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Ostrowski Type Inequalities Involving Conformable Integrals Via Convex Functions*

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Abstract

Through this research article we prove the Ostrowski inequality for conformable integrals.

1 Introduction

Ostrowski presented an interesting integral inequality connected with differentiable mapping in 1938:

Let $I \subseteq \mathbb{R}$ be an interval, I° be the interior of I and $h : I^\circ \to \mathbb{R}$ be a real-valued function. Then the classical Ostrowski inequality [30]

$$\left| h(x) - \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} h(x)dx \right| \le \left[\frac{1}{4} + \frac{(x - \frac{z_1+z_2}{2})^2}{(z_2 - z_1)^2} \right] (z_2 - z_1)||h'||_\infty$$

holds for all $x \in [z_1, z_2]$ with the best possible constant $1/4$ if $z_1, z_2 \in I^\circ$ with $z_1 < z_2$ such that $|h'(x)| \le M$ for all $x \in [z_1, z_2]$.

Many endeavours have been made on generalizations, extensions and variants of Ostrowski type inequalities for the past 60 years (see [4, 13, 14, 15, 16, 20, 24, 33, 36]).

Definition 1 *Let $z_1, z_2 \in \mathbb{R}$ with $z_1 < z_2$. Then a real-valued function $h : [z_1, z_2] \to \mathbb{R}$ is said to be convex if*

$$h[(1 - r)\mu_1 + r\mu_2] \le (1 - r)h(\mu_1) + rh(\mu_2)$$

$\forall\ \mu_1, \mu_2 \in [z_1, z_2]$ and $r \in [0, 1]$. h is said to be concave if $-h$ is convex.

It is well known that the convexity is one of the most important tools to discover inequalities, many generalizations for the convexity can be found in the literature [7, 8, 9, 10, 22, 31, 34, 35, 37, 38].

Let $0 < \beta \le 1$, $r > 0$ and $g : [0, \infty) \to \mathbb{R}$ be a real-valued function. Then the conformable derivative $D_\beta(g)(r)$ of g at r [23] is defined by

$$D_\beta(g)(r) := \frac{d_\beta g(r)}{d_\beta r} = \lim_{\epsilon \to 0} \frac{g(r + \epsilon r^{1-\beta}) - g(r)}{\epsilon}. \tag{1}$$

g is said to be conformable differentiable if the limit in (1) exists. The conformal derivative at 0 is defined by $D_\beta(g)(0) = \lim_{r \to 0^+} D_\beta(g)(r)$.

Let $z_1, z_2, \lambda, c \in \mathbb{R}$ be constants, and h_1 and h_2 be differentiable at $r > 0$. Then the following formulas can be found in the literature [23]:

$$\frac{d_\beta}{d_\beta r}\left(r^\lambda\right) = \lambda r^{\lambda - \beta},\ \frac{d_\beta}{d_\beta r}(c) = 0,$$

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$$\begin{aligned}\frac{d_\beta}{d_\beta r}\left(z_1h_1(r)+z_2h_2(r)\right) &= z_1\frac{d_\beta}{d_\beta r}(h_1(r))+z_2\frac{d_\beta}{d_\beta r}(h_2(r)),\\ \frac{d_\beta}{d_\beta r}\left(h_1(r)h_2(r)\right) &= h_1(r)\frac{d_\beta}{d_\beta r}(h_2(r))+h_2(r)\frac{d_\beta}{d_\beta r}(h_1(r)),\\ \frac{d_\beta}{d_\beta r}\left(\frac{h_1(r)}{h_2(r)}\right) &= \frac{h_2(r)\frac{d_\beta}{d_\beta r}(h_1(r))-h_1(r)\frac{d_\beta}{d_\beta r}(h_2(r))}{(h_2(r))^2},\\ \frac{d_\beta}{d_\beta r}\left(h_1(h_2(r))\right) &= h_1'(h_2(r))\frac{d_\beta}{d_\beta r}(h_2(r))\end{aligned}$$

if h_1 is differentiable at $h_2(r)$. In addition,

$$\frac{d_\beta}{d_\beta r}\left(h_1(r)\right)=r^{1-\beta}\frac{d}{dr}(h_1(r))$$

if h_1 is differentiable.

Let $\beta\in(0,1]$ and $0\le z_1<z_2$. Then the function $g:[z_1,z_2]\to\mathbb{R}$ is said to be conformable integrable if

$$\int_{z_1}^{z_2}g(x)d_\beta x:=\int_{z_1}^{z_2}g(x)x^{\beta-1}dx$$

exists. All conformable integrable functions on $[z_1,z_2]$ is denoted by $L_\beta([z_1,z_2])$. Note that

$$I_\beta^{z_1}(h_1)(r)=I_1^{z_1}(r^{\beta-1}h_1)=\int_{z_1}^{r}\frac{h_1(x)}{x^{1-\beta}}dx$$

for all $\beta\in(0,1]$, where the integral is the usual Riemann improper integral.

For theory and applications for conformable integrals and derivatives we refer the readers to refer the article [1, 2, 3, 4, 5, 6, 11, 17, 18, 19, 21, 25, 26, 27, 28].

Anderson [12] established the conformable integral version of the Hermite-Hadamard inequality

$$\frac{\beta}{z_2^\beta-z_1^\beta}\int_{z_1}^{z_2}h(x)d_\beta x\le\frac{h(z_1)+h(z_2)}{2},$$

for the conformable differentiable function $h:[z_1,z_2]\to\mathbb{R}$ with $\beta\in(0,1]$ and $D_\beta(h)$ is increasing. Moreover, if h is decreasing on $[z_1,z_2]$, then

$$h\left(\frac{z_1+z_2}{2}\right)\le\frac{\beta}{z_2^\beta-z_1^\beta}\int_{z_1}^{z_2}h(x)d_\beta x.$$

The main purpose of the article is to present the Ostrowski type inequalities for conformable integrals by use of convexity.

2 Main Results

In order to establish our main results we need a lemma, which we present in this section.

