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Coupon Coloring Of Some Classes Of Graphs*

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Abstract

Let G be a graph with no isolated vertices. A k-coupon coloring of G is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G, and is denoted by $\chi_c(G)$. In this paper, we investigate the coupon coloring of graphs generated by applying various unary operations on various graph classes.

1 Introduction

All graphs we considered are finite, simple and undirected. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The neighbourhood of the vertex $x \in V(G)$ is the set of all vertices that are adjacent to x. The degree of a vertex x in a graph is the number of vertices adjacent to x. The minimum degree of a graph is denoted as $\delta(G)$. A pendant vertex is a vertex with degree 1 and a universal vertex is a vertex that is adjacent to all other vertices of the graph. For standard graph terminology, we in general follow [1, 13]. Graph coloring is one of the important and fertile areas in the field of Graph Theory. Apart from various coloring problems like list coloring, star coloring, acyclic coloring, Chen et al. [2] introduced coupon coloring in the year 2015. Let G be a graph with no isolated vertices. A k-coupon coloring of G is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G, and is denoted by $\chi_c(G)$ [2]. Clearly, $\chi_c(G) \leq \delta(G)$ for any graph G. An example of coupon coloring is shown in Figure 1. In a k-coloring c, a vertex v is considered to be a bad vertex if its neighborhood lacks vertices of every color from $[k]$ and it is clear that there are no bad vertices in a coupon coloring. A vertex v is said to have a property S if there are two different colors in the neighborhood of v [10]. Let G be a graph with no isolated vertices. In [15], Cockayne et al. introduced the concept of the total

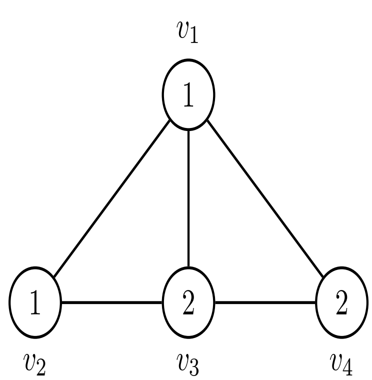

Figure 1: An example of coupon coloring.

domatic number of graphs. A subset S of the vertex set $V(G)$ of a graph G is a total dominating set if every

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vertex of G is adjacent to at least one vertex from S. The maximum number of disjoint total dominating sets is called the total domatic number. In coupon coloring, every color class must be a total dominating set in the graph. Thus, the coupon coloring number is also referred to as the total domatic number [8]. The coupon coloring numbers of wheels, cycles, unicyclic and bicyclic graphs, complete graphs, and complete k-partite graphs were determined by Shi et al. [10]. Additionally, coupon coloring has been examined in [12, 3] and [8]. The coupon coloring number of a few binary products, including the lexicographic and Cartesian products is studied in [6, 11, 9]. Also, in [16], the authors have demonstrated that for every $k \geq 3$, it is NP-complete to decide whether $d_t(G) \geq k$, where G is a split graph. Considering the literature survey on coupon coloring, is it possible to find the coupon coloring number for other graph classes, as well as for unary operations in graphs? This motivated us to explore the coupon coloring of graphs generated by applying various unary operations on them.

In Section 2, we explore the coupon coloring number for friendship graphs and split graphs, and in Section 3, we examine the impact of coupon coloring after unary operations on graphs.

2 Coupon Coloring Number of Split Graphs

The following observation and lemma will be helpful to prove our main results.

Observation 1 *Let G be a graph having a pendant vertex. Then $\chi_c(G) = 1$.*

Lemma 1 ([10]) *Let G be a cycle on n vertices. Then*

$$\chi_c(G) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 1 *A split graph is one whose vertex set can be partitioned as the disjoint union of an independent set and a clique [7].*

Definition 2 *A complete split graph $CS(n, \alpha)$, is a graph on n vertices consisting of a clique on $n - \alpha$ vertices and an independent set on the remaining $\alpha (1 \leq \alpha \leq n-1)$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set [5].*

Theorem 1 *Let $G = CS(m+n, n)$ be a complete split graph having m vertices in the clique and n vertices in the independent set. Then*

$$\chi_c(G) = \begin{cases} \lfloor \frac{m+n}{2} \rfloor & m \geq n, \\ m & m < n. \end{cases}$$

Proof. Let $G = CS(m+n, n)$ be a complete split graph having m vertices in the clique and n vertices in the independent set. Let $\{v_1, v_2, \ldots, v_m\}$ be the vertices in the clique and $\{v_{m+1}, v_{m+2}, \ldots, v_{m+n}\}$ be the vertices in the independent part.

