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A Generalized Best Proximity Point Theorem That Characterizes Metric Completeness*

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Abstract

This paper presents generalization of a best proximity point theorem by utilizing a Suzuki-type setting, which extends Banach's contraction principle to the scenario of non-self mappings. Furthermore, our results define the completeness of the metric space with some different sufficient conditions.

1 Introduction

Functional analysis is a significant field of mathematics that can be primarily classified into two categories: Linear and Non-linear. Non-linear functional analysis has emerged as a distinct and valuable field due to the complex nature of our world, where even small inputs can lead to significant outputs and vice versa. The theory of fixed points pertains to the conditions which ensure the existence of points x in a set X that satisfy an equation of the form $Tx = x$, where T is a self mapping defined on subsets of metric spaces, normed linear spaces, or topological vector spaces. Fixed point theory offers indispensable techniques for resolving issues that arise in diverse areas of mathematical analysis. The following theorem, known as Banach's fixed point theorem, proved in 1922, is a fundamental result in fixed point theory and had a significant impact on various aspects of non-linear functional analysis.

Theorem 1 ([1]) *Let (X, d) be a complete metric space and let T be a contraction map on X, i.e., $T : X \to X$ such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq r \cdot d(x, y) \text{ where } r \in [0, 1).$$

Then the self map T has a unique fixed point.

There are a lot of intriguing variations and extensions of the above result in the literature; refer to [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

All these results, however, have self mappings as their underlying mappings. Given a non-self mapping $T : A \to B$, it is not guaranteed that there will always exist a fixed point. Therefore, it is customary to search for an element x that is in some sense closest to Tx. Theorems on best approximation and best proximity points are applicable in this context. However, on one hand where the best approximation theorems ensure the existence of approximate solutions, it is possible that these outcomes do not necessarily provide optimal solutions. On the other hand, the best proximity point theorems, offer sufficient conditions which ensure the existence of approximate solutions that are also optimal. For a non-self mapping $T : A \to B$, the best proximity point is an optimal approximate solution of the equation $Tx = x$ which satisfies the condition $d(x, Tx) = d(A, B)$. Furthermore, it is evident that best proximity point theorems arise as a logical extension of fixed point theorems, because best proximity point is essentially a fixed point when the mapping being considered is a self mapping. Numerous best proximity point theorems for various types of contractions have been studied in references [12, 13, 14, 15, 16, 17, 18, 19, 20].

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In order to expand the application of fixed point theory to larger classes of problems, best proximity points play a crucial role. Their applications in various fields highlight their importance in addressing practical challenges where exact solutions are unattainable. When a mapping does not have a fixed point, the best proximity points provide solutions for optimization problems where the objective is to minimize the distance between elements in different sets. In control systems, best proximity points can be used to design controllers that bring the system's state as close as possible to the desired state when exact control is not achievable. Similarly, best proximity points have direct and indirect applications in various fields, including applied mathematics, game theory, engineering, economics, and computer science.

In 2010, S. S. Basha proved the following best proximity theorem, which in turn allowed extensions of Banach's contraction principle to the case of non-self mappings.

Theorem 2 ([16]) *Let A and B be non-empty, closed subsets of a complete metric space. Let $T : A \to B$ and $S : B \to A$ be non-self mappings satisfying the following conditions:*

(A) T is a contraction with contraction constant α.

(B) S is non-expansive.

(C) $d(Tx, Sy) < d(x, y)$ whenever $d(x, y) > d(A, B)$ for $x \in A$ and $y \in B$.

Then, the mapping T has a best proximity point $u \in A$ and the mapping S has a best proximity point $v \in B$ such that $d(u, v) = d(A, B)$. Further, if x_0 is any fixed element in $A, x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$, then the sequence $\langle x_{2n} \rangle$ converges to a best proximity point of T and the sequence $\langle x_{2n+1} \rangle$ converges to a best proximity point of S.

The purpose of this paper is to prove a best proximity point theorem, which is a generalization of Theorem 2 and also characterizes the completeness of the metric space. In Section 2, we present the proof of the best proximity point theorem using Suzuki-type setting, which extends the scope of Theorem 2. In addition to this, we investigate a number of examples that highlight the significance of conditions (1), (2), and (3). Lastly, Section 3 addresses the metric completeness.

2 Best Proximity Point Theorem

In this section, we prove the following theorem, which is a generalization of Theorem 2.

