin{aligned}
\frac{z\sum_{j=1}^{n}\alpha_j p_j(z)}{p(z)} - \frac{c}{k-\gamma} &= \sum_{j=1}^{n}\alpha_j\left(\frac{z}{z-z_j}\right) - \frac{k+a}{k-\gamma} \\
&= \sum_{j=1}^{n}\alpha_j\left(\frac{zw_j}{zw_j-1}\right) - \frac{k+a}{k-\gamma} \\
&= \frac{1}{2}\sum_{j=1}^{n}\alpha_j\left(1 - \frac{1+zw_j}{1-zw_j}\right) - \frac{k+a}{k-\gamma} \\
&= -\frac{1}{2}\sum_{j=1}^{n}\alpha_j\left(\frac{1+zw_j}{1-zw_j}\right) + \left(\frac{1}{2} - \frac{k}{k-\gamma}\right) - \frac{a}{k-\gamma} \\
&= -\frac{1}{2}\sum_{j=1}^{n}\alpha_j\left(\frac{1+zw_j}{1-zw_j}\right) - \frac{1}{2}\left(\frac{k+\gamma}{k-\gamma}\right) - \left(\frac{a}{k-\gamma}\right),
\end{aligned}$$

which is equivalent to

$$\mathrm{Re}\left(\frac{z\sum_{j=1}^{n}\alpha_j p_j(z)}{p(z)} - \frac{c}{k-\gamma}\right) = -\frac{1}{2}\sum_{j=1}^{n}\mathrm{Re}\ \alpha_j\left(\frac{1+zw_j}{1-zw_j}\right) - \frac{1}{2}\mathrm{Re}\left(\frac{k+\gamma}{k-\gamma}\right) - \mathrm{Re}\left(\frac{a}{k-\gamma}\right).$$

It is easy to verify that for any complex number γ with $|\gamma| < k$, $k > 0$, $\mathrm{Re}\left(\frac{k+\gamma}{k-\gamma}\right) > 0$, and in the same way, since $|zw_j| < 1$, and each $\alpha_j > 0$ for $1 \leq j \leq n$, we have $\mathrm{Re}\,\alpha_j\left(\frac{1+zw_j}{1-zw_j}\right) > 0$. Further since $a \geq 0$ and $|\gamma| < k$, we have $\mathrm{Re}\left(\frac{a}{k-\gamma}\right) \geq 0$. Thus we have shown that

$$\mathrm{Re}\left(\frac{z\sum_{j=1}^{n}\alpha_j p_j(z)}{p(z)} - \frac{c}{k-\gamma}\right) < 0,$$

which establishes the conclusion of the lemma. ■

Setting $\alpha_j = \frac{1}{n}$, $1 \leq j \leq n$ in (3), as an immediate consequence, we obtain the following analogous result of Lemma 2 for the derivative of a complex polynomial.

Corollary 5 *Let $p(z) = \prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n having no zero in $|z| < k$, $k > 0$. Then the polynomial*

$$cp(z) + (\gamma - k)zp'(z)$$

has no zero in $|z| < k$, $k > 0$ for any complex number γ with $|\gamma| \leq k$, $c \geq nk$.

4 Proof of Theorems

Proof of Theorem 1. Since $p(z)$ has all its zeros in $|z| < k$, $k \leq 1$, by Gauss-Lucas theorem, $p'(z)$ has all its zeros in $|z| < k$, $k \leq 1$. This implies that $S(z) = z^{n-1}p'\left(\frac{1}{z}\right)$ has no zero in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$. Note that

$$\begin{aligned}
T(z) &= nz^{n-1}p\left(\frac{1}{z}\right) - \frac{k+1}{2}z^{n-2}p'\left(\frac{1}{z}\right) + k\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)z^{n-1}p\left(\frac{1}{z}\right) \\
&= n\left\{1 + \frac{k}{n}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right\}z^{n-1}p\left(\frac{1}{z}\right) - \frac{k+1}{2}z^{n-2}p'\left(\frac{1}{z}\right).
\end{aligned}$$

