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Refinement Of Turán's Inequality For The Generalized Derivative Of A Polynomial*

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Abstract

Let $P(z)$ be a polynomial of degree at most n having all its zeros in $|z| \leq k,\ k \geq 1$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ be any non-zero n-dimensional vector of non-negative real numbers $\gamma_1, \gamma_2, \cdots, \gamma_n$. Then the extreme value of the modulus of the generalized derivative $P^\gamma(z)$ of $P(z)$ over $|z| = 1$ is related to the extreme value of $|P(z)|$ over $|z| = 1$ through the Turán type inequality

$$\max_{|z|=1} |P^\gamma(z)| \geq \frac{\Lambda}{1+k^n} \max_{|z|=1} |P(z)|,$$

where $\Lambda = \gamma_1 + \gamma_2 + \cdots + \gamma_n$. In this paper, we shall prove a refinement of the above inequality by involving the coefficients of $P(z)$. The obtained result, besides refining the above result, also generalizes a relevant inequality for classical derivative.

1 Introduction

Let $\mathcal{P}_n$ denote the space of all polynomials of degree at most n over the field $\mathbb{C}$ of complex numbers. The study of extremal properties for polynomials is a classical topic in analysis. One of the fundamental results in this regard is attributed to S. Bernstein [1] which states that if $P \in \mathcal{P}_n$ is of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

Equality in (1) holds for $P(z) = \alpha z^n,\ 0 \neq \alpha \in \mathbb{C}$. Inequality (1) can be sharpened for the polynomials with restricted zeros. For instance the following sharp inequality was conjectured by Erdös and later proved by Lax [7] that if $P \in \mathcal{P}_n$ is of degree n such that $P(z) \neq 0$ for $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

On the other hand, if we assume that all the zeros of $P(z)$ lie in $|z| \leq 1$, then the following inequality, reverse to that of (2), was obtained by Turán in [18] who proved that if $P \in \mathcal{P}_n$ is of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{3}$$

Inequalities (2) and (3) are sharp and equality in both holds for the polynomials whose zeros all lie on the unit circle. As a generalization of (3) to the circles with arbitrary radius less than or equal to one, Malik [8] considered the class of polynomials $P(z)$ having all zeros in $|z| \leq k,\ k \leq 1$ and proved that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{4}$$

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The result is best possible and equality holds for $P(z) = (z+k)^n$. Clearly for $k = 1$ inequality (4) reduces to inequality (3). The other case $k \geq 1$ of inequality (4) was settled by Govil [6] who established for $P \in \mathcal{P}_n$ with all zeros in $|z| \leq k,\ k \geq 1$, that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{5}$$

Inequality (5) is sharp and the extremal polynomial is $P(z) = z^n + k^n$. Many generalizations and extensions of inequalities (4) and (5) have been obtained by different authors over the last few decades. More recently Rather et al. [14], with the help of generalized form of Schwarz's Lemma, have involved the coefficient of the underlying polynomial and established a refinement of inequality (5). In fact, they proved that if $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\begin{aligned} \max_{|z|=1} |P'(z)| \quad \geq \quad & n \left\{ \frac{|a_0| + |a_n| k^{n+1}}{|a_0|(1+k^{n+1}) + |a_n|(k^{n+1}+k^{2n})} \right\} \max_{|z|=1} |P(z)| \\ & + \left\{ \frac{(|a_0| + k^{n-1}|a_n|)(k^{n+1} - h(k))}{|a_0|(k^{n+1}+1) + |a_n|(k^{2n}+k^{n+1})} \right\} |a_1|, \end{aligned} \tag{6}$$

where

$$h(k) = \begin{cases} k^{n-3}, & n > 2, \\ k, & n = 2. \end{cases}$$

The above result is best possible and equality holds for $P(z) = z^n + k^n$. The study of the relationship between the supremum norm of a complex polynomial and that of its derivative is a compelling topic in the field of approximation theory and the above mentioned inequalities are the fundamental ones in this field. Consequently, extensions of such polynomial inequalities, where the classical derivative is replaced by polar and generalized derivatives are explored in the literature, (for reference see [2, 3, 11, 12]). Continuing this exploration, we first introduce the concept of generalized derivative of a polynomial due to Sz-Nagy [16] as follows.