Case 1: $m \geq n$. Since each color should appear atleast twice it follows that $\chi_c(G) \leq \lfloor \frac{m+n}{2} \rfloor$. Now we prove $\chi_c(G) \geq \lfloor \frac{m+n}{2} \rfloor$ by providing a $\lfloor \frac{m+n}{2} \rfloor$-coupon coloring. Define $c : V(G) \to [\lfloor \frac{m+n}{2} \rfloor]$ by

$$c(v_i) = i \quad \text{for } i = 1, 2, \ldots, \lfloor \frac{m+n}{2} \rfloor$$

and

$$c(v_{\lfloor \frac{m+n}{2} \rfloor + i}) = i \quad \text{for } i = 1, 2, \ldots, \lfloor \frac{m+n}{2} \rfloor.$$

The remaining vertices can be colored using any colors from $\{1, 2, \ldots, \lfloor \frac{m+n}{2} \rfloor\}$. Clearly c is a coupon coloring of G. Therefore $\chi_c(G) = \lfloor \frac{m+n}{2} \rfloor$.

Case 2: $m < n$. Since $\delta(G) = m$, there does not exist an $m+1$-coupon coloring. Let $c : V(G) \to [m]$ such that $c(v_i) = i$ for $i = 1, 2, \ldots, m$. Remaining there are n vertices in the graph uncolored. Assign colors in such a way that $c(v_{m+i}) = i$ for all $i = 1, 2, \ldots, m$ and the remaining vertices can be colored using any colors from $\{1, 2, \ldots, m\}$. Since $m < n$, such a coloring is possible. Thus m-coupon coloring exists. Therefore $\chi_c(G) = m$. ■

Theorem 2 *Let G be a split graph having m vertices in the clique and n vertices in the independent set such that $diam(G)$ is 3. Then $\chi_c(G) \leq \lfloor \frac{m}{2} \rfloor$.*

Proof. Let G be a split graph having m vertices in the clique and n vertices in the independent set such that $diam(G)$ is 3. Suppose $\{v_1, v_2, \ldots, v_m\}$ are the vertices of clique and $\{v_{m+1}, v_{m+2}, \ldots, v_{m+n}\}$ are the vertices of independent set. We prove that $\delta(G) \leq \lfloor \frac{m}{2} \rfloor$. Suppose $\delta(G) = k > \lfloor \frac{m}{2} \rfloor$. We know that $deg(v_{m+l}) < deg(v_h)$ for all $l = 1, 2, \ldots, n$ and $h = 1, 2, \ldots, m$. So $\delta(G) = deg(v_{m+l})$ for some l. Let v_{m+i} be an arbitrary vertex such that $deg(v_{m+i}) = k$. Thus v_{m+i} is adjacent to k vertices in clique. Remaining there are $m - k$ vertices in clique, not adjacent to v_{m+i}. Also $m - k \leq \lfloor \frac{m}{2} \rfloor$. Consider another arbitrary vertex v_{m+j}. Suppose there exist an edge joining v_{m+j} and any of k vertices that is adjacent to v_{m+i}. Then $diam(G) = 2$. Hence there is no edge joining v_{m+j} and neighbours of v_{m+i}. Now $deg(v_{m+j}) \leq m - k$. Thus $deg(v_{m+j}) < k$. From this we get $deg(v_{m+j}) < deg(v_{m+i})$, a contradiction since $deg(v_{m+i}) = \delta(G)$. Thus $\delta(G) \leq \lfloor \frac{m}{2} \rfloor$. We know that $\chi_c(G) \leq \delta(G)$, thus $\chi_c(G) \leq \lfloor \frac{m}{2} \rfloor$. ■

Definition 3 *Friendship graph F_n for $n \geq 2$ is a graph constructed by joining n copies of the cycle graph C_3 with a common vertex [14].*

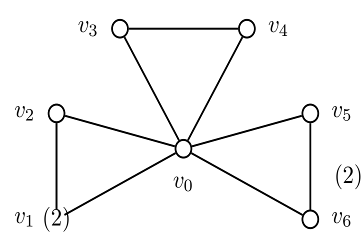

Figure 2: An example for friendship graph.