On $|z|=1$,

$$\begin{aligned}\frac{zT(z)}{S(z)} &= \frac{n\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}z^n p\left(\frac{1}{z}\right)-\frac{k+1}{2}z^{n-1}p'\left(\frac{1}{z}\right)}{z^{n-1}p'\left(\frac{1}{z}\right)}\\ &= \frac{n\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}zp\left(\frac{1}{z}\right)}{p'\left(\frac{1}{z}\right)}-\frac{k+1}{2}.\end{aligned}$$

Therefore,

$$\left|\frac{T(z)}{S(z)}\right|=\left|\frac{n\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}p\left(\bar{z}\right)}{\bar{z}p'\left(\bar{z}\right)}-\frac{k+1}{2}\right| \tag{4}$$

on $|z|=1$. Since $p(z)$ has all its zeros in $|z|<k$, $k\le 1$, from Lemma 1, we have

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right)>\frac{n}{1+k}\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}$$

on $|z|=1$.

Equivalently

$$\operatorname{Re}\left(\frac{zp'(z)}{n\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}p(z)}\right)>\frac{1}{1+k} \tag{5}$$

on $|z|=1$. It is easy to verify that if $\operatorname{Re}(z)>\frac{1}{1+k}$, then $\left|\frac{1}{z}-\frac{k+1}{2}\right|<\frac{k+1}{2}$. Using this property in (5), we obtain

$$\left|\frac{n\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}p(z)}{zp'(z)}-\frac{k+1}{2}\right|<\frac{k+1}{2}.$$

Replacing z by $\bar{z}$ in the above inequality, we get

$$\left|\frac{n\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}p(\bar{z})}{\bar{z}p'(\bar{z})}-\frac{k+1}{2}\right|<\frac{k+1}{2} \tag{6}$$

on $|z|=1$. Then by using equation (4) in inequality (6), we have

$$\left|\frac{T(z)}{S(z)}\right|<\frac{k+1}{2}$$

on $|z|=1$. This completes the proof of Theorem 1. ■

Proof of Theorem 2. Since $U(z)$ and $S(z)$ satisfy

$$\left|\frac{U(z)}{S(z)}\right|<\frac{k+1}{2k^2} \tag{7}$$

on $|z|\le\frac{1}{k}$, $\frac{1}{k}\ge 1$, $S(z)\ne 0$ in $|z|\le\frac{1}{k}$, $\frac{1}{k}\ge 1$. Then $z^{n-1}S\left(\frac{1}{z}\right)$ has all its zeros in $|z|<k$, $k\le 1$. Therefore, $\left(\frac{z}{k^2}\right)^{n-1}S\left(\frac{k^2}{z}\right)$ has all its zeros in $|z|<k^3$, $k^3\le 1$.

Now for any complex number z on $|z|=k$, the complex number $\frac{k^2}{z}$, a point in $|z|\le\frac{1}{k}$, $\frac{1}{k}\ge 1$, also lies on $|z|=k$, $k\le 1\le\frac{1}{k}$. So, it gives by (7) that

$$\left|U\left(\frac{k^2}{z}\right)\right|<\frac{k+1}{2k^2}\left|S\left(\frac{k^2}{z}\right)\right|$$

holds for any z on $|z| = k$, which further gives

$$\left| \left(\frac{z}{k^2}\right)^{n-1} U\left(\frac{k^2}{z}\right) \right| < \frac{k+1}{2k} \left| \left(\frac{z}{k^2}\right)^{n} S\left(\frac{k^2}{z}\right) \right|$$

on $|z| = k$.

Also the polynomial $\left(\frac{z}{k^2}\right)^n S\left(\frac{k^2}{z}\right)$ has all its zeros in $|z| < k^3 \leq k \leq 1$. Then by Rouche's theorem, the polynomial

$$\left(\frac{z}{k^2}\right)^{n-1} U\left(\frac{k^2}{z}\right) + \frac{k+1}{2k}\left(\frac{z}{k^2}\right)^{n} S\left(\frac{k^2}{z}\right) = \frac{n(k+1)}{2k} p\left(\frac{z}{k^2}\right)$$

has all its zeros in $|z| < k$, $k \leq 1$. This completes the proof of Theorem 2. ■