Definition 1 *If $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n, then for an n-tuple $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ of non-negative real numbers (not all zero), the generalized derivative of $P(z)$, usually denoted by $P^{\gamma}(z)$, is given as*

$$P^{\gamma}(z) = P(z) \sum_{j=1}^{n} \frac{\gamma_j}{z - z_j} = \sum_{j=1}^{n} \gamma_j P_j(z), \tag{7}$$

where $P_j(z) = a_n \prod_{i=1, i\neq j}^{n}(z - z_i)$. The ordinary derivative $P'(z)$ of $P(z)$ can be obtained from $P^{\gamma}(z)$ by choosing $\gamma = (1, 1, \cdots, 1)$.

The appearance of generalized derivative of a polynomial in the literature has motivated many researchers to verify and establish the above mentioned polynomial inequalities for the generalized derivative as well (see [2, 3]). In this spirit, Bhat et al. [3] have recently proved the following extensions of inequalities (4) and (5) to the generalized derivative. They proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, then for $0 \neq \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$,

$$\max_{|z|=1} |P^{\gamma}(z)| \geq \frac{\Lambda}{1+k} \max_{|z|=1} |P(z)|, \quad k \leq 1, \tag{8}$$

and

$$\max_{|z|=1} |P^{\gamma}(z)| \geq \frac{\Lambda}{1+k^n} \max_{|z|=1} |P(z)|, \quad k \geq 1, \tag{9}$$

where $\Lambda = \sum_{j=1}^{n} \gamma_j$. Inequality (8) is sharp and equality holds if $P(z) = (z+k)^n$.

2 Main Results

In this paper, we have established the following result which simultaneously provides a refinement of inequality (9) and a generalization of inequality (6) from classical to the generalized derivative. In fact, we prove

Theorem 1 *If $P(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$ of degree $n \geq 2$, has all its zeros in $|z| \leq k$, $k \geq 1$, then for each $0 \neq \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$,*

$$\begin{aligned}\max_{|z|=1} |P^{\gamma}(z)| \quad &\geq \quad \Lambda \left\{ \frac{|a_0| + |a_n|k^{n+1}}{|a_0|(1 + k^{n+1}) + |a_n|(k^{n+1} + k^{2n})} \right\} \max_{|z|=1} |P(z)| \\ &\quad + \left\{ \frac{(|a_0| + k^{n-1}|a_n|)(k^{n+1} - h(k))}{|a_0|(1 + k^{n+1}) + |a_n|(k^{n+1} + k^{2n})} \right\} |P^{\gamma}(0)|, \end{aligned} \tag{10}$$

where $h(k) = k^{n-3}$ for $n > 2$, $h(k) = k$ for $n = 2$ and $\Lambda = \sum_{j=1}^{n} \gamma_j$.

Remark 1 *If we take $\gamma = (1, 1, \cdots, 1)$ in Theorem 1, we obtain inequality (6).*

By choosing $\gamma = (0, 0, \cdots, \gamma_j, \cdots, 0)$, with $\gamma_j = 1$ for $1 \leq j \leq n$ in (10), we obtain the following result

Corollary 1 *If all the zeros $z_1, z_2, \cdots, z_n$ of $P(z) = \sum_{j=0}^{n} a_j z^j$ lie in $|z| \leq k$, $k \geq 1$, then for each j, $1 \leq j \leq n$,*

$$\max_{|z|=1} \left| \frac{P(z)}{z - z_j} \right| \geq \frac{|a_0| + |a_n|k^{n+1}}{|a_0|(1 + k^{n+1}) + |a_n|(k^{n+1} + k^{2n})} \max_{|z|=1} |P(z)| + \left\{ \frac{(|a_0| + k^{n-1}|a_n|)(k^{n+1} - h(k))}{|a_0|(k^{n+1} + 1) + |a_n|(k^{2n} + k^{n+1})} \right\} \frac{|a_0|}{k}.$$