Theorem 3 *Let $G = F_n$ be a friendship graph with $2n + 1$ vertices and $3n$ edges. Then $\chi_c(G) = 1$.*

Proof. Let G be the friendship graph with $2n + 1$ vertices and $3n$ edges. We know that $\chi_c(G) \leq \delta(G)$. Therefore $\chi_c(G) \leq 2$. Let $\{v_0, v_1, \ldots, v_{2n}\}$ be the vertices of G and let v_0 be the universal vertex. Now the remaining vertices along with v_0 form n C_3 cycles such as $v_0v_1v_2v_0$, $v_0v_3v_4v_0$, $v_0v_5v_6v_0$, $v_0v_7v_8v_0, \ldots$, $v_0v_{2n-1}v_{2n}v_0$. Assume that there exists a 2-coupon coloring. Now consider the cycle $v_0v_1v_2$. Assign colors to v_0 and v_2 such that the neighbors of v_1 will be assigned with 2 colors. Let $c(v_0) = 1$ and $c(v_2) = 2$. Now consider the vertex v_2, the neighbors are v_0 and v_1. Since $c(v_0) = 1$, we see that $c(v_1) = 2$. Similarly, all other vertices have the color 2. Also the neighbors of v_0 are v_i where $1 \leq i \leq 2n$. All the neighbors of v_0 are colored with 2. Thus v_0 does not have a neighbor with color 1. Hence 2-coupon coloring is not possible. So all vertices can be colored using 1 color. ■

3 Unary Operations

Unary operations create a new graph from a single initial graph. Here we consider two operations such as total graph and subdivision operation.

3.1 Total Graph $T(G)$

Definition 4 *The total graph $T(G)$ of the graph G whose set of vertices is the union of the set of vertices and set of edges of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident [4].*

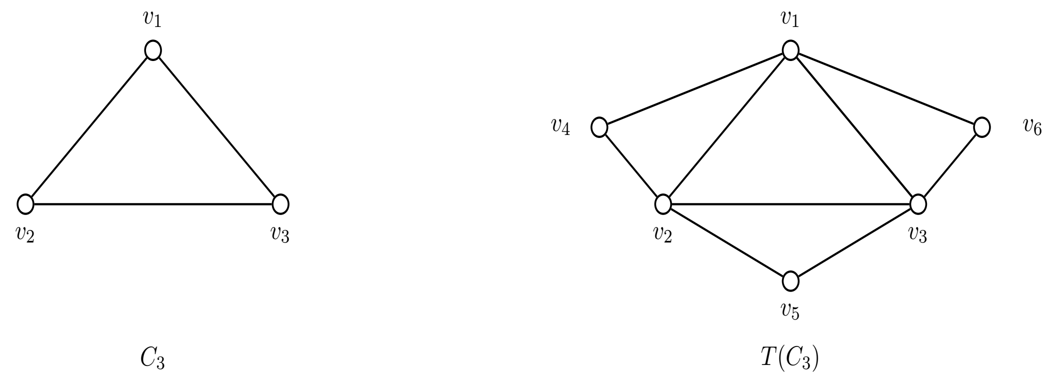

Figure 3: Total graph of C_3.

For example consider the cycle graph on 3 vertices C_3.

Theorem 4 *Let G be a cycle on n vertices. Then*

$$\chi_c(T(G)) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let G be a cycle on n vertices and suppose $v_1 v_2 \ldots v_n v_1$ be the cycle. Now in $T(G)$ label the newly added vertices in such a way that the vertex that is adjacent to v_i and v_{i+1} is denoted as v_{n+i} for $i = 1, 2, \ldots, n-1$ and label the vertex that is adjacent with v_n and v_1 as v_{2n}. Here $\delta(T(G)) = 2$. Thus $\chi_c(T(G)) \leq 2$.