Proof of Theorem 4. Since $p(z)$ has no zero in $|z| < k$, $k > 0$, by Lemma 2,

$$cp(z) + (\gamma - k)z \sum_{j=1}^{n} \alpha_j p_j(z) \tag{8}$$

has no zero in $|z| < k$, $k > 0$ for any complex number γ with $|\gamma| \leq k$. Then, from (8), we have

$$\gamma z \sum_{j=1}^{n} \alpha_j p_j(z) \neq kz \sum_{j=1}^{n} \alpha_j p_j(z) - cp(z)$$

for $|\gamma| \leq k$. For any fixed z, we choose the argument of γ to get

$$\left| \gamma z \sum_{j=1}^{n} \alpha_j p_j(z) \right| \neq \left| kz \sum_{j=1}^{n} \alpha_j p_j(z) - cp(z) \right|,$$

which implies

$$\left| \gamma z \sum_{j=1}^{n} \alpha_j p_j(z) \right| < \left| kz \sum_{j=1}^{n} \alpha_j p_j(z) - cp(z) \right| \tag{9}$$

for $|z| < k$. Otherwise, a contradiction occurs for small values of γ.

Taking $|\gamma| = k$ and $|z| \to k$, we have

$$k^2 \left| \sum_{j=1}^{n} \alpha_j p_j(z) \right| \leq \left| kz \sum_{j=1}^{n} \alpha_j p_j(z) - cp(z) \right| \tag{10}$$

for $|z| = k$. Then, for any α with $|\alpha| < k$, inequality (10) becomes

$$k|\alpha| \left| \sum_{j=1}^{n} \alpha_j p_j(z) \right| < \left| kz \sum_{j=1}^{n} \alpha_j p_j(z) - cp(z) \right|$$

on $|z| = k$. Now by Rouche's theorem, the polynomial

$$k\alpha \sum_{j=1}^{n} \alpha_j p_j(z) - \left\{ kz \sum_{j=1}^{n} \alpha_j p_j(z) - cp(z) \right\}$$

has the same number of zeros as $kz \sum_{j=1}^{n} \alpha_j p_j(z) - cp(z)$ has in $|z| < k$, $k > 0$. But due to (9), we have

$$kz \sum_{j=1}^{n} \alpha_j p_j(z) - cp(z) \neq 0$$

in $|z| < k$, $k > 0$. Therefore, the polynomial

$$cp(z) + (\alpha - z)k \sum_{j=1}^{n} \alpha_j p_j(z)$$

has no zero for each point z in $|z| < k$, $k > 0$ for any α with $|\alpha| < k$. This completes the proof of Theorem 4. ■

5 Conclusion

Mathematicians are always curious if necessary conditions are sufficient for some results. In this paper, we deal with some interesting results concerning the analogues of the necessary and sufficient conditions for the existence of the zeros of some classes of complex polynomials having all their zeros inside or outside an open disc. One of the results obtained generalizes the well-known theorem due to Laguerre and the techniques used could implicate further potential work in similar or related results.

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References

[1] A. L. Cauchy, Exercises de Mathématique, Année de Bure Fréres, Paris, 1829.

[2] A. Cohn, Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise, Math. Zeitschr, 14(1922), 110–138.

[3] G. Eneström, Härledning af an allmän formel för antalet pensionärer, som vid en godtyeklig tidpunkt förefinnas inom en sluten pensionslcassa, Öfversigt af Vetenskaps-Akademiens Förhandlinger, 50(1893), 405–415.

[4] N. K. Govil and Q. I. Rahman, On the Eneström-Kakeya theorem, Tohoku Math. J., 20(1968), 126–136.

[5] P. Kumar and R. Dhankhar, On the location of zeros of polynomials, Complex Anal. Oper. Theory, 16(2022).

[6] E. Laguerre, Nouvelles annales de mathématiques, Oeuvres Série, 2(1878), 97–101.

[7] M. Marden, Geometry of Polynomials, Amer. Math. Soc., Providence, 3(1966).

[8] G. V. Milovanović, D. S. Mitrinović and D. S. Rassias, Topics in Polynomials: Extremal Properties, Inequalities, Zeros, World Scientific Publishing Co., Singapore, 1994.

[9] N. A. Rather, I. Dar and A. Iqbal, Some inequalities for polynomials with restricted zeros, Ann. Univ. Ferrara., 67(2021), 183–189.

[10] I. Schur, Über Polynome, die nur in Innern des Einheitkreis verschwinden, Reine. Angew. Math., 148(1918), 122–145.