Since $k \geq 1$, we see that for $n \geq 2$, we have $k^{n+1} - h(k) \geq 0$. Further note that

$$\frac{|a_0| + |a_n|k^{n+1}}{|a_0|(1 + k^{n+1}) + |a_n|(k^{n+1} + k^{2n})} \geq \frac{1}{1 + k^n},$$

if and only if

$$|a_0|k + |a_n|k^n \leq |a_0| + |a_n|k^{n+1},$$

or equivalently

$$\left| \frac{a_0}{a_n} \right| \leq k^n,$$

which is true by virtue of Viéte's formula. Using the above observation in Theorem 1, we obtain the following refinement of inequality (9).

Corollary 2 *If $P(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$ of degree $n \geq 2$, has all its zeros in $|z| \leq k$, $k \geq 1$, then for each $0 \neq \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, $\gamma_j \geq 0 \ \forall \ j = 1, 2, \cdots, n$,*

$$\max_{|z|=1} |P^{\gamma}(z)| \geq \frac{\Lambda}{1 + k^n} \max_{|z|=1} |P(z)| + \left\{ \frac{(|a_0| + k^{n-1}|a_n|)(k^{n+1} - h(k))}{|a_0|(k^{n+1} + 1) + |a_n|(k^{2n} + k^{n+1})} \right\} |P^{\gamma}(0)|, \tag{11}$$

where $h(k) = k^{n-3}$ for $n > 2$ and $h(k) = k$ for $n = 2$.

Taking $\gamma = (0, 0, \cdots, \gamma_j, \cdots, 0)$, with $\gamma_j = 1$ for $1 \leq j \leq n$ in (11), we get the following result.

Corollary 3 *If all the zeros $z_1, z_2, \cdots, z_n$ of $P(z) = \sum_{j=0}^{n} a_j z^j$ lie in $|z| \leq k$, $k \geq 1$, then for each j, $1 \leq j \leq n$ we have*

$$\max_{|z|=1} \left| \frac{P(z)}{z - z_j} \right| \geq \frac{1}{1 + k^n} \max_{|z|=1} |P(z)| + \left\{ \frac{(|a_0| + k^{n-1}|a_n|)(k^{n+1} - h(k))}{|a_0|(k^{n+1} + 1) + |a_n|(k^{2n} + k^{n+1})} \right\} \frac{|a_0|}{k}.$$

Computational Analysis: Consider the cubic polynomial $P(z) = (z+2)(z+3/2)(z+i)$. Then clearly $P(z)$ has all its zeros in $|z| \leq 2$. From inequality (9), we see that for $k = 2$ and $\gamma = (1/2, 1/3, 2)$,

$$\max_{|z|=1} |P^{\gamma}(z)| \geq 3.8564,$$

whereas Theorem 1 gives

$$\max_{|z|=1} |P^{\gamma}(z)| \geq 10.2124,$$

which is a significant improvement over the bound obtained from inequality (9).

3 Lemmas

For the proof of the main result, we require the following lemmas. The first lemma is due to Frappier et al. [5].

Lemma 1 *If $P(z)$ is a polynomial of degree n, then for $R \geq 1$*

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)|, \quad n \geq 2,$$

and

$$\max_{|z|=R} |P(z)| \leq R \max_{|z|=1} |P(z)| - (R-1)|P(0)|, \quad n = 1.$$

The following lemma is due to Diaz-Barrero and Egozcue [4].

Lemma 2 *All the zeros of $P^{\gamma}(z)$ lie in the convex hull of the zeros of $P(z)$.*

We also have the following well known generalization of Schwarz‘s Lemma, ([17, p. 212]).

Lemma 3 *If $f(z)$ is analytic in $|z| \leq 1$, with $f(0) = a$ and $|f(z)| \leq M$ for $|z| \leq 1$, then*

$$|f(z)| \leq M \frac{M|z| + |a|}{|z||a| + M}, \quad for \quad |z| \leq 1.$$

The next lemma is the generalization of Bernstein‘s inequality, (see [11, p. 510]).