Case 1: If n is even. Assign colors $c : V(G) \to [2]$ such that $c(v_i) = 1$ when i is odd and $c(v_i) = 2$ when i is even, where $i = 1, 2, \ldots, 2n$. Thus 2-coupon coloring exists. Therefore $\chi_c(T(G)) = 2$.

Case 2: If n is odd. Suppose 2-coupon coloring is possible. Consider the vertex $v_{n+1}, \mathcal{N}(v_{n+1}) = \{v_1, v_2\}$. Let $c : V(G) \to [2]$ such that $c(v_i) = a \in \{1, 2\}$ and $c(v_2) = b \in \{1, 2\}$, $a \neq b$. Without loss of generality assume that $c(v_1) = 1$ and $c(v_2) = 2$. Similarly, consider v_{n+2}. Since $c(v_2) = 2$, we have $c(v_3) = 1$. In this way we get $c(v_i) = 1$ when i is odd and $c(v_i) = 2$ when i is even, for $i = 1, 2, ..., n$. Thus $c(v_1) = c(v_n) = 1$. Consider the vertex v_{2n} that is adjacent with v_1 and v_n. Here arises a contradiction. Thus $\chi_c(T(G)) = 1$. ■

Theorem 5 *Let G be a path on n vertices. Then $\chi_c(T(G)) = 1$.*

Proof. Let G be a path with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $v_1 v_2 \ldots v_n$ be the path. In $T(G)$ label the newly add vertices in such a way that the vertex adjacent with v_i and v_{i+1} is denoted as v_{n+i}. Here $\delta(T(G)) = 2$. Thus $\chi_c(T(G)) \leq 2$. Suppose 2-coupon coloring is possible. Consider the vertex v_{n+1} and the neighbors of v_{n+1} are v_1 and v_2. Let $c : V(G) \to [2]$ such that $c(v_i) = a \in \{1, 2\}$ and $c(v_2) = b \in \{1, 2\}$, $a \neq b$. Without loss of generality, let $c(v_1) = 1$ and $c(v_2) = 2$. Similarly consider v_{n+2}. Since $c(v_2) = 2$, we get $c(v_3) = 1$. In this way we get $c(v_i) = 1$ when i is odd and $c(v_i) = 2$ when i is even. Consider the neighbors of v_1. We get $c(v_{n+1}) = 1$. Now the neighbors of the vertex v_2 are v_1, v_3, v_{n+1}, v_{n+2}. Since $c(v_1) = c(v_{n+1}) = c(v_3) = 1$, $c(v_{n+2}) = 2$. From this it follows that $c(v_{n+i}) = 1$ when i is odd and $c(v_{n+i}) = 2$ when i is even. Suppose n is even, then $c(v_n) = 2$, $c(v_{n-1}) = 1$ and $c(v_{n+n-1}) = 1$. Since neighbors of c_n colored with the same color, 2-coupon coloring is not possible. Similarly in the case when n is odd. Thus $\chi_c(T(G)) = 1$. ■

Theorem 6 *Let $G = K_{m,n}$ be a complete bipartite graph, where $m, n \geq 2$. Then $\chi_c(T(G)) = 2$.*

Proof. Let G be a complete bipartite graph with $V(G) = V_1 \bigcup V_2$ such that there are m vertices in V_1 and n vertices in V_2. Since $\delta(T(G)) = 2$, $\chi_c(T(G)) \leq 2$. Without loss of generality assume $m \leq n$. Now, label the vertices of V_1 as v_i, $1 \leq i \leq m$ and the vertices of V_2 as v_j, $m+1 \leq j \leq m+n$. Consider the newly added vertices and label the vertex sharing edges with v_i and v_j as v_{ni+j}. Now we prove that 2-coupon coloring is possible. Define $c : V(G) \to [2]$ in such a way that $c(v_i) = 1$ for all $i \in \{1, 2, \cdots, m\}$ and $c(v_j) = 2$ for all $j \in \{m+1, m+2, \cdots, m+n\}$. Hence every newly added vertex satisfy the property S. Now for every vertex v_{ni+j}, where $1 \leq i \leq m,\ m+1 \leq j \leq m+n$,

$$c(v_{ni+j}) = \begin{cases} 2 & \text{if } i+j \text{ is even,} \\ 1 & \text{if } i+j \text{ is odd.} \end{cases}$$

