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Stringent Bounds For The Non-Zero Bernoulli Numbers*

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Abstract

We present new sharper lower and upper bounds for the non-zero Bernoulli numbers B_k using Euler's formula for the Riemann zeta function. In particular, we determine the best possible constants α and β such that the double inequality

$$\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\alpha)}<|B_{2k}|<\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\beta)},$$

holds for $k=1,2,3,....$ Our main results refine the existing bounds of $|B_{2k}|$ in the literature.

1 Introduction

The classical Bernoulli numbers frequently occur in mathematical analysis and other branches of mathematics and science. These fascinating numbers, which form a sequence of rational numbers, were discovered by the Swiss mathematician Jacob Bernoulli. They are denoted by B_k and may be defined by ([1, p. 804], [10, p. 3])

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!},\ |x|<2\pi.$$

On the one hand, all the odd-indexed Bernoulli numbers B_{2k+1}, where $k\in\{0,1,2,...\}$ are zero except $B_1=-1/2$. On the other hand, all the even-indexed Bernoulli numbers B_{2k} are non-zero and they alternate in sign, i.e., for $k=1,2,...$, the numbers B_{2k} are respectively given by $1/6$, $-1/30$, $1/42$, $-1/30$, $5/66$, $-691/2730$, $7/6$, $-3617/510,....$ Hence,

$$|B_{2k}|=(-1)^{k+1}B_{2k},\ k=1,2,3,....$$

The bounds for even-indexed Bernoulli numbers play a vital role in the theory of inequalities; thus, bounding these Bernoulli numbers has been the topic of interest.

The double inequality

$$\frac{2\cdot(2k)!}{(2\pi)^{2k}}<|B_{2k}|<\frac{2\cdot(2k)!}{(2\pi)^{2k}}\cdot\frac{2^{2k-1}}{(2^{2k-1}-1)}=\frac{(2k)!}{\pi^{2k}(2^{2k-1}-1)};\ k=1,2,... \tag{1}$$

appeared in [1, p. 805]. In 1989, D. J. Leeming [7, p. 128] established that

$$4\sqrt{\pi k}\left(\frac{k}{\pi e}\right)^{2k}<|B_{2k}|<5\sqrt{\pi k}\left(\frac{k}{\pi e}\right)^{2k};\ k\geq 2. \tag{2}$$

As stated in [3] the left inequality of (2) was already established by A. Laforgia [6, p. 2] in 1980. By utilizing the Fourier representation of $B_{2k}(x)$ at $x=0$, i.e.,

$$B_{2k}=\frac{(-1)^{k-1}\cdot 2\cdot(2k)!}{(2\pi)^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}, \tag{3}$$

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C. D'Aniello [3] obtained

$$\frac{2\cdot(2k)!}{(2\pi)^{2k}}\left(1+\frac{1}{2^{2k}}\right)<|B_{2k}|;\ k=1,2,.... \tag{4}$$

Further, using the Stirling formula [1], the inequality

$$4\sqrt{\pi k}\left(\frac{k}{\pi e}\right)^{2k}\left(1+\frac{1}{2^{2k}}\right)<|B_{2k}|;\ k=1,2,...$$

was also obtained in [3]. In the same paper [3], a relation [1, p. 807]

$$|B_{2k}|=\frac{2\cdot(2k)!}{\pi^{2k}}\frac{1}{(2^{2k}-1)}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k}};\ k=1,2,... \tag{5}$$

was used to establish

$$\frac{2\cdot(2k)!}{\pi^{2k}}\frac{1}{(2^{2k}-1)}<|B_{2k}|;\ k=1,2,.... \tag{6}$$

Among all the lower bounds of $|B_{2k}|$ listed above, the lower bound in (6) is stringent. Thus, by combining (1) and (6) we have

$$\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}<|B_{2k}|<\frac{(2k)!}{\pi^{2k}(2^{2k-1}-1)};\ k=1,2,.... \tag{7}$$

For $k\in\mathbb{N}$ (the set of positive integers), H. Alzer [2] showed that the constants $\theta=0$ and $\delta=2+\frac{\ln(1-6/\pi^2)}{\ln 2}$ such that

$$\frac{2\cdot(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\theta-2k}}\le|B_{2k}|\le\frac{2\cdot(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\delta-2k}} \tag{8}$$

are the best possible. The upper bound in (8) is sharper than that in (7). Another upper bound for $|B_{2k}|$ was given by H.-F. Ge [5] as follows:

$$|B_{2k}|\le\frac{2(2^{2n}-1)}{2^{2n}}\zeta(2n)\frac{(2k)!}{\pi^{2k}(2^{2k}-1)};\ k\ge n \text{ and } n\in\mathbb{N}, \tag{9}$$

where ζ is the Riemann zeta function.