Lemma 4 *Let $F(z)$ be a polynomial of degree n having all its zeros in the closed unit disc and $f(z)$ be a polynomial of degree $\leq n$ such that $|f(z)| \leq |F(z)|$ for $|z| = 1$. Then,*

$$|f'(z)| \leq |F'(z)| \quad for\ 1 \leq |z| < \infty.$$

The above inequality is best possible and equality holds at some point outside the closed unit disc if and only if $f(z) = e^{i\lambda}F(z)$ for some $\lambda \in \mathbb{R}$.

The following lemma is due to the authors with $k = 1$, (see [2, Lemma 9]).

Lemma 5 *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \neq 0$, we have*

$$|Q^{\gamma}(z)| \leq |P^{\gamma}(z)| \quad for\ |z| = 1,$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

We also need the following lemma due to Rather et al. [12].

Lemma 6 *If $P(z)$ is a polynomial of degree n, then for $|z| = 1$*

$$|P^{\gamma}(z)| + |Q^{\gamma}(z)| \geq \Lambda |P(z)|,$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

4 Proof of Main Result

By hypothesis $P(z)$ has all its zeros in $|z| \le k,\ k \ge 1$. Therefore all the zeros of $F(z) = P(kz)$ lie in $|z| \le 1$. Thus applying Lemma 5 to the polynomial $F(z)$, we get

$$|G^{\gamma}(z)| \le |F^{\gamma}(z)| \ for \ |z| = 1, \tag{12}$$

where $G(z) = z^n \overline{F\left(\frac{1}{\bar{z}}\right)}$. From inequality (12), it follows that a zero ω_j with $|\omega_j| = 1$ and multiplicity m_j of $F^{\gamma}(z)$ will also be a zero of $G^{\gamma}(z)$ of multiplicity at least m_j. Thus we can write

$$F^{\gamma}(z) = \psi(z) F_1(z) \tag{13}$$

and

$$G^{\gamma}(z) = \psi(z) G_1(z), \tag{14}$$

where

$$\psi(z) = \begin{cases} 1 & \text{if } F^{\gamma}(z) \ne 0 \text{ on } |z| = 1, \\ \prod_{j=1}^{p} (z - \omega_j)^{m_j} & \text{if } F^{\gamma}(z) \text{ has } p \text{ zeros on } |z| = 1,\ p \le n, \end{cases}$$

and $F_1(z) \ne 0$ for $|z| = 1$. From (12)–(14), we obtain

$$|G_1(z)| \le |F_1(z)| \tag{15}$$

for those points on $|z| = 1$, which are not the zeros of $\psi(z)$. Further, for the points on $|z| = 1$, where $\psi(z)$ vanishes, inequality (15) follows from Lemma 4 and repeated application of Gauss-Lucas theorem together with inequalities (12)–(14).

Now $F(z)$ has all its zeros in $|z| \le 1$, therefore by Lemma 2, all the zeros of $F^{\gamma}(z)$ lie in $|z| \le 1$. Thus the function

$$\phi(z) = \frac{G_1(z)}{F_1(z)}$$

is analytic in $|z| > \rho$, for some ρ with $0 < \rho < 1$ and accordingly the function $g(z) = \phi\left(\frac{1}{z}\right)$ is analytic in $|z| < \frac{1}{\rho},\ \frac{1}{\rho} > 1$, with

$$g(0) = \phi(\infty) = \lim_{z \to \infty} \phi(z) = \lim_{z \to \infty} \frac{G^{\gamma}(z)}{F^{\gamma}(z)} \tag{16}$$

Now, we have $F(z) = P(kz)$, therefore

$$F^{\gamma}(z) = P(kz) \sum_{j=1}^{n} \frac{\gamma_j}{z - \frac{z_j}{k}} = a_n k^n \prod_{i=1}^{n} \left(z - \frac{z_i}{k}\right) \sum_{j=1}^{n} \frac{\gamma_j}{z - \frac{z_j}{k}} = a_n k^n \Lambda z^{n-1} + \cdots lower\ terms.$$