Lemma 1 *Let $z_1,z_2\in\mathbb{R}^+$ and $h[z_1,z_2]\to\mathbb{R}$ be a differentiable function on (z_1,z_2). If $D_\beta(h)\in L_\beta([z_1,z_2])$ where $\beta\in(0,1]$, then the following identity holds:*

$$h(x)-\frac{\beta}{z_2^\beta-z_1^\beta}\int_{z_1}^{z_2}h(s)d_\beta s=\frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left[I_1+I_2\right],$$

where

$$I_1=\int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1+tz_2)^{2\beta-1}-z_1^{\beta}((1-t)z_1+tz_2)^{\beta-1}\big)D_\beta(h)((1-t)z_1+tz_2)t^{1-\beta}d_\beta t,$$

$$I_2=\int_{\frac{x-z_1}{z_2-z_1}}^{1}\big(((1-t)z_1+tz_2)^{2\beta-1}-z_2^{\beta}((1-t)z_1+tz_2)^{\beta-1}\big)D_\beta(h)((1-t)z_1+tz_2)t^{1-\beta}d_\beta t.$$

Proof. Integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1+tz_2)^{2\beta-1}-z_1^{\beta}((1-t)z_1+tz_2)^{\beta-1}\big)D_\beta(h)((1-t)z_1+tz_2)dt\\
&= \int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1+tz_2)^{\beta}-z_1^{\beta}\big)h'((1-t)z_1+tz_2)dt\\
&= \big(((1-t)z_1+tz_2)^{\beta}-z_1^{\beta}\big)\frac{h((1-t)z_1+tz_2)}{z_2-z_1}\Bigg|_0^{\frac{x-z_1}{z_2-z_1}}\\
&\quad -\int_0^{\frac{x-z_1}{z_2-z_1}}\beta((1-t)z_1+tz_2)^{\beta-1}(z_2-z_1)\frac{h((1-t)z_1+tz_2)}{z_2-z_1}dt\\
&= \frac{x^\beta-z_1^\beta}{z_2-z_1}h(x)-\frac{\beta}{z_2-z_1}\int_{z_1}^{x}h(s)d_\beta s
\end{aligned}$$

where, we have used the change of variable. And similarly we get

$$\begin{aligned}
I_2 &= \int_{\frac{x-z_1}{z_2-z_1}}^{1}\big(((1-t)z_1+tz_2)^{2\beta-1}-z_2^{\beta}((1-t)z_1+tz_2)^{\beta-1}\big)D_\beta(h)((1-t)z_1+tz_2)dt\\
&= \int_{\frac{x-z_1}{z_2-z_1}}^{1}\big(((1-t)z_1+tz_2)^{\beta}-z_2^{\beta}\big)h'((1-t)z_1+tz_2)dt\\
&= \big(((1-t)z_1+tz_2)^{\beta}-z_2^{\beta}\big)\frac{h((1-t)z_1+tz_2)}{z_2-z_1}\Bigg|_{\frac{x-z_1}{z_2-z_1}}^{1}\\
&\quad -\int_{\frac{x-z_1}{z_2-z_1}}^{1}\beta((1-t)z_1+tz_2)^{\beta-1}(z_2-z_1)\frac{h((1-t)z_1+tz_2)}{z_2-z_1}dt\\
&= \frac{z_2^\beta-x^\beta}{z_2-z_1}h(x)-\frac{\beta}{z_2-z_1}\int_{x}^{z_2}h(s)d_\beta s.
\end{aligned}$$

This completes the proof. ■

Theorem 1 *Let $z_1, z_2 \in \mathbb{R}^+$ and $h : [z_1, z_2] \to \mathbb{R}$ be a differentiable function on (z_1, z_2) such that $D_\beta(h) \in L_\beta([z_1, z_2])$ where $\beta \in (0, 1]$. If $|h'(x)|$ is convex, then the following inequality holds:*

$$\left|h(x)-\frac{\beta}{z_2^\beta-z_1^\beta}\int_{z_1}^{z_2}h(s)ds\right|\le\frac{z_2-z_1}{z_2^\beta-z_1^\beta}\big[(\mathfrak{A}_1+\mathfrak{B}_1)|h'(z_1)|+(\mathfrak{A}_2+\mathfrak{B}_2)|h'(z_2)|\big],$$

where

$$\mathfrak{A}_1 = \left(\frac{1}{4}-\frac{(z_2-x)^4}{4(z_2-z_1)^4}\right)z_1^\beta+\left(\frac{1}{12}-\frac{(z_2-x)^3}{3(z_2-z_1)^3}+\frac{(z_2-x)^4}{4(z_2-z_1)^4}\right)z_1^{\beta-1}z_2$$

$$+\left(\frac{1}{12}-\frac{(z_2-x)^3}{3(z_2-z_1)^3}+\frac{(z_2-x)^4}{4(z_2-z_1)^3}\right)z_2^{\beta-1}z_1+\left(\frac{1}{12}-\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right.$$
$$\left.-\frac{(z_2-x)^4}{4(z_2-z_1)^4}+\frac{2(z_2-x)^3}{3(z_2-z_1)^3}\right)z_2^{\beta}-\left(\frac{1}{2}-\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right)z_1^{\beta},$$

$$\mathfrak{B}_1=\left(\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right)z_2^{\beta}-\left(\frac{(z_2-x)^3}{3(z_2-z_1)^3}\right)z_1^{\beta}-\left(\frac{1}{6}-\frac{(x-z_1)^2}{2(z_2-z_1)^2}+\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_2^{\beta},$$

$$\begin{aligned}\mathfrak{A}_2 \quad = \quad & \left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}+\frac{(x-z_1)^4}{14(z_2-z_1)^4}-\frac{2(x-z_1)^3}{3(z_2-z_1)^3}\right)z_1^{\beta}+\left(\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right.\\ & \left.-\frac{(x-z_1)^4}{4(z_2-z_1)^4}\right)z_1^{\beta-1}z_2+\left(\frac{(x-z_1)^3}{3(z_2-z_1)^3}-\frac{(x-z_1)^4}{4(z_2-z_1)^4}\right)z_2^{\beta-1}z_1\\ & +\left(\frac{(x-z_1)^4}{4(z_2-z_1)^4}\right)z_2^{\beta}-\left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)z_1^{\beta},\end{aligned}$$

$$\mathfrak{B}_2=\left(\frac{1}{2}-\frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)z_2^{\beta}-\left(\frac{1}{6}-\frac{(x-z_1)^2}{2(z_2-z_1)^2}+\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_1^{\beta}-\left(\frac{1}{3}-\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_2^{\beta}.$$