The upper bound in (7) has been nicely sharpened, viz. in (8). However, to the best of the author's knowledge and belief, the lower bound in (7) has not been presented in a refined form. In fact, from (3) and (5) we immediately have the following proposition.

Proposition 1 *(i) For $m,k\in\mathbb{N}$, we have*

$$\frac{2\cdot(2k)!}{(2\pi)^{2k}}\sum_{n=1}^{m}\frac{1}{n^{2k}}<|B_{2k}|. \tag{10}$$

(ii) For $m=0,1,2,...$ and $k\in\mathbb{N}$ we have

$$\frac{2\cdot(2k)!}{\pi^{2k}}\frac{1}{(2^{2k}-1)}\sum_{n=0}^{m}\frac{1}{(2n+1)^{2k}}<|B_{2k}|. \tag{11}$$

It is not difficult to prove that the lower bound in (11) is finer than that in (10) for $m\ge1$. Putting $m=2$ and $m=0$ in (10) and (11) respectively we get the inequalities (4) and (6). To obtain sharper lower bounds, one should increase the value of m in (10) and (11). For example, by putting $m=3$ and $m=1$ in (10) and (11) respectively we get

$$\frac{2\cdot(2k)!}{(2\pi)^{2k}}\left(1+\frac{1}{2^{2k}}+\frac{1}{3^{2k}}\right)<|B_{2k}|;\ k=1,2,...$$

and

$$\frac{2\cdot(2k)!}{\pi^{2k}}\frac{1}{(2^{2k}-1)}\left(1+\frac{1}{3^{2k}}\right)<|B_{2k}|;\ k=1,2,.... \tag{12}$$

Despite this, we will present more stringent and convincing lower bounds for $|B_{2k}|$ in the next section. Our main aim is to refine the double inequality (8) by establishing sharp bounds for $|B_{2k}|$.

2 Main Results

Let us first define $\mathcal{P}_0 = \{p_0 = 2,\ p_1 = 3,\ p_2 = 5,\ p_3 = 7,\ p_4 = 11,\ p_5 = 13,\ p_6 = 17, ...\}$ as the set of all positive prime integers and $\mathcal{P} = \{p_1 = 3,\ p_2 = 5,\ p_3 = 7,\ p_4 = 11,\ p_5 = 13,\ p_6 = 17, ...\}$ as the set of all positive odd prime integers.

We start with the following basic inequality which will motivate us to establish the main result of the paper.

Proposition 2 *The inequality*

$$\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}\prod_{n=1}^{m}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right)<|B_{2k}|, \tag{13}$$

holds for $m,k\in\mathbb{N}$ and $p_n\in\mathcal{P}$.

Proof. Using Euler's fabulous product formula [4] for the zeta function ζ, we have

$$\zeta(2k)=\left(\frac{2^{2k}}{2^{2k}-1}\right)\prod_{n\in\mathbb{N}}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right),$$

where $k\in\mathbb{N},\ p_n\in\mathcal{P}$. This implies

$$\zeta(2k)>\left(\frac{2^{2k}}{2^{2k}-1}\right)\prod_{n=1}^{m}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right).$$

It is given in [10, p. 5, (1.14)] that

$$|B_{2k}|=\frac{2\cdot(2k)!}{(2\pi)^{2k}}\zeta(2k),\ k\in\mathbb{N}.$$

Hence

$$\frac{\pi^{2k}(2^{2k}-1)}{2\cdot(2k)!}|B_{2k}|>\prod_{n=1}^{m}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right),$$

which gives the desired result. ■

In particular, for $m=1,2$, Proposition 2 yields respectively the following inequalities:

$$\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-1)}<|B_{2k}|,\ k\in\mathbb{N} \tag{14}$$

and

$$\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-1)}\frac{5^{2k}}{(5^{2k}-1)}<|B_{2k}|,\ k\in\mathbb{N}. \tag{15}$$

It is easy to check that the inequalities (14) and (15) are refinements of the inequality (12). In Proposition 2, we also observe that the larger the value of m, the sharper is the corresponding inequality.

