Applied Mathematics E-Notes, 26(2026), 223-232 © ISSN 1607-2510
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New Inequalities For Log-Convex Functions And Their Applications*

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Received 26 July 2024

Abstract

Let $f:[0,1]\longrightarrow(0,+\infty)$ be a log-convex function and $0\le\lambda\le1$. Then, for every positive integer m, we prove that

$$\begin{aligned} & f^m(\lambda)+r_0^m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)\\ & +r_m\left(\left((f(0)f(1))^{\frac{m}{4}}-f^{\frac{m}{2}}(0)\right)^2\chi_{\left(0,\frac{1}{2}\right]}(\lambda)+\left((f(0)f(1))^{\frac{m}{4}}-f^{\frac{m}{2}}(1)\right)^2\chi_{\left(\frac{1}{2},1\right]}(\lambda)\right)\\ \le\ & ((1-\lambda)f(0)+\lambda f(1))^m, \end{aligned}$$

where $r_0=\min\{\lambda,1-\lambda\}$, $r_m=\min\{(m+1)r_0^m,(1-r_0)^m-r_0^m\}$ and χ_I stands for the characteristic function. As a consequence of this inequality, we present generalized refinements of the difference between the arithmetic-power mean and arithmetic-geometric mean inequalities for scalars. Further, we give some improvements of Young-type inequalities for the traces, determinants and p-norms of positive τ-measurable operators.

1 Introduction and Preliminaries

Convex functions are widely used in various fields of mathematics, such as functional analysis, optimization theory, mathematical economics, etc. Their properties have received a lot of attention from researchers in recent years, see for instance [18] for an exhaustive bibliography.

The theory of convex functions is closely connected to the theory of mathematical inequalities. Indeed, any convex function $f:I\longrightarrow\mathbb{R}$ defined on a real interval I satisfies the following inequality

$$f\Big((1-\lambda)a+\lambda b\Big)\le(1-\lambda)f(a)+\lambda f(b), \tag{1}$$

for all $a,b\in I$ and $\lambda\in[0,1]$.

It is widely known that a twice differentiable function $f:I\longrightarrow\mathbb{R}$ is convex if and only if $f''(t)\ge0$ for all $t\in I$, where f'' designates the second derivative of f (see [23, Theorem 1.11]). Indeed, thanks to this characterization, we can show that the function $f_{a,b}(\lambda)=a^{1-\lambda}b^{\lambda}$ ($\lambda\in[0,1]$) is convex. So we can infer that

$$f_{a,b}(\lambda)\le(1-\lambda)f_{a,b}(0)+\lambda f_{a,b}(1),\ \ \lambda\in[0,1].$$

This is equivalent to the famous Young's inequality, which is formulated as follows:

$$a^{1-\lambda}b^{\lambda}\le(1-\lambda)a+\lambda b, \tag{2}$$

*Mathematics Subject Classifications: 26D07; 47A63

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for all $a, b > 0$ and all $0 \leq \lambda \leq 1$. Note that the inequality (2) becomes equality if and only if $a = b$.

Young's inequality has many important applications in various fields of mathematics, particularly in operator theory and the theory of inequalities. Over the years, it has been refined by several authors. Its first refinement, referred to in the literature as the squared version, was established by Hirzallah and Kittaneh in [8], as follows:

$$\left(a^{1-\lambda}b^{\lambda}\right)^2 + r_0^2(a-b)^2 \leq ((1-\lambda)a + \lambda b)^2, \tag{3}$$

for every $a, b > 0$ and every $\lambda \in [0,1]$, where $r_0 = \min\{\lambda, 1-\lambda\}$.

Later, Kittaneh and Al-Manasrah [15] provided the following refinement of Young's inequality:

$$a^{1-\lambda}b^{\lambda} + r_0\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\lambda)a + \lambda b, \tag{4}$$

where $a, b > 0$, $\lambda \in [0,1]$ and $r_0 = \min\{\lambda, 1-\lambda\}$.

In [24], Zhao and Wu obtained the following refinement of inequality (4):

$$\begin{aligned} a^{1-\lambda}b^{\lambda} + r_0(\sqrt{a} - \sqrt{b})^2 + r_1\left(\left(\sqrt[4]{ab} - \sqrt{a}\right)^2 \chi_{\left(0,\frac{1}{2}\right]}(\lambda) + \left(\sqrt[4]{ab} - \sqrt{b}\right)^2 \chi_{\left(\frac{1}{2},1\right]}(\lambda)\right) \\ \leq (1-\lambda)a + \lambda b, \end{aligned} \tag{5}$$

where $a, b > 0$, $\lambda \in [0,1]$, $r_0 = \min\{\lambda, 1-\lambda\}$, $r_1 = \min\{2r_0, 1-2r_0\}$ and χ_J $(J \subset \mathbb{R})$ is the characteristic function defined by:

$$\chi_J(x) = \begin{cases} 1 & \text{if } x \in J, \\ 0 & \text{if } x \notin J. \end{cases}$$