Proof. Let $\varphi_1(y)=y^{\beta-1}$ and $\varphi_2(y)=-y^{\beta}, y>0, \beta\in(0,1]$ clearly the functions φ_1 and φ_2 are convex. Now using Lemma 1, the convexity of φ_1, φ_2 and convexity of $|h'|$, we have

$$\begin{aligned}& \left|h(x)-\frac{\beta}{z_2^{\beta}-z_1^{\beta}}\int_{z_1}^{z_2}h(s)d_{\beta}s\right|\\ \leq\ & \frac{z_2-z_1}{z_2^{\beta}-z_1^{\beta}}\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1+tz_2)^{\beta}-z_1^{\beta}\right)|h'((1-t)z_1+tz_2)|dt\right.\\ & \left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\left(z_2^{\beta}-((1-t)z_1+tz_2)^{\beta}\right)|h'((1-t)z_1+tz_2)|dt\right)\\ =\ & \frac{z_2-z_1}{z_2^{\beta}-z_1^{\beta}}\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1+tz_2)^{\beta-1}((1-t)z_1+tz_2)-z_1^{\beta}\right)|h'((1-t)z_1+tz_2)|dt\right.\\ & \left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\left(z_2^{\beta}-((1-t)z_1+tz_2)^{\beta}\right)|h'((1-t)z_1+tz_2)|dt\right)\\ \leq\ & \frac{z_2-z_1}{z_2^{\beta}-z_1^{\beta}}\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^{\beta}\right)|h'((1-t)z_1+tz_2)|dt\right.\\ & \left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\left(z_2^{\beta}-((1-t)z_1^{\beta}+tz_2^{\beta})\right)|h'((1-t)z_1+tz_2)|dt\right)\\ \leq\ & \frac{z_2-z_1}{z_2^{\beta}-z_1^{\beta}}\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^{\beta}\right)[(1-t)|h'(z_1)|+t|h'(z_2)|]dt\right.\\ & \left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\left(z_2^{\beta}-((1-t)z_1^{\beta}+tz_2^{\beta})\right)[(1-t)|h'(z_1)|+t|h'(z_2)|]dt\right)\\ \leq\ & \frac{z_2-z_1}{z_2^{\beta}-z_1^{\beta}}\left[(\mathfrak{A}_1+\mathfrak{B}_1)|h'(z_1)|+(\mathfrak{A}_2+\mathfrak{B}_2)|h'(z_2)|\right].\end{aligned}$$

■

Corollary 1 *Let $x=(z_1+z_2)/2$. Then Theorem 1 leads to*

$$\left|h\left(\frac{z_1+z_2}{2}\right)-\frac{\beta}{z_2^\beta-z_1^\beta}\int_{z_1}^{z_2}h(s)d_\beta s\right|\le\frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left[(\mathfrak{A}_1+\mathfrak{B}_1)|h'(z_1)|+(\mathfrak{A}_2+\mathfrak{B}_2)|h'(z_2)|\right],\tag{1}$$

where

$$\mathfrak{A}_1=\frac{15}{64}z_1^\beta+\frac{11}{192}z_1^{\beta-1}z_2+\frac{11}{192}z_2^{\beta-1}z_1+\frac{5}{192}z_2^\beta-\frac{3}{8}z_1^\beta,$$
$$\mathfrak{B}_1=\frac{1}{8}z_2^\beta-\frac{1}{24}z_1^\beta-\frac{1}{12}z_2^\beta,$$
$$\mathfrak{A}_2=\frac{11}{192}z_1^\beta+\frac{5}{192}z_1^{\beta-1}z_2+\frac{5}{192}z_2^{\beta-1}z_1+\frac{1}{64}z_2^\beta-\frac{1}{8}z_1^\beta,$$
$$\mathfrak{B}_2=\frac{3}{8}z_2^\beta-\frac{1}{12}z_1^\beta-\frac{7}{24}z_2^\beta.$$

Remark 1 *Let $\beta=1$. The inequality (1) leads to*

$$\left|h\left(\frac{z_1+z_2}{2}\right)-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}h(s)ds\right|\le\frac{z_2-z_1}{8}\left[|h'(z_1)|+|h'(z_2)|\right],$$

which was proved by U. S. Kirmaci in [29].

Theorem 2 *Let $z_1,z_2\in\mathbb{R}^+$ and $h:[z_1,z_2]\to\mathbb{R}$ be a differentiable function on (z_1,z_2) such that $D_\beta(h)\in L_\beta([z_1,z_2])$ where $\beta\in(0,1]$. If $|h'|^q$ is convex for $q>1$ and $q^{-1}+p^{-1}=1$, then the following inequality holds:*

$$\begin{aligned}\left|h(x)-\frac{\beta}{z_2^\beta-z_1^\beta}\int_{z_1}^{z_2}h(s)d_\beta s\right| \quad\le\quad & \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\Big[(\mathfrak{A}(\beta))^{1-\frac{1}{q}}\left\{[\mathfrak{A}_1(\beta)|h'(z_1)|^q+\mathfrak{A}_2(\beta)|h'(z_2)|^q]\right\}^{\frac{1}{q}}\\ &+(\mathfrak{B}(\beta))^{1-\frac{1}{q}}\left\{[\mathfrak{B}_1(\beta)|h'(z_1)|^q+\mathfrak{B}_2(\beta)|h'(z_2)|^q]\right\}^{\frac{1}{q}}\Big],\end{aligned}$$

where

$$\begin{aligned}\mathfrak{A}(\beta)\quad=\quad&\left(\frac{1}{3}-\frac{(z_2-x)^3}{3(z_2-z_1)^3}\right)z_1^\beta+\left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}-\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_1^{\beta-1}z_2\\&+\left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}-\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_2^{\beta-1}z_1+\frac{(x-z_1)^3}{3(z_2-z_1)^3}z_2^\beta-\frac{x-z_1}{z_2-z_1}z_1^\beta,\end{aligned}$$

$$\mathfrak{B}(\beta)=\frac{z_2-x}{z_2-z_1}z_2^\beta-\frac{(z_2-x)^2}{2(z_2-z_1)^2}z_1^\beta-\left(\frac{1}{6}-\frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)z_2^\beta,$$

$$\begin{aligned}\mathfrak{A}_1(\beta)\quad=\quad&\left(\frac{1}{4}-\frac{(z_2-x)^4}{4(z_2-z_1)^4}\right)z_1^\beta+\left(\frac{1}{12}-\frac{(z_2-x)^3}{3(z_2-z_1)^3}+\frac{(z_2-x)^4}{4(z_2-z_1)^4}\right)z_1^{\beta-1}z_2\\&+\left(\frac{1}{12}-\frac{(z_2-x)^3}{3(z_2-z_1)^3}+\frac{(z_2-x)^4}{4(z_2-z_1)^3}\right)z_2^{\beta-1}z_1+\left(\frac{1}{12}-\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right.\\&\left.-\frac{(z_2-x)^4}{4(z_2-z_1)^4}+\frac{2(z_2-x)^3}{3(z_2-z_1)^3}\right)z_2^\beta-\left(\frac{1}{2}-\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right)z_1^\beta,\end{aligned}$$