Remark 1 *The statement of Proposition 2 can be reformulated as follows:*
If m is a non-negative integer, then

$$\frac{2\cdot(2k)!}{(2\pi)^{2k}}\prod_{n=0}^{m}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right)<|B_{2k}|, \tag{16}$$

where $k\in\mathbb{N}$ and $p_n\in\mathcal{P}_0$.

Here, (16) generalizes the inequality (6). The next Proposition shows that the lower bounds in Proposition 2 are sharper than those in Proposition 1.

Proposition 3 *For $k\in\mathbb{N}$ and $p_n\in\mathcal{P}$, it is true that*

$$\prod_{n=1}^{m}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right)>\sum_{n=0}^{m}\frac{1}{(2n+1)^{2k}},$$

Proof. For $k\in\mathbb{N}$ and $p_n\in\mathcal{P}$, we write

$$\prod_{n=1}^{m}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right)=\prod_{n=1}^{m}\left(\frac{1}{1-\frac{1}{p_n^{2k}}}\right),$$

i.e.,

$$\prod_{n=1}^{m}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right)=\left(\frac{1}{1-\frac{1}{3^{2k}}}\right)\cdot\left(\frac{1}{1-\frac{1}{5^{2k}}}\right)\cdot\left(\frac{1}{1-\frac{1}{7^{2k}}}\right)\cdot\left(\frac{1}{1-\frac{1}{11^{2k}}}\right)\cdots\left(\frac{1}{1-\frac{1}{p_m^{2k}}}\right).$$

Making use of

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots,\ |x|<1,$$

we get

$$\begin{aligned}\prod_{n=1}^{m}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right) &= \left(1+\frac{1}{3^{2k}}+\frac{1}{9^{2k}}+\frac{1}{3^{6k}}+\cdots\right)\times\left(1+\frac{1}{5^{2k}}+\frac{1}{5^{4k}}+\frac{1}{5^{6k}}+\cdots\right)\\ &\times\left(1+\frac{1}{7^{2k}}+\frac{1}{7^{4k}}+\frac{1}{7^{6k}}+\cdots\right)\times\left(1+\frac{1}{11^{2k}}+\frac{1}{11^{4k}}+\frac{1}{11^{6k}}+\cdots\right)\\ &\times\cdots\times\left(1+\frac{1}{p_m^{2k}}+\frac{1}{p_m^{4k}}+\frac{1}{p_m^{6k}}+\cdots\right).\end{aligned}$$

Thus

$$\begin{aligned}\prod_{n=1}^{m}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right) &= \left(1+\frac{1}{3^{2k}}+\frac{1}{5^{2k}}+\frac{1}{7^{2k}}+\frac{1}{9^{2k}}\cdots+\frac{1}{(2m+1)^{2k}}+\cdots+\frac{1}{p_m^{2k}}\right)\\ &+\text{sum of numbers greater than zero}\\ &>1+\frac{1}{3^{2k}}+\frac{1}{5^{2k}}+\frac{1}{7^{2k}}+\frac{1}{9^{2k}}\cdots+\frac{1}{(2m+1)^{2k}}=\sum_{n=0}^{m}\frac{1}{(2n+1)^{2k}}.\end{aligned}$$

The proof is completed. ■

Let L_{2k} be the lower bound of $|B_{2k}|$ established in (13). For $m=1$, we list the approximate numerical values $|B_{2k}|-L_{2k}$ in the below table.

From these values, we infer that the lower bound in (13) is sufficiently sharp even for $m=1$. One should get sharper and sharper lower bounds for larger values of m.

Table 1: Numerical values of $|B_{2k}| - L_{2k}$ for $m = 1$.