Theorem 3 *Let A and B be non-empty closed subsets of a complete metric space (X, d). Let $T : A \to B$ and $S : B \to A$ be two non-self mappings. Define a non-increasing function ϕ from $[0, 1)$ to $(\frac{1}{2}, 1]$ by the rule*

$$\phi(\alpha) = \begin{cases} 1 & \text{if } 0 \le \alpha < (\sqrt{5}-1)/2, \\ (1-\alpha)\alpha^{-2} & \text{if } (\sqrt{5}-1)/2 \le \alpha < 2^{-1/2}, \\ (1+\alpha)^{-1} & \text{if } 2^{-1/2} \le \alpha < 1. \end{cases}$$

Assume that

(1) For $x, y \in A$, there exists $\alpha \in [0, 1)$ such that

$$\phi(\alpha) \cdot d(x, STx) \le d(x, y) \text{ implies } d(Tx, Ty) \le \alpha \cdot d(x, y). \tag{1}$$

(2) For $x', y' \in B$,

$$\frac{1}{2} \cdot d(x', TSx') \le d(x', y') \text{ implies } d(Sx', Sy') \le d(x', y'). \tag{2}$$

(3) For $x \in A$, $x' \in B$,

$$d(x, x') > d(A, B) \text{ implies } d(Tx, Sx') < d(x, x'). \tag{3}$$

Then, the mapping T and S have best proximity points $u \in A$ and $v \in B$ respectively such that $d(u, v) = d(A, B)$.

Proof. Since $\phi(\alpha) \leq 1, \phi(\alpha) \cdot d(x, STx) \leq d(x, STx)$ holds for every $x \in A$, and by hypothesis (1), we get

$$d(Tx, TSTx) \leq \alpha \cdot d(x, STx). \tag{4}$$

Since $x \in A, Tx \in B$ and by the similar arguments above $\frac{1}{2} \cdot d(Tx, TSTx) \leq d(Tx, TSTx)$, and by hypothesis (2) and (4), we get

$$d(STx, STSTx) \leq \alpha \cdot d(x, STx). \tag{5}$$

Consider an arbitrary element $x_0 \in A$ such that $T(x_{2n}) = x_{2n+1}$, $S(x_{2n+1}) = x_{2n+2}$ for all $n \geq 0$, $n \in \mathbb{Z}$. Define two sequences $\langle x_{2n} \rangle$ and $\langle x_{2n+1} \rangle$ in A and B respectively such that $x_{2n} = (ST)^n x_0$ and $x_{2n+1} = T(ST)^n x_0$. Then using (4) and (5), one can easily show that

$$d(x_{2n}, x_{2n+2}) \leq \alpha^n \cdot d(x_0, x_2), d(x_{2n+1}, x_{2n+3}) \leq \alpha^{n+1} \cdot d(x_0, x_2),$$

and so $\sum_{n=0}^{\infty} d(x_{2n}, x_{2n+2}) < \infty$ and $\sum_{n=0}^{\infty} d(x_{2n+1}, x_{2n+3}) < \infty$. This shows that $\langle (ST)^n x_0 \rangle$ and $\langle T(ST)^n x_0 \rangle$ both are Cauchy sequences in A and B respectively and completeness implies that $\langle (ST)^n x_0 \rangle$ and $\langle T(ST)^n x_0 \rangle$ converge to some point $u \in A$ and $v \in B$ respectively. Repeating the same procedure, one can easily show that for any element $y_0 \in B$, $\langle (TS)^n y_0 \rangle$ and $\langle S(TS)^n y_0 \rangle$ both are Cauchy sequences in B and A respectively. Next, we show that

$$d(Tx, v) \leq \alpha \cdot d(x, u) \text{ for all } x \in A \setminus \{u\}, \tag{6}$$

and

$$d(Sy, u) \leq d(y, v) \text{ for all } y \in B \setminus \{v\}. \tag{7}$$

Since $\langle x_{2n} \rangle$ converges to $u \in A$, we see that, for $x \in A \setminus \{u\}$ there exists $m \in \mathbb{N}$ such that

$$d(x_{2n}, u) \leq \frac{d(x, u)}{3} \text{ for all } n \in \mathbb{N} \text{ with } n \geq m.$$