$$\mathfrak{B}_1=\left(\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right)z_2^\beta-\left(\frac{(z_2-x)^3}{3(z_2-z_1)^3}\right)z_1^\beta-\left(\frac{1}{6}-\frac{(x-z_1)^2}{2(z_2-z_1)^2}+\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_2^\beta,$$

$$\begin{aligned}\mathfrak{A}_2(\beta) &= \left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}+\frac{(x-z_1)^4}{4(z_2-z_1)^4}-\frac{2(x-z_1)^3}{3(z_2-z_1)^3}\right)z_1^\beta+\left(\frac{(x-z_1)^3}{3(z_2-z_1)^3}-\frac{(x-z_1)^4}{4(z_2-z_1)^4}\right)z_1^{\beta-1}z_2\\ &\quad+\left(\frac{(x-z_1)^3}{3(z_2-z_1)^3}-\frac{(x-z_1)^4}{4(z_2-z_1)^4}\right)z_2^{\beta-1}z_1+\left(\frac{(x-z_1)^4}{4(z_2-z_1)^4}\right)z_2^\beta-\left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)z_1^\beta,\end{aligned}$$

$$\mathfrak{B}_2=\left(\frac{1}{2}-\frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)z_2^\beta-\left(\frac{1}{6}-\frac{(x-z_1)^2}{2(z_2-z_1)^2}+\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_1^\beta-\left(\frac{1}{3}-\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_2^\beta.$$

Proof. Using Lemma 1, it follows that

$$\begin{aligned}&\left|h(x)-\frac{\beta}{z_2^\beta-z_1^\beta}\int_{z_1}^{z_2}h(s)d_\beta s\right|\\ \leq\;& \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1+tz_2)^{2\beta-1}-z_1^\beta((1-t)z_1+tz_2)^{\beta-1}\big)\right.\\ &\times D_\beta(h)((1-t)z_1+tz_2)dt+\int_{\frac{x-z_1}{z_2-z_1}}^1\big(((1-t)z_1+tz_2)^{2\beta-1}-z_2^\beta\\ &\left.\times(((1-t)z_1+tz_2))^{\beta-1}\big)D_\beta(h)((1-t)z_1+tz_2)dt\right|\\ =\;& \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1+tz_2)^\beta-z_1^\beta\big)h'((1-t)z_1+tz_2)dt\right.\\ &\left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\big(z_2^\beta-((1-t)z_1+tz_2)^\beta\big)h'((1-t)z_1+tz_2)dt\right|\\ =\;& \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1+tz_2)^{\beta-1}((1-t)z_1+tz_2)-z_1^\beta\big)h'((1-t)z_1+tz_2)dt\right.\\ &\left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\big(z_2^\beta-((1-t)z_1+tz_2)^\beta\big)h'((1-t)z_1+tz_2)dt\right|\\ \leq\;& \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\big)h'((1-t)z_1+tz_2)dt\right.\\ &\left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\big(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\big)h'((1-t)z_1+tz_2)dt\right|.\end{aligned}$$

Now, making use of power mean inequality we get

$$\begin{aligned}&\int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\big)|h'((1-t)z_1+tz_2)|dt\\ \leq\;&\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\big)dt\right)^{1-\frac{1}{q}}\\ &\times\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\big(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\big)|h'((1-t)z_1+tz_2)|^qdt\right)^{\frac{1}{q}}\end{aligned}$$

and similarly, we have

$$\int_{\frac{x-z_1}{z_2-z_1}}^1\big(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\big)|h'((1-t)z_1+tz_2)|dt$$

$$\begin{aligned}
&\leq \left(\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)|h'((1-t)z_1+tz_2)|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Now by the convexity of $|h'|^q$ from above, we have

$$\begin{aligned}
&\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)|h'((1-t)z_1+tz_2)|^q dt\\
\leq\ &\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)[(1-t)|h'(z_1)|^q+t|h'(z_2)|^q]dt\\
=\ &|h'(z_1)|^q\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)(1-t)dt\\
&+|h'(z_2)|^q\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)tdt\\
=\ &\mathfrak{A}_1(\beta)|h'(z_1)|^q+\mathfrak{A}_2(\beta)|h'(z_2)|^q
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)|h'((1-t)z_1+tz_2)|^q dt\\
\leq\ &\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)[(1-t)|h'(z_1)|^q+t|h'(z_2)|^q]dt\\
=\ &|h'(z_1)|^q\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)(1-t)dt\\
&+|h'(z_2)|^q\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)tdt\\
=\ &\mathfrak{B}_1(\beta)|h'(z_1)|^q+\mathfrak{B}_2(\beta)|h'(z_2)|^q
\end{aligned}$$

where, we have used the identities

$$\begin{aligned}
&\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)dt\\
=\ &\left(\frac{1}{3}-\frac{(z_2-x)^3}{3(z_2-z_1)^3}\right)z_1^\beta+\left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}-\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_1^{\beta-1}z_2\\
&+\left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}-\frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_2^{\beta-1}z_1+\frac{(x-z_1)^3}{3(z_2-z_1)^3}z_2^\beta-\frac{x-z_1}{z_2-z_1}z_1^\beta\\
=\ &\mathfrak{A}(\beta)
\end{aligned}$$

and

$$\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)dt=\frac{z_2-x}{z_2-z_1}z_2^\beta-\frac{(z_2-x)^2}{2(z_2-z_1)^2}z_1^\beta-\left(\frac{1}{6}-\frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)z_2^\beta=\mathfrak{B}(\beta).$$