	$m = 1$				
k	1	2	3	4	5
$\lvert B_{2k}\rvert - L_{2k}$	$1.46... \times 10^{-2}$	$7.15... \times 10^{-5}$	$1.74... \times 10^{-6}$	$9.13... \times 10^{-8}$	$8.02... \times 10^{-9}$
	$m = 1$				
k	6	7	8	9	10
$\lvert B_{2k}\rvert - L_{2k}$	$1.05... \times 10^{-9}$	$1.92... \times 10^{-10}$	$4.66... \times 10^{-11}$	$1.43... \times 10^{-11}$	$5.11... \times 10^{-12}$

Corollary 1 *Suppose that a is any real number and $k \in \mathbb{N}$. Then*

$$\frac{2 \cdot (2k)!}{\pi^{2k}(2^{2k} - 1)} \frac{a^{2k}}{(a^{2k} - 1)} < |B_{2k}| \quad \textit{if } a \geq 3. \tag{17}$$

Proof. For $a \geq 3$, it is obvious that

$$\frac{a^{2k}}{(a^{2k} - 1)} \leq \frac{3^{2k}}{(3^{2k} - 1)}.$$

Then by (14), we get the inequality (17). ■

Remark 2 *Applying the same argument as in the proof of Corollary 1, for any real number a and $m, k \in \mathbb{N}$, we have*

$$\frac{2 \cdot (2k)!}{\pi^{2k}(2^{2k} - 1)} \frac{3^{2k}}{(3^{2k} - 1)} \frac{5^{2k}}{(5^{2k} - 1)} \cdots \frac{p_{m-1}^{2k}}{(p_{m-1}^{2k} - 1)} \frac{a^{2k}}{(a^{2k} - 1)} < |B_{2k}| \tag{18}$$

if $a \geq p_m$, where $p_i \in \mathcal{P}$.

Table 1 shows that the inequality (14) is very sharp. It is of particular interest and it motivates us to ask the natural question: What can be the best possible constants α and β such that the double inequality

$$\frac{2 \cdot (2k)!}{\pi^{2k}(2^{2k} - 1)} \frac{3^{2k}}{(3^{2k} - \alpha)} < |B_{2k}| < \frac{2 \cdot (2k)!}{\pi^{2k}(2^{2k} - 1)} \frac{3^{2k}}{(3^{2k} - \beta)},$$

holds for $k = 1, 2, ...$?

The answer to the above question is provided in the following theorem, which is our main finding.

Theorem 1 *The best possible constants α and β satisfying the double inequality*

$$\frac{2 \cdot (2k)!}{\pi^{2k}(2^{2k} - 1)} \frac{3^{2k}}{(3^{2k} - \alpha)} < |B_{2k}| < \frac{2 \cdot (2k)!}{\pi^{2k}(2^{2k} - 1)} \frac{3^{2k}}{(3^{2k} - \beta)}, \quad k \in \mathbb{N} \tag{19}$$

are 1 and $9\left(1 - \frac{8}{\pi^2}\right) \approx 1.704875$ respectively.

Proof. Utilizing the same relation

$$\zeta(2k) = \frac{(2\pi)^{2k}}{2(2k)!} |B_{2k}|, \;\; k \in \mathbb{N}$$

as in the proof of Proposition 2, the double inequality (19) can be equivalently written as

$$\frac{2^{2k}}{(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\alpha)}<\zeta(2k)<\frac{2^{2k}}{(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\beta)},$$

i.e.,

$$\alpha<3^{2k}\left[1-\frac{1}{\left(1-\frac{1}{2^{2k}}\right)\zeta(2k)}\right]<\beta,\ k\in\mathbb{N}.$$

We define

$$h(x)=3^{x}\left[1-\frac{1}{\left(1-\frac{1}{2^{x}}\right)\zeta(x)}\right],\ x=\{2,4,6,8,...\}.$$

However, we prove the monotonicity of $h(x)$ for $x\in\mathbb{N}-\{1\}$. Consider $g(x)=h(x)-h(x+1)$. Then

$$g(x)=3^{x}\left[1-\frac{1}{\left(1-\frac{1}{2^{x}}\right)\zeta(x)}\right]-3^{x+1}\left[1-\frac{1}{\left(1-\frac{1}{2^{x+1}}\right)\zeta(x+1)}\right]:=3^{x}\cdot t(x),$$

where

$$\begin{aligned}t(x)&=\frac{3}{\left(1-\frac{1}{2^{x+1}}\right)\zeta(x+1)}-\frac{1}{\left(1-\frac{1}{2^{x}}\right)\zeta(x)}-2\\&=\frac{3\left(1-\frac{1}{2^{x}}\right)\zeta(x)-\left(1-\frac{1}{2^{x+1}}\right)\zeta(x+1)}{\left(1-\frac{1}{2^{x+1}}\right)\zeta(x+1)\cdot\left(1-\frac{1}{2^{x}}\right)\zeta(x)}-2.\end{aligned}$$