Then, we have

$$\begin{aligned}
\phi(\alpha) \cdot d(x_{2n}, STx_{2n}) &\leq d(x_{2n}, x_{2n+2}) \\
&\leq d(x_{2n}, u) + d(u, x_{2n+2}) \\
&\leq \frac{2 \cdot d(x, u)}{3} \\
&= d(x, u) - \frac{d(x, u)}{3} \\
&\leq d(x, u) - d(x_{2n}, u) \\
&\leq d(x_{2n}, x),
\end{aligned}$$

and so by hypothesis (1), we have $d(x_{2n+1}, Tx) \leq \alpha \cdot d(x_{2n}, x)$ for $n \geq m$. As $n \to \infty$, we get

$$d(Tx, v) \leq \alpha \cdot d(x, u) \text{ for all } x \in A \setminus \{u\},$$

which is the condition (6). Similarly, we can obtain condition (7).

Furthermore, we show that there exist $k, k' \in \mathbb{N}$ such that

$$(ST)^k u = u,\ T(ST)^k u = v,\ (TS)^{k'} v = v, \text{ and } S(TS)^{k'} v = u, \tag{8}$$

where $(ST)^k = STST...ST$ (k copies) and $(TS)^{k'} = TSTS...TS$ (k' copies).

Let us assume that it is not happening, i.e., $(ST)^k u \neq u$, $T(ST)^k u \neq v$, $(TS)^{k'} v \neq v$, and $S(TS)^{k'} v \neq u$ for all $k, k' \in \mathbb{N}$. Then by using the method of induction, first we show the following:

$$d(T(ST)^k u, v) \leq \alpha^k \cdot d(STu, u) \text{ and } d((ST)^{k+1} u, u) \leq \alpha^k \cdot d(STu, u) \text{ for all } k \in \mathbb{N}, \tag{9}$$

and

$$d(S(TS)^{k'}v,u) \leq \alpha^{k'-1} \cdot d(TSv,v) \text{ and } d((TS)^{k'+1}v,v) \leq \alpha^{k'} \cdot d(TSv,v) \text{ for all } k' \in \mathbb{N}. \tag{10}$$

To show (9), from (6) and (7), we get

$$d(T(ST)u,v) \leq \alpha \cdot d(STu,u)$$

and

$$d((ST)^2u,u) \leq d(TSTu,v) \leq \alpha \cdot d(STu,u).$$

Also suppose that $d(T(ST)^k u,v) \leq \alpha^k \cdot d(STu,u)$ and $d((ST)^{k+1}u,u) \leq \alpha^k \cdot d(STu,u)$, then we have

$$d(T(ST)^{k+1}u,v) \leq \alpha \cdot d((ST)^{k+1}u,u) \leq \alpha^{k+1} \cdot d(STu,u)$$

and

$$d((ST)^{k+2}u,u) \leq \alpha \cdot d((ST)^{k+1}u,u) \leq \alpha^{k+1} \cdot d(STu,u).$$

Hence by induction we establish condition (9). Similarly, by using induction we can obtain condition (10).

Moreover, applying the conditions (9) and (10) to appropriate situations, we find a contradiction to our assumption in the following cases:

Case 1 *For $0 \leq \alpha < (\sqrt{5}-1)/2$, we have $\phi(\alpha) = 1$, $\alpha^2 + \alpha - 1 < 0$ and $2\alpha^2 < 1$. Now, if*

$$d(u,(ST)^2u) < \phi(\alpha) \cdot d((ST)^2u,(ST)^3u) = d((ST)^2u,(ST)^3u),$$

we get

$$\begin{aligned} d(u,STu) &\leq d(u,(ST)^2u) + d((ST)^2u,STu) \\ &< d((ST)^2u,(ST)^3u) + d((ST)^2u,STu) \\ &\leq \alpha^2 \cdot d(u,STu) + \alpha \cdot d(u,STu) \\ &= (\alpha^2+\alpha) \cdot d(u,STu) \\ &< d(u,STu), \end{aligned}$$

which is a contradiction. Hence, $d(u,(ST)^2u) \geq \phi(\alpha) \cdot d((ST)^2u,(ST)^3u)$ and so by hypothesis (1), we get

$$d(Tu,T(ST)^2u) \leq \alpha \cdot d(u,(ST)^2u) \leq \alpha^2 \cdot d(u,STu).$$

Thus

$$\begin{aligned} d(u,STu) &\leq d(u,(ST)^3u) + d((ST)^3u,STu) \\ &\leq \alpha^2 \cdot d(u,STu) + \alpha^2 \cdot d(u,STu) \\ &= 2\alpha^2 \cdot d(u,STu) \\ &< d(u,STu), \end{aligned}$$

which is a contradiction.