Also

$$\begin{aligned} G^{\gamma}(z) &= G(z) \sum_{j=1}^{n} \frac{\gamma_j}{z - \frac{k}{\bar{z}_j}} \\ &= k^n Q\left(\frac{z}{k}\right) \sum_{j=1}^{n} \frac{\gamma_j}{z - \frac{k}{\bar{z}_j}} \left[\because G(z) = z^n \overline{F\left(\frac{1}{\bar{z}}\right)} = k^n Q\left(\frac{z}{k}\right),\ Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}\right] \\ &= k^n \bar{a}_n \prod_{i=1}^{n} \left(1 - \frac{z\bar{z}_i}{k}\right) \sum_{j=1}^{n} \frac{\gamma_j}{z - \frac{k}{\bar{z}_j}} \\ &= -\bar{a}_0 \Lambda z^{n-1} + \cdots \ lower\ terms. \end{aligned}$$

So that from (16),

$$g(0) = -\frac{\bar{a}_0}{a_n k^n}.$$

Further, from (15) we have $|\phi(z)| \leq 1$ for $|z| = 1$, therefore $|g(z)| \leq 1$ for $|z| = 1$. Applying Lemma 3 to the function $g(z)$, we obtain

$$|g(z)| \leq \frac{|z| + \left|\frac{a_0}{a_n k^n}\right|}{\left|\frac{a_0}{a_n k^n}\right| |z| + 1} \; for \; |z| \leq 1,$$

i.e.,

$$|g(re^{i\theta})| \leq \frac{|a_n|k^n r + |a_0|}{|a_0|r + |a_n|k^n}, \; 0 < r \leq 1 \; and \; 0 \leq \theta < 2\pi,$$

Since $g(z) = \phi(1/z)$, we get

$$\left|\phi\left(\frac{1}{r}e^{-i\theta}\right)\right| \leq \frac{|a_n|k^n r + |a_0|}{|a_0|r + |a_n|k^n}, \; 0 < r \leq 1 \; and \; 0 \leq \theta < 2\pi,$$

i.e.,

$$|\phi(Re^{-i\theta})| \leq \frac{|a_n|k^n + |a_0|R}{|a_0| + |a_n|k^n R}, \; R \geq 1 \; and \; 0 \leq \theta < 2\pi,$$

or,

$$|G_1(Re^{-i\theta})| \leq \frac{|a_n|k^n + |a_0|R}{|a_0| + |a_n|k^n R}|F_1(Re^{-i\theta})|, \; R \geq 1 \; and \; 0 \leq \theta < 2\pi,$$

which by using (13) and (14) implies

$$|G^\gamma(z)| \leq \frac{|a_n|k^n + |a_0||z|}{|a_0| + |a_n|k^n|z|}|F^\gamma(z)|, \; |z| \geq 1. \tag{17}$$

Further $F(z) = P(kz)$, therefore

$$F^\gamma(z) = F(z)\sum_{j=1}^{n}\frac{\gamma_j}{z - \frac{z_j}{k}} = P(kz)\sum_{j=1}^{n}\frac{k\gamma_j}{kz - z_j} = kP^\gamma(kz). \tag{18}$$

Also

$$G^\gamma(z) = G(z)\sum_{j=1}^{n}\frac{\gamma_j}{z - \frac{k}{\bar{z}_j}} = G(z)\sum_{j=1}^{n}\frac{\gamma_j\bar{z}_j}{z\bar{z}_j - k} = k^n Q\left(\frac{z}{k}\right)\sum_{j=1}^{n}\frac{\gamma_j\bar{z}_j}{z\bar{z}_j - k} = k^{n-1}Q\left(\frac{z}{k}\right)\sum_{j=1}^{n}\frac{\gamma_j\bar{z}_j}{\frac{z\bar{z}_j}{k} - 1}$$

i.e.

$$G^\gamma(z) = k^{n-1}Q^\gamma\left(\frac{z}{k}\right). \tag{19}$$

Using (18) and (19) in (17), we get

$$k^{n-2}|Q^\gamma(z/k)| \leq \frac{|a_n|k^n + |a_0||z|}{|a_0| + |a_n|k^n|z|}|P^\gamma(kz)|, \; |z| \geq 1.$$