Now using Euler's thinking [4, p. 15], we have

$$\left(1-\frac{1}{2^{x}}\right)\zeta(x)=1+\frac{1}{3^{x}}+\frac{1}{5^{x}}+\frac{1}{7^{x}}+\frac{1}{9^{x}}+\cdots=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{x}}.$$

Therefore

$$\begin{aligned}t(x)&=\frac{3\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)^{x}-\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)^{x+1}}{\left[\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)^{x+1}\right]\left[\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)^{x}\right]}-2\\&=\frac{2\sum_{n=0}^{\infty}\left(\frac{3n+1}{2n+1}\right)\left(\frac{1}{2n+1}\right)^{x}}{\sum_{n=0}^{\infty}\sum_{l=0}^{n}\left(\frac{1}{2l+1}\right)^{x+1}\left(\frac{1}{2n-2l+1}\right)^{x}}-2\\&:=\frac{2\sum a_n}{\sum b_n}-2.\end{aligned}$$

The actual terms of the series $\sum b_n$ are given by

$$\begin{aligned}\sum b_n\ &=\ 1+\frac{(4/3)}{3^{x}}+\frac{(6/5)}{5^{x}}+\frac{(8/7)}{7^{x}}+\frac{(13/9)}{9^{x}}+\frac{(12/11)}{11^{x}}+\frac{(14/13)}{13^{x}}+\frac{(24/15)}{15^{x}}\\&\quad+\frac{(18/17)}{17^{x}}+\frac{(20/19)}{19^{x}}+\frac{(32/21)}{21^{x}}+\cdots.\end{aligned}$$

Similarly, the terms of the series $\sum a_n$ can be written as

$$\sum a_n\ =\ 1+\frac{(4/3)}{3^{x}}+\left[\frac{(6/5)}{5^{x}}+\frac{(1/5)}{5^{x}}\right]+\left[\frac{(8/7)}{7^{x}}+\frac{(2/7)}{7^{x}}\right]+\frac{(13/9)}{9^{x}}+\left[\frac{(12/11)}{11^{x}}+\frac{(4/11)}{11^{x}}\right]$$

$$+\left[\frac{(14/13)}{13^x}+\frac{(5/13)}{13^x}\right]+\left[\frac{(24/15)}{15^x}-\frac{(2/15)}{15^x}\right]+\left[\frac{(18/17)}{17^x}+\frac{(7/17)}{17^x}\right]$$
$$+\left[\frac{(20/19)}{19^x}+\frac{(8/19)}{19^x}\right]+\left[\frac{(32/21)}{21^x}-\frac{(1/21)}{21^x}\right]+\cdots,$$

i.e.,

$$\sum a_n=\sum b_n+\left[\frac{(1/5)}{5^x}+\frac{(2/7)}{7^x}+\frac{(4/11)}{11^x}+\frac{(5/13)}{13^x}-\frac{(2/15)}{15^x}+\frac{(7/17)}{17^x}+\frac{(8/19)}{19^x}-\frac{(1/21)}{21^x}+\cdots\right].$$

From this, it is obvious that $\sum a_n>\sum b_n$. So we get $t(x)>0$ and hence $g(x)>0$, which implies that $h(x)$ is strictly decreasing for $x\in\mathbb{N}-\{1\}$. Thus, $h(x)$ is strictly decreasing for $x\in\{2,4,6,\cdots\}$. Consequently, $\alpha=\lim_{x\to\infty^-}h(x)=1$ and $\beta=h(2)=9\left(1-\frac{8}{\pi^2}\right)$. This completes the proof. ■

Remark 3 *The lower bound in (19) is clearly stronger than that in (8). The upper bound in (19) is also stronger than that in (8). Because*

$$\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\beta)}\le\frac{2\cdot(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\delta-2k}}$$
$$\Longleftrightarrow\frac{1}{(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\beta)}\le\frac{1}{(2^{2k}-2^{\delta})}$$
$$\Longleftrightarrow\beta\cdot(2^{2k}-1)\le(2^{\delta}-1)\cdot3^{2k}.$$