Case 1: $i+j$ is even. Then i and j have the same parity. Suppose i and j are odd. Thus $c(v_{ni+j}) = 2$. Consider the vertices v_i and v_j, v_i and v_{ni+j} are neighbouring vertices of v_j, thus $c(v_i) = 1$ and $c(v_{ni+j}) = 2$. Now let $j = m+k$, $1 \leq k \leq n$. Since G is complete bipartite, v_i is adjacent to v_{m+k} for all $k = 1, 2, \ldots, n$. Thus v_i is adjacent to either v_{m+k-1} or v_{m+k+1} or both. Let $v_l = v_{m+k-1}$ or v_{m+k+1}. In both cases l is even and i is odd. So the vertex adjacent to both v_i and v_l will get the color 1. That is $c(v_{ni+l}) = 1$. v_j and v_{ni+l} are neighboring vertices of v_i and $c(v_j) = 2$, $c(v_{ni+j}) = 1$. Thus in this case 2-coupon coloring is possible. Similarly we can prove 2-coupon coloring is possible in the case when i and j are even.

Case 2: $i+j$ is odd. Either one among i or j is odd. Thus $c(v_{ni+j}) = 1$. Also $c(v_j) = 2$. Thus neighbors of v_i has 2 colors. Similarly as Case 1. v_i is connected to either v_{m+k-1} or v_{m+k+1} or both. In this case $c(v_{ni+j}) = 2$. Hence neighboring vertices v_j has 2 colors. Thus 2-coupon coloring is possible. Hence $\chi_c(T(G)) = 2$. ■

Theorem 7 *Let G be a graph of order n having a universal vertex. Then $\chi_c(T(G)) = 1$.*

Proof. Let G be a graph with n vertices having a universal vertex and let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of G. Suppose v_1 is the universal vertex. After adding a vertex to each edge and connecting the end vertices of that edge, we label those vertices joining v_1 and v_i as v_{n+i-1} where $i = 2, 3, \ldots, n$. Remaining additional vertices (if exists) can be labelled as $v_{2n+g},\ g = 0, 1, \ldots$. Since $\delta(T(G)) = 2$, $\chi_c(T(G)) \leq 2$. Suppose there exists a 2-coupon coloring, $c : V(G) \to [2]$. Consider the vertex v_{n+1}, which shares edges with v_1 and v_2. Also its neighbors are v_1 and v_2 then $c(v_i) = a \in \{1, 2\}$ and $c(v_2) = b \in \{1, 2\}$, $a \neq b$. Without loss of generality, assume that $c(v_1) = 2$ and $c(v_2) = 1$. Consider v_{n+2}, v_1 and v_3 are its neighbors. Since $c(v_1) = 2$, we get $c(v_3) = 1$ such that neighbors of v_{n+2} is assigned with 2 colors. Similarly,

$$c(v_2) = c(v_3) = \cdots = c(v_n) = 1.$$

Let v_i and v_k be two arbitrary vertices such that $i, k \neq 1$.

Case 1 : Suppose there exists an edge joining v_i and v_k in G. Then there exists a vertex v_{2n+g}, where $g \in \{0, 1, \ldots \frac{n(n-3)}{2}\}$, that shares edges with v_i and v_k. Since $c(v_i) = c(v_k) = 1$, a 2-coupon coloring is not possible.

Case 2 : Suppose there exist no edge joining v_i and v_k. Consider the vertex that shares edges with v_1 and v_i. That is, v_{n+i-1}. We have $c(v_1) = 2$, $c(v_i) = 1$ where v_1 and v_{n+i-1} are the neighboring vertices of v_i. Thus, $c(v_{n+i-1}) = 1$ for all $i = 2, 3, \ldots, n$. Similarly, this holds for all other vertices. The neighbors of v_1 are not assigned with 2 colors. Hence, a 2-coupon coloring does not exist. Therefore, $\chi_c(T(G)) = 1$. ■

Corollary 1 *Let G be a split graph having diameter 2. Then $\chi_c(T(G)) = 1$.*

Proof. Split graph having diameter 2 has at least one universal vertex. Then by Theorem 7, $\chi_c(T(G)) = 1$. ■

3.2 Subdivision Operation $S(G)$

Definition 5 *$S(G)$ is obtained by splitting each edge of G by introducing a new vertex [13].*

For example, consider a cycle graph on 3 vertices.