Case 2 *For $(\sqrt{5}-1)/2 \leq \alpha < 2^{-1/2}$, we have $\phi(\alpha) = (1-\alpha)\alpha^{-2}$ and $2\alpha^2 < 1$. If $d(u,(ST)^2u) < \phi(\alpha) \cdot d((ST)^2u,(ST)^3u)$, we get*

$$\begin{aligned} d(u,STu) &\leq d(u,(ST)^2u) + d((ST)^2u,STu) \\ &< \phi(\alpha) \cdot d((ST)^2u,(ST)^3u) + d((ST)^2u,STu) \\ &\leq \phi(\alpha) \cdot \alpha^2 \cdot d(u,STu) + \alpha \cdot d(u,STu) \\ &= d(u,STu), \end{aligned}$$

which is a contradiction. Hence, $d(u,(ST)^2u) \geq \phi(\alpha) \cdot d((ST)^2u,(ST)^3u)$. As in the earlier case, we can prove

$$d(u,STu) \leq 2\alpha^2 \cdot d(u,STu) < d(u,STu),$$

which is again a contradiction.

Case 3 *For $2^{-1/2} \leq \alpha < 1$, we have $\phi(\alpha) = (1+\alpha)^{-1}$. Now, we claim that*

$$\text{either } \phi(\alpha) \cdot d(x_{2n}, x_{2n+2}) \leq d(x_{2n}, u) \text{ or } \phi(\alpha) \cdot d(x_{2n+2}, x_{2n+4}) \leq d(x_{2n+2}, u). \tag{11}$$

Suppose condition (11) does not hold, then

$$\begin{aligned} d(x_{2n}, x_{2n+2}) &\leq d(x_{2n}, u) + d(u, x_{2n+2}) \\ &< \phi(\alpha) \cdot d(x_{2n}, x_{2n+2}) + \phi(\alpha) \cdot d(x_{2n+2}, x_{2n+4}) \\ &\leq \phi(\alpha) \cdot (1+\alpha) d(x_{2n}, x_{2n+2}) \\ &= d(x_{2n}, x_{2n+2}), \end{aligned}$$

which is a contradiction. Hence, either $\phi(\alpha) \cdot d(x_{2n}, x_{2n+2}) \leq d(x_{2n}, u)$ or $\phi(\alpha) \cdot d(x_{2n+2}, x_{2n+4}) \leq d(x_{2n+2}, u)$. Then hypothesis (1) will imply

$$d(x_{2n+1}, Tu) \leq \alpha \cdot d(x_{2n}, u) \text{ or } d(x_{2n+3}, Tu) \leq \alpha \cdot d(x_{2n+2}, u),$$

making limit as $n \to \infty$, we get

$$d(v, Tu) \leq \alpha \cdot d(u, u) \text{ or } d(v, Tu) \leq \alpha \cdot d(u, u).$$

This implies that $d(v, Tu) = 0$ or $Tu = v$. Similarly, by employing the same procedure, we can shown that $Sv = u$, and thus $STu = Sv = u$, $TSv = Tu = v$, which is again contrary to our assumption.

Thus, we conclude that the condition (8) is true. Therefore in all the cases, there exist $k, k' \in \mathbb{N}$ such that $(ST)^k u = u$, $T(ST)^k u = v$, $(TS)^{k'} v = v$, and $S(TS)^{k'} v = u$. Moreover $\langle (ST)^n u \rangle$ and $\langle T(ST)^n u \rangle$ are Cauchy sequences in A and B respectively, so we have $STu = u$ and consequently $Tu = v$. Similarly $\langle (TS)^n v \rangle$ and $\langle S(TS)^n v \rangle$ are Cauchy sequences in B and A respectively, $TSv = v$ and consequently $Sv = u$.

In order to show u and v are proximity points of A and B respectively, it is enough to show $d(u, Tu) = d(A, B)$ and $d(v, Sv) = d(A, B)$. Suppose if possible $d(u, v) > d(A, B)$, then using hypothesis (3), we get

$$d(u, v) = d(Sv, Tu) < d(v, u),$$

which is a contradiction. Therefore, $d(u, v) = d(A, B)$. Hence

$$d(u, Tu) = d(u, v) = d(A, B),$$

and

$$d(v, Sv) = d(v, u) = d(A, B).$$

This shows that u and v are proximity points of T and S respectively. This completes the proof. ■

Here, we give an example which shows that the conditions (1) and (2) are generalization of the conditions (A) and (B) in Theorem 2.