The last inequality holds for all $k\in\mathbb{N}$ due to

$$\frac{\beta}{(2^{\delta}-1)}=\frac{9\left(1-\frac{8}{\pi^2}\right)}{4\cdot2^{\frac{\ln(1-6/\pi^2)}{\ln 2}}-1}=\frac{9\left(1-\frac{8}{\pi^2}\right)}{4\left(1-\frac{6}{\pi^2}\right)-1}=3.$$

If we set

$$E_1(k)=\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\alpha)}-\frac{2\cdot(2k)!}{(2\pi)^{2k}(1-2^{-2k})}=\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}\left(\frac{3^{2k}}{3^{2k}-1}-1\right)$$

and

$$E_2(k)=\frac{2\cdot(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\delta-2k}}-\frac{2\cdot(2k)!}{\pi^{2k}(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\beta)}$$
$$=\frac{2\cdot(2k)!}{\pi^{2k}}\left(\frac{1}{(2^{2k}-2^{\delta})}-\frac{3^{2k}}{(2^{2k}-1)(3^{2k}-\beta)}\right),$$

then the validity of our claim in Remark 3 is also demonstrated in Figure 1.

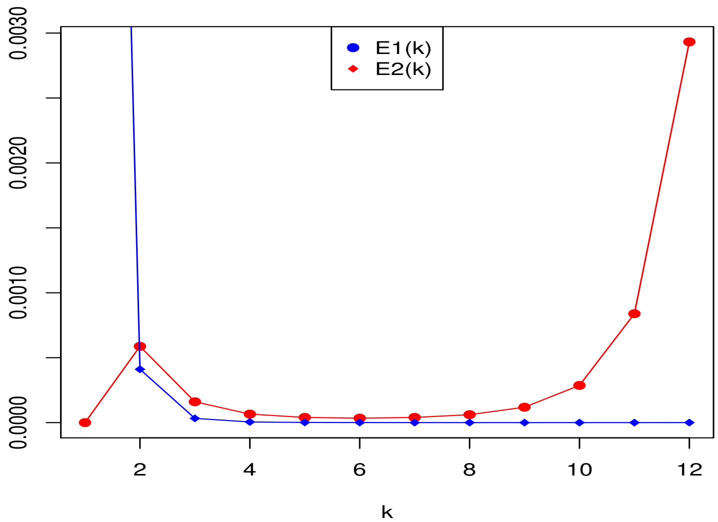

Figure 1: Plots of $E_1(k)$ and $E_2(k)$ showing the difference between corresponding lower and upper bounds of $|B_{2k}|$ in (8) and (19) for $k = 1, 2, 3, \cdots 12$.

Lastly, it is not difficult to show or verify that an upper bound in (19) is stronger than that in (9).

3 Conclusion

In this paper, several bounds for the non-zero Bernoulli numbers have been obtained. The double inequality (19) is of particular interest and it improves all the existing bounds of non-zero Bernoulli numbers in the literature. Due to the appearance of the Bernoulli numbers in the series expansions of trigonometric and hyperbolic functions, the author believes that inequality (19) will play a crucial role in obtaining fine inequalities involving these functions. Further, it should be noted that a function $h(x)$ in the proof of Theorem 1 can be shown strictly decreasing for all real numbers $x \geq 2$ by another method which may be applied to prove the following conjecture.

Conjecture 1. The double inequality

$$\prod_{n=1}^{m-1}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right) \times \frac{p_m^{2k}}{(p_m^{2k}-\alpha')} < \frac{\pi^{2k}(2^{2k}-1)}{2\cdot(2k)!}|B_{2k}| < \prod_{n=1}^{m-1}\left(\frac{p_n^{2k}}{p_n^{2k}-1}\right) \times \frac{p_m^{2k}}{(p_m^{2k}-\beta')}, \text{ where } m, k \in \mathbb{N},$$

$m \geq 2$ and $p_n \in \mathcal{P}$ holds with the best possible constants

$$\alpha' = 1 \quad \text{and} \quad \beta' = p_m^2\left[1 - \frac{8}{\pi^2}\prod_{n=1}^{m-1}\frac{p_n^2}{(p_n^2-1)}\right].$$

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