Hence, we have the result in (4). ■

Corollary 2 *Let $x = (z_1 + z_2)/2$. Then Theorem 2 leads to*

$$\left|h\left(\frac{z_1+z_2}{2}\right) - \frac{\beta}{z_2^\beta - z_1^\beta}\int_{z_1}^{z_2} h(s)d_\beta s\right| \leq \frac{z_2-z_1}{z_2^\beta - z_1^\beta}\Big[(\mathfrak{A}(\beta))^{1-\frac{1}{q}}\left\{[\mathfrak{A}_1(\beta)|h'(z_1)|^q + \mathfrak{B}_1(\beta)|h'(z_2)|^q]\right\}^{\frac{1}{q}} + (\mathfrak{B}(\beta))^{1-\frac{1}{q}}\left\{[\mathfrak{A}_2(\beta)|h'(z_1)|^q + \mathfrak{B}_2(\beta)|h'(z_2)|^q]\right\}^{\frac{1}{q}}\Big], \tag{2}$$

where

$$\mathfrak{A}(\beta) = \frac{7}{24}z_1^\beta + \frac{1}{12}z_1^{\beta-1}z_2 + \frac{1}{12}z_2^{\beta-1}z_1 + \frac{1}{24}z_2^\beta - \frac{1}{2}z_1^\beta,$$
$$\mathfrak{B}(\beta) = \frac{1}{2}z_2^\beta - \frac{1}{8}z_1^\beta - \frac{3}{8}z_2^\beta,$$
$$\mathfrak{A}_1(\beta) = \frac{15}{64}z_1^\beta + \frac{11}{192}z_1^{\beta-1}z_2 + \frac{11}{192}z_2^{\beta-1}z_1 + \frac{5}{192}z_2^\beta - \frac{3}{8}z_1^\beta,$$
$$\mathfrak{B}_1(\beta) = \frac{1}{8}z_2^\beta - \frac{1}{24}z_1^\beta - \frac{1}{12}z_2^\beta,$$
$$\mathfrak{A}_2(\beta) = \frac{11}{192}z_1^\beta + \frac{5}{192}z_1^{\beta-1}z_2 + \frac{5}{192}z_2^{\beta-1}z_1 + \frac{1}{64}z_2^\beta - \frac{1}{8}z_1^\beta,$$
$$\mathfrak{B}_2(\beta) = \frac{3}{8}z_2^\beta - \frac{1}{12}z_1^\beta - \frac{7}{24}z_2^\beta.$$

Remark 2 *Let $\beta = 1$. The inequality (2) leads to*

$$\left|h\left(\frac{z_1+z_2}{2}\right) - \frac{1}{z_2-z_1}\int_{z_1}^{z_2} h(s)ds\right|$$
$$\leq (z_2-z_1)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{|h'(z_1)|^q + 2|h'(z_2)|^q}{24}\right)^{\frac{1}{q}} + \left(\frac{|h'(z_2)|^q + 2|h'(z_1)|^q}{24}\right)^{\frac{1}{q}}\right],$$

which was proved by U. S. Kirmaci in [29].

Theorem 3 *Let $z_1, z_2 \in \mathbb{R}^+$ and $h : [z_1, z_2] \to \mathbb{R}$ be a differentiable function on (z_1, z_2) such that $D_\beta(h) \in L_\beta([z_1, z_2])$ where $\beta \in (0, 1]$. If $|h'|^q$ is convex for $q > 1$ and $q^{-1} + p^{-1} = 1$, then the following inequality holds:*

$$\left|h(x) - \frac{\beta}{z_2^\beta - z_1^\beta}\int_{z_1}^{z_2} h(s)d_\beta s\right|$$
$$\leq \frac{z_2-z_1}{z_2^\beta - z_1^\beta}\Bigg[(\mathfrak{A}_1(\beta,p))^{\frac{1}{p}}\left(\left(\frac{1}{2} - \frac{(z_2-x)^2}{2(z_2-z_1)^2}\right)|h'(z_1)|^q + \left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)|h'(z_2)|^q\right)^{\frac{1}{q}}$$
$$+(\mathfrak{A}_2(\beta,p))^{\frac{1}{p}}\left(\left(\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right)|h'(z_1)|^q + \left(\left(\frac{1}{2} - \frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)|h'(z_2)|^q\right)^{\frac{1}{q}}\right)\Bigg],$$

where

$$\mathfrak{A}_1(\beta,p) = \int_0^{\frac{x-z_1}{z_2-z_1}} \left(((1-t)z_1^{\beta-1} + tz_2^{\beta-1})((1-t)z_1 + tz_2) - z_1^\beta\right)^p dt,$$
$$\mathfrak{A}_2(\beta,p) = \int_{\frac{x-z_1}{z_2-z_1}}^1 \left(z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta)\right)^p dt.$$

Proof. Using Lemma 1, it follows that

$$\begin{aligned}
&\left|h(x)-\frac{\beta}{z_2^\beta-z_1^\beta}\int_{z_1}^{z_2}h(s)d_\beta s\right|\\
\leq\ & \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1+tz_2)^{2\beta-1}-z_1^\beta((1-t)z_1+tz_2)^{\beta-1}\right)\right.\\
&\times D_\beta(h)((1-t)z_1+tz_2)dt+\int_{\frac{x-z_1}{z_2-z_1}}^1\left(((1-t)z_1+tz_2)^{2\beta-1}-z_2^\beta\right.\\
&\left.\left.\times(((1-t)z_1+tz_2))^{\beta-1}\right)D_\beta(h)((1-t)z_1+tz_2)dt\right|\\
=\ & \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1+tz_2)^{\beta}-z_1^\beta\right)h'((1-t)z_1+tz_2)dt\right.\\
&\left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\left(z_2^\beta-((1-t)z_1+tz_2)^\beta\right)h'((1-t)z_1+tz_2)dt\right|\\
=\ & \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1+tz_2)^{\beta-1}((1-t)z_1+tz_2)-z_1^\beta\right)h'((1-t)z_1+tz_2)dt\right.\\
&\left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\left(z_2^\beta-((1-t)z_1+tz_2)^\beta\right)h'((1-t)z_1+tz_2)dt\right|\\
\leq\ & \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)h'((1-t)z_1+tz_2)dt\right.\\
&\left.+\int_{\frac{x-z_1}{z_2-z_1}}^1\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)h'((1-t)z_1+tz_2)dt\right|.
\end{aligned}$$

Now by the Hölder inequality

$$\begin{aligned}
&\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)|h'((1-t)z_1+tz_2)|dt\\
\leq&\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{x-z_1}{z_2-z_1}}|h'((1-t)z_1+tz_2)|^q dt\right)^{\frac{1}{q}}\\
\leq&\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^{\frac{x-z_1}{z_2-z_1}}[(1-t)|h'(z_1)|^q+t|h'(z_2)|^q]dt\right)^{\frac{1}{q}}\\
=&(\mathfrak{A}_1(\beta,p))^{\frac{1}{p}}\left(\left(\frac{1}{2}-\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right)|h'(z_1)|^q+\left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)|h'(z_2)|^q\}\right)^{\frac{1}{q}}
\end{aligned}$$

and similarly, we have

$$\int_{\frac{x-z_1}{z_2-z_1}}^1\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)|h'((1-t)z_1+tz_2)|dt$$

$$
\begin{aligned}
&\leq \left(\int_{\frac{x-z_1}{z_2-z_1}}^{1} \left(z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta)\right)^p dt\right)^{\frac{1}{p}} \left(\int_{\frac{x-z_1}{z_2-z_1}}^{1} |h'((1-t)z_1 + tz_2)|^q dt\right)^{\frac{1}{q}} \\
&\leq \left(\int_{\frac{x-z_1}{z_2-z_1}}^{1} \left(z_2^\beta - ((1-t)z_1 + tz_2)^\beta\right)^p dt\right)^{\frac{1}{p}} \left(\int_{\frac{x-z_1}{z_2-z_1}}^{1} [(1-t)|h'(z_1)|^q + t|h'(z_2)|^q] dt\right)^{\frac{1}{q}} \\
&= (\mathfrak{A}_2(\beta,p))^{\frac{1}{p}} \left(\left(\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right)|h'(z_1)|^q + \left(\frac{1}{2} - \frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)|h'(z_2)|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