Example 1 *Consider the Euclidean space $\mathbb{R}^2$ with the metric d is defined as*

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|,$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let

$$A = \{(0,0), (4,0), (0,4), (4,5), (5,4)\}$$

and

$$B = \{(-1,-1), (-4,0), (0,-4), (-4,-5), (-5,-4)\}$$

be two non-empty closed subsets of $\mathbb{R}^2$. *Let* $T : A \to B$ *and* $S : B \to A$ *be defined as*

$$T(x_1, x_2) = \begin{cases} (-1,-1) & \text{if } |x_1| + |x_2| \text{ is even,} \\ (-|x_1|, 0) & \text{if } |x_1| + |x_2| \text{ is odd and } |x_1| < |x_2|, \\ (0, -|x_2|) & \text{if } |x_1| + |x_2| \text{ is odd and } |x_1| > |x_2|, \end{cases}$$

and

$$S(y_1, y_2) = \begin{cases} (0,0) & \text{if } |y_1| + |y_2| \text{ is even,} \\ (|y_1|, 0) & \text{if } |y_1| + |y_2| \text{ is odd and } |y_1| < |y_2|, \\ (0, |y_2|) & \text{if } |y_1| + |y_2| \text{ is odd and } |y_1| > |y_2|. \end{cases}$$

Table 1: Condition (1) for Example 1

x	y	$d(x,y)$	$d(x,STx)$	$d(Tx,Ty)$
(0,0)	(0,4)	4	0	0
(0,0)	(4,0)	4	0	0
(0,0)	(4,5)	9	0	4
(0,0)	(5,4)	9	0	4
(0,4)	(0,0)	4	4	0
(0,4)	(4,0)	8	4	0
(0,4)	(4,5)	5	4	4
(0,4)	(5,4)	5	4	4
(4,0)	(0,0)	4	4	0
(4,0)	(0,4)	8	4	0
(4,0)	(4,5)	5	4	4
(4,0)	(5,4)	5	4	4
(4,5)	(0,0)	9	9	4
(4,5)	(0,4)	5	9	4
(4,5)	(4,0)	5	9	4
(4,5)	(5,4)	2	9	8
(5,4)	(0,0)	9	9	4
(5,4)	(0,4)	5	9	4
(5,4)	(4,0)	5	9	4
(5,4)	(4,5)	2	9	8

Table 2: Condition (2) for Example 1

x'	y'	$d(x',y')$	$d(x',TSx')$	$d(Sx',Sy')$
(-1,-1)	(0,-4)	4	0	0
(-1,-1)	(-4,0)	4	0	0
(-1,-1)	(-4,-5)	7	0	4
(-1,-1)	(-5,-4)	7	0	4
(0,-4)	(-1,-1)	4	4	0
(0,-4)	(-4,0)	8	4	0
(0,-4)	(-4,-5)	5	4	4
(0,-4)	(-5,-4)	5	4	4
(-4,0)	(-1,-1)	4	4	0
(-4,0)	(0,-4)	8	4	0
(-4,0)	(-4,-5)	5	4	4
(-4,0)	(-5,-4)	5	4	4
(-4,-5)	(-1,-1)	7	7	4
(-4,-5)	(0,-4)	5	7	4
(-4,-5)	(-4,0)	5	7	4
(-4,-5)	(-5,-4)	2	7	8
(-5,-4)	(-1,-1)	7	7	4
(-5,-4)	(0,-4)	5	7	4
(-5,-4)	(-4,0)	5	7	4
(-5,-4)	(-4,-5)	2	7	8

From Tables 1, 2 and 3, we observe following facts:

- $d(A,B) = 2$.
- *If* $(x,y) \neq ((4,5),(5,4))$ *and* $(x,y) \neq ((5,4),(4,5))$, *then there exists* $\alpha = \frac{4}{5} \in [0,1)$ *such that* $\phi(\alpha) \cdot d(x,STx) \leq d(x,y)$ *and* $d(Tx,Ty) \leq \alpha \cdot d(x,y)$.
- *If* $(x,y) = ((4,5),(5,4))$ *or* $(x,y) = ((5,4),(4,5))$, *then* $\phi(\alpha) \cdot d(x,STx) > d(x,y)$ *for every* $\alpha \in [0,1)$.
- *If* $(x',y') \neq ((-4,-5),(-5,-4))$ *and* $(x',y') \neq ((-5,-4),(-4,-5))$, *then* $\frac{1}{2} \cdot d(x',TSx') \leq d(x',y')$ *and* $d(Sx',Sy') \leq d(x',y')$.