In 2015, Al-Manasrah and Kittaneh [2] gave the following generalization of inequalities (3) and (4):

Theorem 1 ([2]) *Let a and b be two positive numbers and $0 \leq \lambda \leq 1$, then for $m = 1, 2, 3, \ldots$ we have*

$$\left(a^{1-\lambda}b^{\lambda}\right)^m + r_0^m\left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \leq ((1-\lambda)a + \lambda b)^m, \tag{6}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$.

Recently, Ighachane and Akkouchi [9] established a new improvement of (6), which also generalizes inequalities (3) and (4), as follows:

Theorem 2 ([9]) *Let a and b be two positive numbers and $0 \leq \lambda \leq 1$, then for $m = 1, 2, 3, \ldots$ we have*

$$r_0^m\left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \leq r_0^m\left(\frac{b^{m+1} - a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right) \leq ((1-\lambda)a + \lambda b)^m - \left(a^{1-\lambda}b^{\lambda}\right)^m, \tag{7}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$.

For an overview of recent progress in this direction, we refer the interested reader to [9, 10, 11, 12, 13, 14, 16, 18, 22] and the references therein.

Recall that, a function $f : I \longrightarrow (0, +\infty)$ is said to be log-convex if the function $\log \circ f$ is convex. In other words, f is log-convex if it satisfies the following inequality

$$f\Big((1-\lambda)a + \lambda b\Big) \leq f^{1-\lambda}(a)f^{\lambda}(b), \tag{8}$$

for all $a, b \in I$ and $\lambda \in [0,1]$. Here, $f^{\beta}(t) = (f(t))^{\beta}$ for every $\beta \in \mathbb{R}$ and every $t \in I$. Notice that from Young's inequality, we obtain that any log-convex function is convex.

This paper aims to refine and extend inequalities (5) and (7) to the following one:

$$\begin{aligned}
& f^m(\lambda) + r_0^m \left(\frac{f^{m+1}(0) - f^{m+1}(1)}{f(0) - f(1)} - (m+1)(f(0)f(1))^{\frac{m}{2}} \right) \\
& + r_m \left(\left((f(0)f(1))^{\frac{m}{4}} - f^{\frac{m}{2}}(0) \right)^2 \chi_{\left(0,\frac{1}{2}\right]}(\lambda) + \left((f(0)f(1))^{\frac{m}{4}} - f^{\frac{m}{2}}(1) \right)^2 \chi_{\left(\frac{1}{2},1\right]}(\lambda) \right) \\
\leq \; & ((1-\lambda)f(0) + \lambda f(1))^m, \qquad (9)
\end{aligned}$$

where $f : [0,1] \longrightarrow (0,+\infty)$ is a log-convex function, $r_0 = \min\{\lambda, 1-\lambda\}$, $r_m = \min\left\{(m+1)r_0^m, (1-r_0)^m - r_0^m\right\}$. In particular, if we take $f(\lambda) = a^{1-\lambda}b^{\lambda}$ ($\lambda \in [0,1]$), with $a, b > 0$ and $a \neq b$, then we obtain an improvement of the inequality (7).

As applications, by choosing some appropriate log-convex functions, we obtain generalized refinements of the difference between the arithmetic-power and arithmetic-geometric means inequalities for scalars. Moreover, we present some new improvements of Young-type inequalities for the traces, determinants and p-norms of positive τ-measurable operators.

The paper is structured as follows: Section 2 is devoted to the proof of the principal inequality (9). Section 3 focuses on its applications to scalar inequalities.

2 New Inequalities for Log-Convex Functions

In this section, we establish the main inequality (9). To do this, we need the following auxiliary lemmas. The first one is known as Jensen's inequality.

Lemma 1 ([23]) *Let $f : I \longrightarrow \mathbb{R}$ be a convex function, $x_k \in I$ and $\lambda_k \geq 0$ $(1 \leq k \leq n)$ such that $\sum_{k=1}^{n} \lambda_k = 1$. Then,*

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k).$$

The second lemma that we need is the next result which was proved by Akkouchi and Ighachane in [1].

Lemma 2 ([1]) *Let m be a positive integer and $\lambda \in [0,1]$. Then, we have*

$$\sum_{k=1}^{m} \binom{m}{k} k \lambda^k (1-\lambda)^{m-k} = m\lambda,$$

where $\binom{m}{k}$ is the binomial coefficient.