■

Corollary 3 *Let $x = (z_1+z_2)/2$. Then Theorem 3 leads to*

$$
\begin{aligned}
&\left|h\left(\frac{z_1+z_2}{2}\right) - \frac{\beta}{z_2^\beta - z_1^\beta}\int_{z_1}^{z_2} h(s) d_\beta s\right| \\
\leq\ & \frac{z_2-z_1}{z_2^\beta - z_1^\beta}\left[(\mathfrak{A}_1(\beta,p))^{\frac{1}{p}}\left(\frac{3|h'(z_1)|^q + |h'(z_2)|^q}{8}\right)^{\frac{1}{q}} + (\mathfrak{A}_2(\beta,p))^{\frac{1}{p}}\left(\frac{|h'(z_1)|^q + 3|h'(z_2)|^q}{8}\right)^{\frac{1}{q}}\right], \qquad (3)
\end{aligned}
$$

where

$$
\mathfrak{A}_1(\beta,p) = \int_0^{\frac{1}{2}} \left(((1-t)z_1^{\beta-1} + tz_2^{\beta-1})((1-t)z_1 + tz_2) - z_1^\beta\right)^p dt,
$$

$$
\mathfrak{A}_2(\beta,p) = \int_{\frac{1}{2}}^{1} \left(z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta)\right)^p dt.
$$

Remark 3 *Let $\beta = 1$. The inequality (3) leads to*

$$
\begin{aligned}
&\left|h\left(\frac{z_1+z_2}{2}\right) - \frac{1}{z_2-z_1}\int_{z_1}^{z_2} h(s) ds\right| \\
\leq\ & \frac{z_2-z_1}{2^{p+1}(p+1)^{\frac{1}{p}}}\left[\left(\frac{3|h'(z_1)|^q + |h'(z_2)|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{|h'(z_1)|^q + 3|h'(z_2)|^q}{8}\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

which was proved by U. S. Kirmaci in [29].

Theorem 4 *Let $z_1, z_2 \in \mathbb{R}^+$ and $h : [z_1, z_2] \to \mathbb{R}$ be a differentiable function on (z_1, z_2) such that $D_\beta(h) \in L_\beta([z_1, z_2])$ where $\beta \in (0, 1]$. If $|h'|^q$ is convex for $q > 1$ and $q^{-1} + p^{-1} = 1$, then the following inequality holds:*

$$
\begin{aligned}
\left|h(x) - \frac{\beta}{z_2^\beta - z_1^\beta}\int_{z_1}^{z_2} h(s) d_\beta s\right| \leq\ & \frac{z_2-z_1}{z_2^\beta - z_1^\beta}\left[\left(\frac{x-z_1}{z_2-z_1}\right)^{\frac{1}{p}} \left(\mathfrak{C}_1(\beta,q)|h'(z_1)|^q + \mathfrak{C}_2(\beta,q)|h'(z_2)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{z_2-x}{z_2-z_1}\right)^{\frac{1}{p}} \left(\mathfrak{C}_3(\beta,q)|h'(z_1)|^q + \mathfrak{C}_4(\beta,q)|h'(z_2)|^q\right)^{\frac{1}{q}}\right], \qquad (4)
\end{aligned}
$$

where

$$
\mathfrak{C}_1(\beta,q) = \int_0^{\frac{x-z_1}{z_2-z_1}} \left(((1-t)z_1^{\beta-1} + tz_2^{\beta-1})((1-t)z_1 + tz_2) - z_1^\beta\right)^q (1-t) dt,
$$

$$
\mathfrak{C}_2(\beta,q) = \int_0^{\frac{x-z_1}{z_2-z_1}} \left(((1-t)z_1^{\beta-1} + tz_2^{\beta-1})((1-t)z_1 + tz_2) - z_1^\beta\right)^q t dt,
$$

$$\mathfrak{C}_3(\beta,q)=\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)^q(1-t)dt,$$

$$\mathfrak{C}_4(\beta,q)=\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)^q tdt.$$

Proof. Using Lemma 1, it follows that

$$\begin{aligned}
&\left|h(x)-\frac{\beta}{z_2^\beta-z_1^\beta}\int_{z_1}^{z_2}h(s)d_\beta s\right|\\
\leq\ & \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1+tz_2)^{2\beta-1}-z_1^\beta((1-t)z_1+tz_2)^{\beta-1}\right)\right.\\
&\times D_\beta(h)((1-t)z_1+tz_2)dt+\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(((1-t)z_1+tz_2)^{2\beta-1}-z_2^\beta\right.\\
&\left.\times(((1-t)z_1+tz_2))^{\beta-1}\right)D_\beta(h)((1-t)z_1+tz_2)dt\Bigg|\\
=\ & \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1+tz_2)^{\beta}-z_1^\beta\right)h'((1-t)z_1+tz_2)dt\right.\\
&\left.+\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1+tz_2)^\beta\right)h'((1-t)z_1+tz_2)dt\right|\\
=\ & \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1+tz_2)^{\beta-1}((1-t)z_1+tz_2)-z_1^\beta\right)h'((1-t)z_1+tz_2)dt\right.\\
&\left.+\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1+tz_2)^\beta\right)h'((1-t)z_1+tz_2)dt\right|\\
\leq\ & \frac{z_2-z_1}{z_2^\beta-z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)h'((1-t)z_1+tz_2)dt\right.\\
&\left.+\int_{\frac{x-z_1}{z_2-z_1}}^{1}\left(z_2^\beta-((1-t)z_1^\beta+tz_2^\beta)\right)h'((1-t)z_1+tz_2)dt\right|.
\end{aligned}$$