Table 3: Condition (3) for Example 1

x	x'	$d(x,x')$	$d(Tx,Sx')$
(0,0)	(-1,-1)	2	2
(0,0)	(-4,0)	4	2
(0,0)	(0,-4)	4	2
(0,0)	(-4,-5)	9	6
(0,0)	(-5,-4)	9	6
(4,0)	(-1,-1)	6	2
(4,0)	(-4,0)	8	2
(4,0)	(0,-4)	8	2
(4,0)	(-4,-5)	13	6
(4,0)	(-5,-4)	13	6
(0,4)	(-1,-1)	6	2
(0,4)	(-4,0)	8	2
(0,4)	(0,-4)	8	2
(0,4)	(-4,-5)	13	6
(0,4)	(-5,-4)	13	6
(4,5)	(-1,-1)	11	4
(4,5)	(-4,0)	13	4
(4,5)	(0,-4)	13	4
(4,5)	(-4,-5)	18	8
(4,5)	(-5,-4)	18	8
(5,4)	(-1,-1)	11	4
(5,4)	(-4,0)	13	4
(5,4)	(0,-4)	13	4
(5,4)	(-4,-5)	18	8
(5,4)	(-5,-4)	18	8

- *If* $(x',y') = ((-4,-5),(-5,-4))$ *or* $(x',y') = ((-5,-4),(-4,-5))$*, then* $\frac{1}{2} \cdot d(x', TSx') > d(x',y')$.

Hence, T *and* S *satisfy the conditions* (1)*,* (2)*, and* (3)*, respectively in Theorem 3. But, since*

$$d(T(4,5),T(5,4)) = 8 > 2 = d((4,5),(5,4)),$$

and

$$d(S(-4,-5),S(-5,-4)) = 8 > 2 = d((-4,-5),(-5,-4)).$$

This shows that T *and* S *do not satisfy the conditions (A) and (B) respectively in Theorem 2. Moreover, it can be noted that* $(0,0)$ *and* $(-1,-1)$ *are best proximity points for mappings* T *and* S *respectively.*

Here, we provide an additional example which illustrate the preceding Theorem 3. This example demonstrates that hypotheses (1), (2), and (3) of Theorem 3 are sufficient to confirm mappings T and S have the best proximity points. It also brings out that uniqueness of the best proximity point is not plausible.

Example 2 *Consider the Euclidean space* $\mathbb{R}^2$ *with the metric* d *is defined as*

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|,$$

where $x = (x_1,x_2)$*,* $y = (y_1,y_2) \in \mathbb{R}^2$*. Let* $A = \{(0,1),(1,0)\}$ *and* $B = \{(0,-1),(-1,0)\}$ *be two non-empty closed subsets of* $\mathbb{R}^2$*. Let* $T : A \to B$ *and* $S : B \to A$ *be defined as*

$$T(x_1,x_2) = (-1,0) \text{ for all } (x_1,x_2) \in A,$$

and

$$S(y_1, y_2) = (0,1) \text{ for all } (y_1, y_2) \in B.$$

Then, all the hypotheses of the aforesaid Theorem are satisfied. Moreover, it can be noted that $(0,1)$ *and* $(1,0)$ *are best proximity points for mapping* T *and* $(0,-1)$ *and* $(-1,0)$ *are best proximity points for mapping* S*.*

The following examples demonstrates that conditions (1) and (2) in the Theorem 3 are indispensable.