We are now in a position to prove our main inequality (9).

Theorem 3 *Let $f : [0,1] \longrightarrow (0,+\infty)$ be a log-convex function and $0 \leq \lambda \leq 1$. Then for every positive integer m, we have*

$$\begin{aligned}
& f^m(\lambda) + r_0^m \left(\frac{f^{m+1}(0) - f^{m+1}(1)}{f(0) - f(1)} - (m+1)(f(0)f(1))^{\frac{m}{2}} \right) \\
& + r_m \left(\left((f(0)f(1))^{\frac{m}{4}} - f^{\frac{m}{2}}(0) \right)^2 \chi_{\left(0,\frac{1}{2}\right]}(\lambda) + \left((f(0)f(1))^{\frac{m}{4}} - f^{\frac{m}{2}}(1) \right)^2 \chi_{\left(\frac{1}{2},1\right]}(\lambda) \right) \\
\leq \; & ((1-\lambda)f(0) + \lambda f(1))^m,
\end{aligned}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$ and $r_m = \min\left\{(m+1)r_0^m, (1-r_0)^m - r_0^m\right\}$.

Proof. The result is trivial when either $\lambda = 0$ or $\lambda = 1$. So, we may assume that $0 < \lambda < 1$. Hence, there are two cases:

Case 1 Suppose that $0 < \lambda \leq \frac{1}{2}$. We claim that

$$((1-\lambda)f(0)+\lambda f(1))^m - \lambda^m \left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)} - (m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-r_m(f(0)f(1))^{\frac{m}{4}} - f^{\frac{m}{2}}(0))^2 \geq f^m(\lambda).$$

We have the following identities

$$\begin{aligned}
&((1-\lambda)f(0)+\lambda f(1))^m - \lambda^m \left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)} - (m+1)(f(0)f(1))^{\frac{m}{2}}\right) \\
&-r_m\left((f(0)f(1))^{\frac{m}{4}} - f^{\frac{m}{2}}(0)\right)^2 \\
= &\sum_{k=0}^{m}\binom{m}{k}\lambda^k(1-\lambda)^{m-k}f^k(1)f^{m-k}(0) - \lambda^m\left(\sum_{k=0}^{m} f^k(1)f^{m-k}(0) - (m+1)(f(0)f(1))^{\frac{m}{2}}\right) \\
&-r_m\left((f(0)f(1))^{\frac{m}{2}} + f^m(0) - 2(f(0)f(1))^{\frac{m}{4}}f^{\frac{m}{2}}(0)\right) \\
= &\sum_{k=0}^{m}\left(\binom{m}{k}\lambda^k(1-\lambda)^{m-k} - \lambda^m\right)f^k(1)f^{m-k}(0) + (m+1)\lambda^m(f(0)f(1))^{\frac{m}{2}} \\
&-r_m\left((f(0)f(1))^{\frac{m}{2}} + f^m(0) - 2(f(0)f(1))^{\frac{m}{4}}f^{\frac{m}{2}}(0)\right) \\
= &\sum_{k=1}^{m}\left(\binom{m}{k}\lambda^k(1-\lambda)^{m-k} - \lambda^m\right)f^k(1)f^{m-k}(0) + \left((1-\lambda)^m - \lambda^m - r_m\right)f^m(0) \\
&+\left((m+1)\lambda^m - r_m\right)(f(0)f(1))^{\frac{m}{2}} + 2r_m(f(0)f(1))^{\frac{m}{4}}f^{\frac{m}{2}}(0) \\
= &\sum_{k=1}^{m}\left(\binom{m}{k}\lambda^k(1-\lambda)^{m-k} - \lambda^m\right)(f^m(1))^{\frac{k}{m}}(f^m(0))^{\frac{m-k}{m}} + \left((1-\lambda)^m - \lambda^m - r_m\right)f^m(0) \\
&+\left((m+1)\lambda^m - r_m\right)(f^m(0))^{\frac{1}{2}}(f^m(1))^{\frac{1}{2}} + 2r_m(f^m(0))^{\frac{3}{4}}(f^m(1))^{\frac{1}{4}}.
\end{aligned}$$