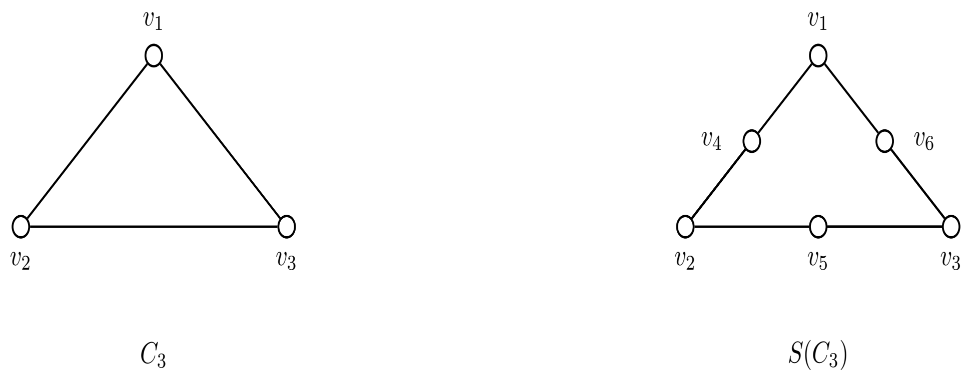

Figure 4: $S(C_3)$.

Theorem 8 *Let G be a cycle on n vertices. Then*

$$\chi_c(S(G)) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $G = C_n$ be a cycle on n vertices. Splitting of each edge will result again in cycle on $2n$ vertices. That is, $\chi_c(S(C_n)) = \chi_c(C_{2n})$. From Lemma 1 we have $\chi_c(S(G)) = 2$ when n is even and $\chi_c(S(G)) = 1$ when n is odd. ■

Theorem 9 *Let G be a path on n vertices. Then $\chi_c S((G)) = 1$.*

Proof. Let $G = P_n$ be a path on n vertices. Splitting of an edge into two edges by adding a vertex in between them results on a path with $2n - 1$ vertices. That is, $\chi_c(S(P_n)) = \chi_c(P_{2n-1})$. ■

Theorem 10 *Let G be a complete bipartite graph with m vertices in one set and n vertices in another set such that $m, n > 1$. Then $\chi_c(S(G)) = 2$.*

Proof. The proof of this theorem is similar to that of Theorem 6. ■

Theorem 11 *Let G be any graph of order n having a universal vertex. Then $\chi_c(S(G)) = 1$.*

Proof. The proof of this theorem is similar to that of Theorem 7. ■

Corollary 2 *Let G be a split graph having diameter 2. Then $\chi_c(S(G)) = 1$.*

Proof. By Theorem 11, $\chi_c(S(G)) = 1$. ■

4 Conclusion

In this paper, we explored coupon-coloring of friendship graphs and split graphs. Also we have investigated the coupon coloring of graphs generated after applying unary operations on various graph classes.

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Graph class	$\|V(G)\|$	$\|V(T(G))\|$,$\|V(S(G))\|$	$\chi_c(G)$	$\chi_c(T(G))$, $\chi_c(S(G))$
C_n	n	$2n$	$\begin{cases} 2 & n \equiv (0 \mod 4) \\ 1 & otherwise \end{cases}$	$\begin{cases} 2 & n\ iseven \\ 1 & n\ isodd \end{cases}$
P_n	n	$2n-1$	1	1
W_n	n	$3n-2$	2	1
K_n	n	$\frac{n(n+1)}{2}$	$\lfloor \frac{n}{2} \rfloor$	1
$K_{1,n}$	$n+1$	$2n+1$	1	1
$K_{m,n}$ $(m,n>1)$	$m+n$	$m(n+1)+n$	$\min\{m,n\}$	2

Table 1: Impact of Total graph operation and Subdivision operation on Coupon coloring number of standard graph classes.

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