Example 3 *Consider the Euclidean space* $\mathbb{R}^2$ *with the metric* d *is defined as*

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|,$$

where $x = (x_1, x_2)$*,* $y = (y_1, y_2) \in \mathbb{R}^2$*. Let* $A = \{(0,0), (0,2)\}$ *and* $B = \{(2,0), (2,2)\}$ *be two non-empty closed subsets of* $\mathbb{R}^2$*. Let* $T : A \to B$ *be defined as*

$$T((0,0)) = (2,2) \quad \text{and} \quad T((0,2)) = (2,0).$$

Let $S : B \to A$ *be defined as*

$$S((2,0)) = (0,0) \quad \text{and} \quad S((2,2)) = (0,2).$$

Table 4: Condition (1) for Example 3

x	y	$d(x,y)$	$d(x,STx)$	$d(Tx,Ty)$
(0,0)	(0,2)	2	2	2
(0,2)	(0,0)	2	2	2

Table 5: Condition (2) for Example 3

x'	y'	$d(x',y')$	$d(x',TSx')$	$d(Sx',Sy')$
(2,0)	(2,2)	2	2	2
(2,2)	(2,0)	2	2	2

Table 6: Condition (3) for Example 3

x	x'	$d(x,x')$	$d(Tx,Sx')$
(0,0)	(2,2)	4	2
(0,2)	(2,0)	4	2

Since $d(A,B) = 2$*. From Table 4, it can be observed that though condition (1) of Theorem 3 do not satisfied, yet conditions (2) and (3) of Theorem 3 holds good (see Tables 5 and 6). Further, it can be noted that mapping* T *has no best proximity point.*

Example 4 *Consider the space* $\mathbb{R}$ *with the usual metric* d *is defined as* $d(x,y) = |x - y|$*, where* $x, y \in \mathbb{R}$*. Let* $A = \{-1, 2\}$ *and* $B = \{0, 1\}$ *be two non-empty closed subsets of* $\mathbb{R}$*. Let* $T : A \to B$ *be defined as*

$$T(-1) = 0 \quad \text{and } T(2) = 1.$$

Let $S : B \to A$ *be defined as*

$$S(0) = 2 \quad \text{and} \quad S(1) = -1.$$

Table 7: Condition (1) for Example 4

x	y	$d(x,y)$	$d(x,STx)$	$d(Tx,Ty)$
-1	2	3	3	1
2	-1	3	3	1

Table 8: Condition (2) for Example 4

x'	y'	$d(x',y')$	$d(x',TSx')$	$d(Sx',Sy')$
0	1	1	1	3
1	0	1	1	3

Since $d(A,B) = 1$*. From Table 8, it can be observed that though condition (2) of Theorem 3 do not satisfied, yet conditions (1) and (3) of Theorem 3 holds good (see Table 7 and 9). Further, it can be noted that mapping* S *has no best proximity point.*

Table 9: Condition (3) for Example 4

x	x'	$d(x,x')$	$d(Tx,Sx')$
-1	1	2	1
2	0	2	1

The following simple example reveals that condition (3) in Theorem 3 is obligatory.

Example 5 *Consider the Euclidean space $\mathbb{R}^2$ with the metric d is defined as*

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

where $x = (x_1,x_2)$, $y = (y_1,y_2) \in \mathbb{R}^2$. Let $A = \{(0,1),(1,0)\}$ and $B = \{(0,5),(5,0)\}$ be two non-empty closed subsets of $\mathbb{R}^2$. Let $T : A \to B$ be defined as

$$T(x_1,x_2) = (5,0) \text{ for all } (x_1,x_2) \in A.$$

Let $S : B \to A$ be defined as

$$S((0,5)) = (1,0) \quad \text{and} \quad S((5,0)) = (0,1).$$

Table 10: Condition (1) for Example 5

x	y	$d(x,y)$	$d(x,STx)$	$d(Tx,Ty)$
(0,1)	(1,0)	1	0	0
(1,0)	(0,1)	1	1	0

Table 11: Condition (2) for Example 5

x'	y'	$d(x',y')$	$d(x',TSx')$	$d(Sx',Sy')$
(0,5)	(5,0)	5	5	1
(5,0)	(0,5)	5	0	1

Table 12: Condition (3) for Example 5

x	x'	$d(x,x')$	$d(Tx,Sx')$
(0,1)	(5,0)	5	5
(1,0)	(0,5)	5	4

Since $d(A,B) = 4$. From Tables 10 and 11, it can be observed that though conditions (1) and (2) of Theorem 3 are satisfied, yet the condition (3) of Theorem 3 does not hold good (see Table 12). Moreover, it can be noted that mapping S has no best proximity point.