So, by the log-convexity of the function f^m, we get

$$\begin{aligned}
&((1-\lambda)f(0)+\lambda f(1))^m - \lambda^m \left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)} - (m+1)(f(0)f(1))^{\frac{m}{2}}\right) \\
&- r_m\left((f(0)f(1))^{\frac{m}{4}} - f^{\frac{m}{2}}(0)\right)^2 \\
&\geq \sum_{k=1}^{m}\left(\binom{m}{k}\lambda^k(1-\lambda)^{m-k} - \lambda^m\right)f^m\left(\frac{k}{m}\right) + \left((1-\lambda)^m - \lambda^m - r_m\right)f^m(0) \\
&+ \left((m+1)\lambda^m - r_m\right)f^m\left(\frac{1}{2}\right) + 2r_m f^m\left(\frac{1}{4}\right) \\
&= \sum_{k=0}^{m+2}\lambda_k f^m\left(x_k\right),
\end{aligned}$$

where

$$x_k = \begin{cases} 0 & \text{if } k = 0, \\ \frac{k}{m} & \text{if } 1 \leq k \leq m, \\ \frac{1}{2} & \text{if } k = m+1, \\ \frac{1}{4} & \text{if } k = m+2, \end{cases}$$

and

$$\lambda_k = \begin{cases} (1-\lambda)^m - \lambda^m - r_m & \text{if } k = 0, \\ \binom{m}{k}\lambda^k(1-\lambda)^{m-k} - \lambda^m & \text{if } 1 \leq k \leq m, \\ (m+1)\lambda^m - r_m & \text{if } k = m+1, \\ 2r_m & \text{if } k = m+2. \end{cases}$$

Clearly, $x_k \geq 0$ and $\lambda_k \geq 0$ for all $k \in \{0, \ldots, m+2\}$. Moreover, $\sum_{k=0}^{m+2} \lambda_k = 1$. Applying Lemmas 1 and 2, and using the fact that f^m is convex, we get

$$\begin{aligned}
& ((1-\lambda)f(0)+\lambda f(1))^m - \lambda^m \left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)} - (m+1)(f(0)f(1))^{\frac{m}{2}} \right) \\
& -r_m \left((f(0)f(1))^{\frac{m}{4}} - f^{\frac{m}{2}}(0) \right)^2 \\
\geq \ & \sum_{k=0}^{m+2} \lambda_k f^m(x_k) \\
\geq \ & f^m \left(\sum_{k=0}^{m+2} \lambda_k x_k \right) \\
= \ & f^m(\lambda).
\end{aligned}$$

Case 2 First note that the function $x \mapsto f(1-x)$ is log-convex in $[0,1]$. Further, if $\frac{1}{2} \leq \lambda \leq 1$, then we have $0 \leq 1-\lambda \leq \frac{1}{2}$. So by replacing $f(x)$ and λ by $f(1-x)$ and $1-\lambda$ respectively in the first case, we obtain the desired result.

This completes the proof. ■

3 Applications

In this section we present some applications of the main inequality (9) to some scalar inequalities.

3.1 Applications to Certain Means Inequalities for Scalars

We begin this subsection by recalling a few definitions and notations. Let $a, b > 0$, $\lambda \in [0,1]$ and $p \in \mathbb{R} \setminus \{0\}$, the arithmetic $a\nabla_\lambda b$, geometric $a\sharp_\lambda b$ and power $a\sharp_{p,\lambda} b$ means of a and b are respectively given by

$$a\nabla_\lambda b := (1-\lambda)a + \lambda b,$$

$$a\sharp_\lambda b := a^{1-\lambda} b^\lambda,$$

$$a\sharp_{p,\lambda} b := ((1-\lambda)a^p + \lambda b^p)^{\frac{1}{p}}.$$

It is widely known that for a fixed λ the function $p \mapsto a\sharp_{p,\lambda} b$ is increasing on $\mathbb{R} \setminus \{0\}$. In particular, we have

$$a\sharp_{p,\lambda} b \leq a\nabla_\lambda b,$$

for every $p \in (-\infty, 0)$. Furthermore, it is known that

$$a\sharp_\lambda b = \lim_{\substack{p \to 0 \\ p \neq 0}} a\sharp_{p,\lambda} b.$$

On the other hand, we can easily show that for every $p \in (-\infty, 0)$, the function $\lambda \mapsto a\sharp_{p,\lambda} b$ is log-convex on $[0,1]$. So, by applying Theorem 3 we obtain the following new lower bound for the difference between the arithmetic and power means.

Corollary 1 *Let $a, b > 0$, $0 \leq \lambda \leq 1$ and $p \in (-\infty, 0)$. Then for every positive integer m, we have*

$$\begin{aligned}
& (a\sharp_{p,\lambda} b)^m + r_0^m \left(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}} \right) \\
& +r_m \left(\left((ab)^{\frac{m}{4}} - a^{\frac{m}{2}} \right)^2 \chi_{(0,\frac{1}{2}]}(\lambda) + \left((ab)^{\frac{m}{4}} - b^{\frac{m}{2}} \right)^2 \chi_{(\frac{1}{2},1]}(\lambda) \right) \\
\leq \ & (a\nabla_\lambda b)^m,
\end{aligned}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$ and $r_m = \min\{(m+1)r_0^m, (1-r_0)^m - r_0^m\}$.