Now by the Hölder inequality

$$\begin{aligned}
&\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)|h'((1-t)z_1+tz_2)|dt\\
\leq\ & \left(\int_0^{\frac{x-z_1}{z_2-z_1}}dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)^q|h'((1-t)z_1+tz_2)|^q dt\right)^{\frac{1}{q}}\\
\leq\ & \left(\int_0^{\frac{x-z_1}{z_2-z_1}}dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^{\frac{x-z_1}{z_2-z_1}}\left(((1-t)z_1^{\beta-1}+tz_2^{\beta-1})((1-t)z_1+tz_2)-z_1^\beta\right)^q[(1-t)|h'(z_1)|^q+t|h'(z_2)|^q]dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$= \left(\frac{x-z_1}{z_2-z_1}\right)^{\frac{1}{p}} \left(\mathfrak{C}_1(\beta,q)|h'(z_1)|^q + \mathfrak{C}_2(\beta,q)|h'(z_2)|^q\right)^{\frac{1}{q}}$$

and similarly, we have

$$\begin{aligned}
&\int_{\frac{x-z_1}{z_2-z_1}}^{1} \left(z_2^\beta - ((1-t)z_1^\beta + tz_2)^\beta\right)|h'((1-t)z_1+tz_2)|dt \\
\leq\ & \left(\int_{\frac{x-z_1}{z_2-z_1}}^{1} dt\right)^{\frac{1}{p}} \left(\int_{\frac{x-z_1}{z_2-z_1}}^{1} \left(z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta)\right)^q |h'((1-t)z_1+tz_2)|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \left(\int_{\frac{x-z_1}{z_2-z_1}}^{1} dt\right)^{\frac{1}{p}} \left(\int_{\frac{x-z_1}{z_2-z_1}}^{1} \left(z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta)\right)^q [(1-t)|h'(z_1)|^q + t|h'(z_2)|^q] dt\right)^{\frac{1}{q}} \\
=\ & \left(\frac{z_2-x}{z_2-z_1}\right)^{\frac{1}{p}} \left(\mathfrak{C}_3(\beta,q)|h'(z_1)|^q + \mathfrak{C}_4|h'(z_2)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

■

Theorem 5 *Let $h:[z_1,z_2]\to\mathbb{R}$ be an β-differentiable function on (z_1,z_2) for $\beta\in(0,1]$. If $D_\beta(h)\in L^1_\beta([z_1,z_2])$ and $|h'|^q$ is concave on $[z_1,z_2]$ for $q>1$, then the following inequality holds:*

$$\left|h(x) - \frac{\beta}{z_2^\beta - z_1^\beta}\int_{z_1}^{z_2} h(s)d_\beta s\right| \leq \frac{z_2-z_1}{z_2^\beta - z_1^\beta}\left[\mathfrak{A}(\beta)\left|h'\left(\frac{\mathfrak{C}_1(\beta)}{\mathfrak{A}(\beta)}\right)\right| + \mathfrak{B}(\beta)\left|h'\left(\frac{\mathfrak{C}_2(\beta)}{\mathfrak{B}(\beta)}\right)\right|\right], \tag{5}$$

where

$$\begin{aligned}
\mathfrak{A}(\beta) &= \left(\frac{1}{3} - \frac{(z_2-x)^3}{3(z_2-z_1)^3}\right)z_1^\beta + \left(\frac{(x-z_1)^2}{2(z_2-z_1)^2} - \frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_1^{\beta-1}z_2 \\
&\quad + \left(\frac{(x-z_1)^2}{2(z_2-z_1)^2} - \frac{(x-z_1)^3}{3(z_2-z_1)^3}\right)z_2^{\beta-1}z_1 + \frac{(x-z_1)^3}{3(z_2-z_1)^3}z_2^\beta - \frac{x-z_1}{z_2-z_1}z_1^\beta,
\end{aligned}$$

$$\mathfrak{B}(\beta) = \frac{z_2-x}{z_2-z_1}z_2^\beta - \frac{(z_2-x)^2}{2(z_2-z_1)^2}z_1^\beta - \left(\frac{1}{6} - \frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)z_2^\beta,$$

$$\begin{aligned}
\mathfrak{C}_1(\beta) &= \left(\frac{1}{4} - \frac{(z_2-x)^4}{4(z_2-z_1)^4}\right)z_1^{\beta+1} + \left(\frac{1}{12} - \frac{(z_2-x)^3}{3(z_2-z_1)^3} + \frac{(z_2-x)^4}{4(z_2-z_1)^4}\right)z_1^\beta z_2 \\
&\quad + \left(\frac{1}{12} - \frac{(z_2-x)^3}{3(z_2-z_1)^3} + \frac{(z_2-x)^4}{4(z_2-z_1)^3}\right)z_2^{\beta-1}z_1^2 + \left(\frac{1}{12} - \frac{(z_2-x)^2}{2(z_2-z_1)^2}\right. \\
&\quad \left. - \frac{(z_2-x)^4}{4(z_2-z_1)^4} + \frac{2(z_2-x)^3}{3(z_2-z_1)^3}\right)z_1 z_2^\beta - \left(\frac{1}{2} - \frac{(z_2-x)^2}{2(z_2-z_1)^2}\right)z_1^{\beta+1} \\
&\quad + \left(\frac{(x-z_1)^2}{2(z_2-z_1)^2} + \frac{(x-z_1)^4}{4(z_2-z_1)^4} - \frac{2(x-z_1)^3}{3(z_2-z_1)^3}\right)z_1^\beta z_2 \\
&\quad + \left(\frac{(x-z_1)^3}{3(z_2-z_1)^3} - \frac{(x-z_1)^4}{4(z_2-z_1)^4}\right)z_1^{\beta-1}z_2^2 + \left(\frac{(x-z_1)^3}{3(z_2-z_1)^3} - \frac{(x-z_1)^4}{4(z_2-z_1)^4}\right)z_2^\beta z_1 \\
&\quad + \left(\frac{(x-z_1)^4}{4(z_2-z_1)^4}\right)z_2^{\beta+1} - \left(\frac{(x-z_1)^2}{2(z_2-z_1)^2}\right)z_2 z_1^\beta,
\end{aligned}$$

$$\begin{aligned}\mathfrak{C}_2(\beta) &= \left(\frac{(z_2-x)^2}{2(z_2-z_1)^2}\right) z_1 z_2^\beta - \left(\frac{(z_2-x)^3}{3(z_2-z_1)^3}\right) z_1^{\beta+1} - \left(\frac{1}{6} - \frac{(x-z_1)^2}{2(z_2-z_1)^2} + \frac{(x-z_1)^3}{3(z_2-z_1)^3}\right) z_1 z_2^\beta \\ &+ \left(\frac{1}{2} - \frac{(x-z_1)^2}{2(z_2-z_1)^2}\right) z_2^{\beta+1} - \left(\frac{1}{6} - \frac{(x-z_1)^2}{2(z_2-z_1)^2} + \frac{(x-z_1)^3}{3(z_2-z_1)^3}\right) z_2 z_1^\beta \\ &- \left(\frac{1}{3} - \frac{(x-z_1)^3}{3(z_2-z_1)^3}\right) z_2^{\beta+1}.\end{aligned}$$