Now replacing z by $ke^{i\theta}$, $0 < \theta \leq 2\pi$, we get

$$k^{n-3}|Q^\gamma(e^{i\theta})| \leq \frac{|a_n|k^{n-1} + |a_0|}{|a_0| + |a_n|k^{n+1}}|P^\gamma(k^2e^{i\theta})|,$$

which implies

$$k^{n-3}\max_{|z|=1}|Q^\gamma(z)| \leq \frac{|a_n|k^{n-1} + |a_0|}{|a_0| + |a_n|k^{n+1}}\max_{|z|=k^2}|P^\gamma(z)|.$$

Now invoking Lemma 1, we get from the above inequality that

$$k^{n-3}\max_{|z|=1}|Q^\gamma(z)| \leq \frac{|a_n|k^{n-1} + |a_0|}{|a_0| + |a_n|k^{n+1}}\left\{k^{2n-2}\max_{|z|=1}|P^\gamma(z)| - (k^{2n-2} - k^{2n-6})|P^\gamma(0)|\right\} \; for \; n > 2$$

and

$$k^{-1}\max_{|z|=1}|Q^{\gamma}(z)| \leq \frac{|a_n|k+|a_0|}{|a_0|+|a_n|k^3}\left\{k^2\max_{|z|=1}|P^{\gamma}(z)|-(k^2-1)|P^{\gamma}(0)|\right\}\ for\ n=2.$$

Or equivalently

$$\max_{|z|=1}|Q^{\gamma}(z)| \leq \frac{|a_n|k^{n-1}+|a_0|}{|a_0|+|a_n|k^{n+1}}\left\{k^{n+1}\max_{|z|=1}|P^{\gamma}(z)|-(k^{n+1}-h(k))|P^{\gamma}(0)|\right\}, \tag{20}$$

where $h(k)=\begin{cases}k^{n-3}, & n>2\\ k, & n=2.\end{cases}$

Now from Lemma 6, we have

$$\max_{|z|=1}|P^{\gamma}(z)|+\max_{|z|=1}|Q^{\gamma}(z)| \geq \Lambda\max_{|z|=1}|P(z)|. \tag{21}$$

Combining (20) and (21), we get

$$\Lambda\max_{|z|=1}|P(z)| \leq \max_{|z|=1}|P^{\gamma}(z)|+\frac{|a_0|+|a_n|k^{n-1}}{|a_0|+k^{n+1}|a_n|}\left\{k^{n+1}\max_{|z|=1}|P^{\gamma}(z)|-\left(k^{n+1}-h(k)\right)|P^{\gamma}(0)|\right\},$$

or

$$\left(1+\frac{|a_0|+|a_n|k^{n-1}}{|a_0|+k^{n+1}|a_n|}k^{n+1}\right)\max_{|z|=1}|P^{\gamma}(z)| \geq \Lambda\max_{|z|=1}|P(z)|+\frac{|a_0|+|a_n|k^{n-1}}{|a_0|+k^{n+1}|a_n|}(k^{n+1}-h(k))|P^{\gamma}(0)|.$$

Equivalently,

$$\begin{aligned}\max_{|z|=1}|P^{\gamma}(z)| \geq \Lambda&\left\{\frac{|a_0|+|a_n|k^{n+1}}{|a_0|(1+k^{n+1})+|a_n|(k^{n+1}+k^{2n})}\right\}\max_{|z|=1}|P(z)|\\ &+\left\{\frac{(|a_0|+k^{n-1}|a_n|)(k^{n+1}-h(k))}{|a_0|(k^{n+1}+1)+|a_n|(k^{2n}+k^{n+1})}\right\}|P^{\gamma}(0)|.\end{aligned}$$

This completes the proof.

5 Conclusion

In this paper, we have provided a refined version of a known inequality concerning the extremal properties of polynomials. By establishing an enhanced bound, we have demonstrated a significant improvement to a known inequality concerning the generalized derivative of a complex polynomial. Our approach highlights the effectiveness of using a generalized form of Schwarz's Lemma in improving the polynomial inequalities. The methodology used in this study is not only applicable to the specific inequality studied, but it also suggests its usefulness in improving similar types of inequalities.

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