Letting $p \to 0$ in Corollary 1, we get the following new lower bound for the difference between the arithmetic and geometric mean.

Corollary 2 *Let $a, b > 0$ and $0 \le \lambda \le 1$. Then for every positive integer m, we have*

$$\begin{aligned} & (a\sharp_\lambda b)^m + r_0^m \left(\frac{b^{m+1} - a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}} \right) \\ & + r_m \left(\left((ab)^{\frac{m}{4}} - a^{\frac{m}{2}} \right)^2 \chi_{\left(0, \frac{1}{2}\right]}(\lambda) + \left((ab)^{\frac{m}{4}} - b^{\frac{m}{2}} \right)^2 \chi_{\left(\frac{1}{2}, 1\right]}(\lambda) \right) \\ \le \ & (a\nabla_\lambda b)^m, \end{aligned}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$ and $r_m = \min\{(m+1)r_0^m, (1-r_0)^m - r_0^m\}$.

3.2 Applications to Symmetric Norms for τ-Measurable Operators

In this subsection, we provide some improvements to certain inequalities due to J. Shao [19], for the traces, determinants and p-norms of positive τ-measurable operators.

We denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$. We designate by I the identity element of $\mathcal{B}(\mathcal{H})$.

Throughout this paper, we consider $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$ as a finite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. That is, $\mathcal{V}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology. For standard facts on von Neumann algebras, we refer the reader to [3, 4, 17, 19, 21] and the references therein.

A trace τ on a von Neumann algebra $\mathcal{V}$ is a map

$$\tau : \mathcal{V}^+ \to [0, \infty],$$

where

$$\mathcal{V}^+ = \{A \in \mathcal{V} : A \ge 0\},$$

such that τ is additive, positively homogeneous, and unitarily invariant; that is,

$$\tau(A) = \tau(U^*AU)$$

for all $A \in \mathcal{V}^+$ and every unitary $U \in \mathcal{V}$.

A trace $\tau : \mathcal{V}^+ \to [0, \infty]$ is called:

1. faithful if for all $A \in \mathcal{V}^+$, $\tau(A) = 0$ implies that $A = 0$;
2. semi-finite if for every $A \in \mathcal{V}^+$ with $\tau(A) > 0$, there exists $0 \le B \le A$ such that
$$0 < \tau(B) < \infty;$$
3. normal if for every increasing net $(A_i)_{i \in I}$ in $\mathcal{V}^+$ with $A_i \uparrow A$, we have
$$\tau(A_i) \uparrow \tau(A).$$
4. A trace τ is called finite if $\tau(I) < \infty$.

Recall that, an operator $A \in \mathcal{B}(\mathcal{H})$ is called affiliated with $\mathcal{V}$ if $U^*AU = A$ for every unitary operator U that belongs to the commutant $\mathcal{V}'$ of $\mathcal{V}$, where U^* is the adjoint operator of U. We say that $A \in \mathcal{B}(\mathcal{H})$ is a τ-measurable operator if there exists $\mu > 0$ such that $\tau(p_{(\mu,+\infty)}(|A|)) < +\infty$, where $|A| = (A^*A)^{\frac{1}{2}}$ is the square root of the operator A^*A and $p_{(\mu,+\infty)}(|A|)$ is the spectral projection of the operator $|A|$ corresponding to the interval $(\mu, +\infty)$ (see [4, 17, 19]).

For $0 < p < +\infty$, we denote by $L_p(\mathcal{V};\tau)$ the set of all τ-measurable operators A affiliated with $\mathcal{V}$ which satisfy

$$\mathfrak{n}_p(A) = \tau\left(|A|^p\right)^{\frac{1}{p}} < +\infty.$$

Note that for $1 \le p < +\infty$, $\Big(L_p(\mathcal{V},\tau);\mathfrak{n}_p\Big)$ is a Banach space (see [17]). In the case where $\mathcal{V} = \mathcal{B}(\mathcal{H})$ is endowed with the standard trace, the corresponding L_p-spaces are the Schatten classes $\mathcal{C}_p(\mathcal{H})$ (see [4]). For more information concerning these sets and their applications, we refer the reader to [4, 17, 19] and the references therein.