Proof. Since $|h'|^q$ is concave so $|h'|$ is also concave (for details see [32]) Now taking into consideration Lemma 1, it follows that

$$\begin{aligned}&\left|h(x) - \frac{\beta}{z_2^\beta - z_1^\beta}\int_{z_1}^{z_2} h(s)d_\beta s\right| \\ \leq\ & \frac{z_2-z_1}{z_2^\beta - z_1^\beta}\left|\int_0^{\frac{x-z_1}{z_2-z_1}} \big(((1-t)z_1^{\beta-1} + tz_2^{\beta-1})((1-t)z_1+tz_2) - z_1^\beta\big) h'((1-t)z_1+tz_2)dt\right. \\ &\left. + \int_{\frac{x-z_1}{z_2-z_1}}^1 \big(z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta)\big) h'((1-t)z_1+tz_2)dt\right|.\end{aligned}$$

and applying Jensen's integral inequality, we have

$$\begin{aligned}&\int_0^{\frac{x-z_1}{z_2-z_1}} \big((((1-t)z_1^{\beta-1} + tz_2^{\beta-1}))(((1-t)z_1+tz_2)) - z_1^\beta\big)|h'(((1-t)z_1+tz_2))|dt \\ \leq\ & \left(\int_0^{\frac{x-z_1}{z_2-z_1}} \big((((1-t)z_1^{\beta-1} + tz_2^{\beta-1}))(((1-t)z_1+tz_2)) - z_1^\beta\big)dt\right) \\ &\times \left|h'\left(\frac{\int_0^{\frac{x-z_1}{z_2-z_1}} \Big(\big((((1-t)z_1^{\beta-1} + tz_2^{\beta-1}))(((1-t)z_1+tz_2)) - z_1^\beta\big)\Big)((1-t)z_1+tz_2)dt}{\int_0^{\frac{x-z_1}{z_2-z_1}} \big((((1-t)z_1^{\beta-1} + tz_2^{\beta-1}))(((1-t)z_1+tz_2)) - z_1^\beta\big)}\right)\right| \\ =\ & \mathfrak{A}(\beta) h'\left(\frac{\mathfrak{C}_1(\beta)}{\mathfrak{A}(\beta)}\right).\end{aligned}$$

Equivalently, we have

$$\begin{aligned}&\int_{\frac{x-z_1}{z_2-z_1}}^1 \Big(z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta)\Big)|h'((1-t)z_1+tz_2)|\,dt \\ \leq\ & \left(\int_{\frac{x-z_1}{z_2-z_1}}^1 (z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta))dt\right)\left|h'\left(\frac{\int_{\frac{x-z_1}{z_2-z_1}}^1 \Big(z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta)\Big)((1-t)z_1+tz_2)dt}{\int_{\frac{x-z_1}{z_2-z_1}}^1 \Big(z_2^\beta - ((1-t)z_1^\beta + tz_2^\beta)\Big)dt}\right)\right| \\ =\ & \mathfrak{B}(\beta) h'\left(\frac{\mathfrak{C}_2(\beta)}{\mathfrak{B}(\beta)}\right).\end{aligned}$$

■

Corollary 4 *Let $x = (z_1+z_2)/2$. Then Theorem 5 leads to*

$$\left|h\left(\frac{z_1+z_2}{2}\right) - \frac{\beta}{z_2^\beta - z_1^\beta}\int_{z_1}^{z_2} h(s)d_\beta s\right| \leq \frac{z_2-z_1}{z_2^\beta - z_1^\beta}\left[\mathfrak{A}(\beta)\left|h'\left(\frac{\mathfrak{C}_1(\beta)}{\mathfrak{A}(\beta)}\right)\right| + \mathfrak{B}(\beta)\left|h'\left(\frac{\mathfrak{C}_2(\beta)}{\mathfrak{B}(\beta)}\right)\right|\right], \tag{6}$$

where

$$\mathfrak{A}(\beta) = \frac{7}{24}z_1^\beta + \frac{1}{12}z_1^{\beta-1}z_2 + \frac{1}{12}z_2^{\beta-1}z_1 + \frac{1}{24}z_2^\beta - \frac{1}{2}z_1^\beta,$$

$$\mathfrak{B}(\beta) = \frac{1}{2}z_2^\beta - \frac{1}{8}z_1^\beta - \frac{3}{8}z_2^\beta,$$

$$\begin{aligned}\mathfrak{C}_1(\beta) \quad = \quad & \frac{15}{64}z_1^{\beta+1} + \frac{11}{192}z_1^\beta z_2 + \frac{11}{192}z_2^{\beta-1}z_1^2 + \frac{5}{192}z_1 z_2^\beta - \frac{3}{8}z_1^{\beta+1} \\ & + \frac{11}{192}z_1^\beta z_2 + \frac{5}{192}z_1^{\beta-1}z_2^2 + \frac{5}{192}z_2^\beta z_1 + \frac{1}{64}z_2^{\beta+1} - \frac{1}{8}z_2 z_1^\beta,\end{aligned}$$

$$\mathfrak{C}_2(\beta) = \frac{1}{8}z_1 z_2^\beta - \frac{1}{24}z_1^{\beta+1} - \frac{1}{12}z_1 z_2^\beta + \frac{3}{8}z_2^{\beta+1} - \frac{1}{12}z_2 z_1^\beta - \frac{7}{24}z_2^{\beta+1}.$$

Remark 4 *Let $\beta = 1$. The inequality (6) leads to*

$$\left|h\left(\frac{z_1+z_2}{2}\right) - \frac{1}{z_2-z_1}\int_{z_1}^{z_2} h(s)ds\right| \le \frac{z_2-z_1}{8}\left[\left|h'\left(\frac{2z_1+z_2}{3}\right)\right| + \left|h'\left(\frac{z_1+2z_2}{3}\right)\right|\right],$$

which was proved by U. S. Kirmaci in [29].

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