The determinant $\mathfrak{D}_\tau(A)$ of an element A of $\mathcal{V}$ is defined by (see [5]):

$$\mathfrak{D}_\tau(A) = \begin{cases} \exp\tau(\log|A|) & \text{if } |A| \text{ is invertible,} \\ \inf_{\varepsilon>0}\mathfrak{D}_\tau(|A|+\varepsilon I) & \text{otherwise.}\end{cases}$$

It is well-known that the determinant $\mathfrak{D}_\tau$ satisfies the following properties (see [5, 6]):

(i) $\mathfrak{D}_\tau(\lambda A) = |\lambda|\mathfrak{D}_\tau(A)$ for all $\lambda \in \mathbb{R}$;

(ii) $\mathfrak{D}_\tau(AB) = \mathfrak{D}_\tau(A)\mathfrak{D}_\tau(B)$ and $\mathfrak{D}_\tau(A) = \mathfrak{D}_\tau(A^*) = \mathfrak{D}_\tau(|A|)$ for all $A, B \in \mathcal{V}$;

(iii) $\mathfrak{D}_\tau(A+B) \ge \mathfrak{D}_\tau(A) + \mathfrak{D}_\tau(B)$ for all $A, B \in \mathcal{V}^+$;

(iv) $\mathfrak{D}_\tau\left(|A|^\alpha\right) = \mathfrak{D}_\tau\left(|A|\right)^\alpha$ for all $A \in \mathcal{V}$ and all $\alpha \in \mathbb{R}^+$;

(v) $\mathfrak{D}_\tau(I) = 1$.

For more details on the determinant $\mathfrak{D}_\tau$, we refer the reader to [5, 6, 7, 19] and the references therein.

The versions of Young's inequality for the trace, determinant and p-norm are respectively formulated as follows (see [5, 6]):

$$\tau\big(A^{1-\lambda}B^\lambda\big) \le \tau\big((1-\lambda)A + \lambda B\big) \quad (A, B \in L_1(\mathcal{V};\tau)), \tag{10}$$

$$\mathfrak{D}_\tau\big(A^{1-\lambda}B^\lambda\big) \le \mathfrak{D}_\tau\big((1-\lambda)A + \lambda B\big) \quad (A, B \in \mathcal{V}^+), \tag{11}$$

$$\mathfrak{n}_p\big(A^{1-\lambda}ZB^\lambda\big) \le (1-\lambda)\mathfrak{n}_p\big(AZ\big) + \lambda\mathfrak{n}_p\big(ZB\big) \quad (A, B \in L_p(\mathcal{V};\tau) \text{ and } Z \in \mathcal{V}), \tag{12}$$

for every $\lambda \in [0,1]$.

In [19], Shao used the inequality (6) to improve the inequalities (10), (11) and (12), as follows:

$$\Big(\tau\left(A^{1-\lambda}B^\lambda\right)\Big)^m + r_0^m\Big((\tau(A))^{\frac{m}{2}} - (\tau(B))^{\frac{m}{2}}\Big)^2 \le \Big(\tau((1-\lambda)A + \lambda B)\Big)^m, \tag{13}$$

$$\Big(\mathfrak{D}_\tau\left(A^{1-\lambda}B^\lambda\right)\Big)^m + r_0^m\left((\mathfrak{D}_\tau(A))^{\frac{m}{2}} - (\mathfrak{D}_\tau(B))^{\frac{m}{2}}\right)^2 \le \Big(\mathfrak{D}_\tau\big((1-\lambda)A + \lambda B\big)\Big)^m, \tag{14}$$

$$\Big(\mathfrak{n}_p\big(A^{1-\lambda}ZB^\lambda\big)\Big)^m + r_0^m\Big(\mathfrak{n}_p\left(AZ\right)^{\frac{m}{2}} - \mathfrak{n}_p\left(ZB\right)^{\frac{m}{2}}\Big)^2 \le \Big[(1-\lambda)\mathfrak{n}_p\left(AZ\right) + \lambda\mathfrak{n}_p\left(ZB\right)\Big]^m. \tag{15}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$.

Using Theorem 3, we obtain the following new refinement of the inequality (14).

Theorem 4 *Let $A, B \in \mathcal{V}^+$ and $0 \le \lambda \le 1$. Then for every positive integer m, we have*

$$\begin{aligned}&\left(\mathfrak{D}_\tau\left(A^{1-\lambda}B^\lambda\right)\right)^m + r_0^m\left(\frac{(\mathfrak{D}_\tau(B))^{m+1} - (\mathfrak{D}_\tau(A))^{m+1}}{\mathfrak{D}_\tau(B) - \mathfrak{D}_\tau(A)} - (m+1)\left(\mathfrak{D}_\tau(AB)\right)^{\frac{m}{2}}\right) \\ &+ r_m\left[\left(\Big[\mathfrak{D}_\tau(AB)\Big]^{\frac{m}{4}} - \Big(\mathfrak{D}_\tau(A)\Big)^{\frac{m}{2}}\right)^2\chi_{\left(0,\frac{1}{2}\right]}(\lambda) + \left(\Big[\mathfrak{D}_\tau(AB)\Big]^{\frac{m}{4}} - \Big(\mathfrak{D}_\tau(B)\Big)^{\frac{m}{2}}\right)^2\chi_{\left(\frac{1}{2},1\right]}(\lambda)\right] \\ &\le \mathfrak{D}_\tau((1-\lambda)A + \lambda B)^m\end{aligned}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$ and $r_m = \min\left\{(m+1)r_0^m, (1-r_0)^m - r_0^m\right\}$.

Proof. It follows from Corollary 2 and Inequality (14) that

$$\begin{aligned}\mathfrak{D}_\tau((1-\lambda)A+\lambda B)^m \geq & \Big[\lambda\mathfrak{D}_\tau(A)+(1-\lambda)\mathfrak{D}_\tau(B)\Big]^m \\ \geq & \Big[(\mathfrak{D}_\tau(A))^\lambda\,(\mathfrak{D}_\tau(B))^{1-\lambda}\Big]^m \\ & + r_0^m\left(\frac{(\mathfrak{D}_\tau(B))^{m+1}-(\mathfrak{D}_\tau(A))^{m+1}}{\mathfrak{D}_\tau(B)-\mathfrak{D}_\tau(A)}-(m+1)\,(\mathfrak{D}_\tau(AB))^{\frac{m}{2}}\right) \\ & + r_m\left[\left([\mathfrak{D}_\tau(A)\mathfrak{D}_\tau(B)]^{\frac{m}{4}}-(\mathfrak{D}_\tau(A))^{\frac{m}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\lambda)\right. \\ & \left.\quad + \left([\mathfrak{D}_\tau(A)\mathfrak{D}_\tau(B)]^{\frac{m}{4}}-(\mathfrak{D}_\tau(B))^{\frac{m}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\lambda)\right].\end{aligned}$$

This gives the desired inequality. ■

It was proved by Shao in [20] that the following functions are log-convex on $[0,1]$:

$$f_1(\lambda)=\mathfrak{n}_p\left(A^{1-\lambda}ZB^\lambda\right) \quad\text{and}\quad f_2(\lambda)=\mathfrak{n}_p\left(A^\lambda ZB^\lambda\right),$$

where $p\in(0,+\infty)$, $A,B\in\mathcal{V}^+$ and $Z\in L_p(\mathcal{V};\tau)$.

By applying Theorem 3 to these functions, we obtain the following new inequalities, which are improvements of inequalities (13) and (15).

Theorem 5 *Let $p\in(0,+\infty)$, $\lambda\in[0,1]$, $A,B\in\mathcal{V}^+$ and $Z\in L_p(\mathcal{V};\tau)$. Then for every positive integer m, we have*

$$\begin{aligned}&\Big(\mathfrak{n}_p\left(A^{1-\lambda}ZB^\lambda\right)\Big)^m \\ &\quad + r_0^m\left(\frac{(\mathfrak{n}_p(ZB))^{m+1}-(\mathfrak{n}_p(AZ))^{m+1}}{\mathfrak{n}_p(ZB)-\mathfrak{n}_p(AZ)}-(m+1)(\mathfrak{n}_p(ZB)\,\mathfrak{n}_p(AZ))^{\frac{m}{2}}\right) \\ & + r_m\left(\left((\mathfrak{n}_p(ZB)\,\mathfrak{n}_p(AZ))^{\frac{m}{4}}-(\mathfrak{n}_p(AZ))^{\frac{m}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\lambda)\right) \\ &\quad + r_m\left(\left((\mathfrak{n}_p(ZB)\,\mathfrak{n}_p(AZ))^{\frac{m}{4}}-(\mathfrak{n}_p(ZB))^{\frac{m}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\lambda)\right) \\ & \leq\Big((1-\lambda)\mathfrak{n}_p(AZ)+\lambda\mathfrak{n}_p(ZB)\Big)^m,\end{aligned}$$

where $r_0=\min\{\lambda,1-\lambda\}$ and $r_m=\min\left\{(m+1)r_0^m,(1-r_0)^m-r_0^m\right\}$.

Taking $p=1$ in Theorem 5, we obtain the following inequality.

Corollary 3 *Let $\lambda\in[0,1]$, $A,B\in\mathcal{V}^+$ and $Z\in L_1(\mathcal{V};\tau)$. Then for every positive integer m, we have*

$$\begin{aligned}&\left(\tau\Big(\left|A^{1-\lambda}ZB^\lambda\right|\Big)\right)^m \\ & + r_0^m\left(\frac{\Big(\tau\big(|ZB|\big)\Big)^{m+1}-\Big(\tau\big(|AZ|\big)\Big)^{m+1}}{\tau\big(|ZB|\big)-\tau\big(|AZ|\big)}-(m+1)\Big(\tau(|ZB|)\,\tau(|AZ|)\Big)^{\frac{m}{2}}\right) \\ & + r_m\left(\left(\Big(\tau(|ZB|)\,\tau(|AZ|)\Big)^{\frac{m}{4}}-\Big(\tau(|AZ|)\Big)^{\frac{m}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\lambda)\right) \\ & + r_m\left(\left(\Big(\tau(|ZB|)\,\tau(|AZ|)\Big)^{\frac{m}{4}}-\Big(\tau(|ZB|)\Big)^{\frac{m}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\lambda)\right) \\ \leq\quad & \Big((1-\lambda)\tau(|AZ|)+\lambda\tau(|ZB|)\Big)^m,\end{aligned}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$ *and* $r_m = \min\left\{(m+1)r_0^m, (1-r_0)^m - r_0^m\right\}$.

The second inequality is as follows.

Theorem 6 *Let* $p \in (0, +\infty)$, $\lambda \in [0,1]$, $A, B \in \mathcal{V}^+$ *and* $Z \in L_p(\mathcal{V};\tau)$. *Then for every positive integer* m, *we have*

$$\begin{aligned}
&\Big(\mathfrak{n}_p\left(A^\lambda Z B^\lambda\right)\Big)^m \\
&+r_0^m \left(\frac{\Big(\mathfrak{n}_p(Z)\Big)^{m+1} - \Big(\mathfrak{n}_p(AZB)\Big)^{m+1}}{\mathfrak{n}_p(Z) - \mathfrak{n}_p(AZB)} - (m+1)\Big(\mathfrak{n}_p(Z)\,\mathfrak{n}_p(AZB)\Big)^{\frac{m}{2}}\right) \\
&+r_m \left(\left(\Big(\mathfrak{n}_p(Z)\,\mathfrak{n}_p(AZB)\Big)^{\frac{m}{4}} - \Big(\mathfrak{n}_p(Z)\Big)^{\frac{m}{2}}\right)^2 \chi_{\left(0,\frac{1}{2}\right]}(\lambda)\right) \\
&+r_m \left(\left(\Big(\mathfrak{n}_p(Z)\,\mathfrak{n}_p(AZB)\Big)^{\frac{m}{4}} - \Big(\mathfrak{n}_p(AZB)\Big)^{\frac{m}{2}}\right)^2 \chi_{\left(\frac{1}{2},1\right]}(\lambda)\right) \\
\leq\ &\Big((1-\lambda)\mathfrak{n}_p(AZB) + \lambda\mathfrak{n}_p(Z)\Big)^m,
\end{aligned}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$ *and* $r_m = \min\left\{(m+1)r_0^m, (1-r_0)^m - r_0^m\right\}$.

The following result is a direct consequence of Theorem 6.

Corollary 4 *Let* $\lambda \in [0,1]$, $A, B \in \mathcal{V}^+$ *and* $Z \in L_1(\mathcal{V};\tau)$. *Then for every positive integer* m, *we have*

$$\begin{aligned}
&\Big(\tau\left(\left|A^\lambda Z B^\lambda\right|\right)\Big)^m \\
&+r_0^m \left(\frac{\Big(\tau\big(|Z|\big)\Big)^{m+1} - \Big(\tau(|AZB|)\Big)^{m+1}}{\tau(|Z|) - \tau(|AZB|)} - (m+1)\Big(\tau(|Z|)\,\tau(|AZB|)\Big)^{\frac{m}{2}}\right) \\
&+r_m \left(\left(\Big(\tau(|Z|)\,\tau(|AZB|)\Big)^{\frac{m}{4}} - \Big(\tau(|Z|)\Big)^{\frac{m}{2}}\right)^2 \chi_{\left(0,\frac{1}{2}\right]}(\lambda)\right) \\
&+r_m \left(\left(\Big(\tau(|Z|)\,\tau(|AZB|)\Big)^{\frac{m}{4}} - \Big(\tau(|AZB|)\Big)^{\frac{m}{2}}\right)^2 \chi_{\left(\frac{1}{2},1\right]}(\lambda)\right) \\
\leq\ &\Big((1-\lambda)\tau(|AZB|) + \lambda\tau(|Z|)\Big)^m,
\end{aligned}$$

where $r_0 = \min\{\lambda, 1-\lambda\}$ *and* $r_m = \min\left\{(m+1)r_0^m, (1-r_0)^m - r_0^m\right\}$.

Acknowledgments. We sincerely thank the referees for their time and effort in reviewing our paper. Their insightful comments and suggestions have greatly improved the quality of our work.

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