Corollary 1 *Let (X,d) be a complete metric space. Let T and S be two self maps on X. Define a function ϕ as in Theorem 3. Assume that*

(1) For $x,y \in A$, there exists $\alpha \in [0,1)$ such that

$$\phi(\alpha) \cdot d(x,STx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq \alpha \cdot d(x,y).$$

(2) For $x,y \in X$,

$$\frac{1}{2} \cdot d(x,TSx) \leq d(x,y) \text{ implies } d(Sx,Sy) \leq d(x,y).$$

(3) For $x,y \in X$,

$$d(x,y) > 0 \text{ implies } d(Tx,Sy) < d(x,y).$$

Then the mapping T and S have a unique common fixed point.

Remark 1 *The well-known Banach's contraction principle is summed up in the previous result.*

3 Metric Completeness

In this section, we discuss the metric completeness.

Theorem 4 *Let (X, d) and (Y, d) be two metric spaces and define a function ϕ as in Theorem 3. Consider the non-self onto maps $T : X \to Y$ and $S : Y \to X$ satisfying the following:*

(1) For $x \in X$, $x' \in Y$,

$$d(x, x') > d(X, Y) \text{ implies } d(Tx, Sx') < d(x, x'). \tag{12}$$

Let $M_{\alpha,\beta}$ be the family of non-self onto mappings $T : X \to Y$ and $S : Y \to X$ satisfying (12) and the following:

(2) For $x, y \in X$, there exist $\alpha \in [0, 1)$ and $\beta \in (0, \phi(\alpha)]$ such that

$$\beta \cdot d(x, STx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \alpha \cdot d(x, y). \tag{13}$$

(3) For $x', y' \in Y$,

$$\frac{1}{2} \cdot d(x', TSy') \leq d(x', y') \text{ implies } d(Sx', Sy') \leq d(x', y'). \tag{14}$$

Let $N_{\alpha,\beta}$ be the family of non-self onto mappings $T : X \to Y$ and $S : Y \to X$ satisfying (12), (13), (14) and the following:

(4) $ST(X)$ and $TS(Y)$ are countably infinite.

(5) Every subset of $ST(X)$ and $TS(Y)$ are closed.

Then, the following statements are equivalent:

(I) *X and Y are complete.*

(II) *Every pair of mappings $T, S \in M_{\alpha,\phi(\alpha)}$ have best proximity points $u \in X$ and $v \in Y$ respectively, such that $Tu = v$ and $Sv = u$.*

(III) *There exist $\alpha \in (0, 1)$ and $\beta \in (0, \phi(\alpha)]$ such that every pair of mappings $T, S \in N_{\alpha,\beta}$ have best proximity points $u \in X$ and $v \in Y$ respectively, such that $Tu = v$ and $Sv = u$.*

Proof. By Theorem 3, (**I**) implies (**II**). Since $N_{\alpha,\beta} \subset M_{\alpha,\phi(\alpha)}$ for $\alpha \in [0, 1)$ and $\beta \in (0, \phi(\alpha)]$, (**II**) implies (**III**). Let us prove (**III**) implies (**I**). We assume that condition (**III**) holds. Suppose if possible, X and Y are not complete. That is, there exists a Cauchy sequence $\langle u_n \rangle$ in X and $\langle v_n \rangle$ in Y which does not converge. Define two functions $f : X \to [0, \infty)$ and $g : Y \to [0, \infty)$ as follows:

$$f(x) = \lim_{n\to\infty} d(x, u_n) \text{ for } x \in X,$$

and

$$g(x') = \lim_{n\to\infty} d(x', v_n) \text{ for } x' \in Y,$$

and connect them by a relation

$$g(Tx) = f(STx) \text{ for all } x \in X \text{ and } f(Sx') = g(TSx') \text{ for all } x' \in Y.$$

We have following observations for functions f and g:

- For $x, y \in X$, $f(x) - f(y) \leq d(x, y) \leq f(x) + f(y)$.
- For $x', y' \in Y$, $g(x') - g(y') \leq d(x', y') \leq g(x') + g(y')$.

- $f(x) > 0$ for all $x \in X$ and $g(x') > 0$ for all $x' \in Y$.
- $\lim_{n\to\infty} f(u_n) = 0$ and $\lim_{n\to\infty} g(v_n) = 0$.

Define two non-self onto mappings $T : X \to Y$ and $S : Y \to X$ satisfying (12), as follows: For each $x \in X$, since $f(x) > 0$ and $\lim_{n\to\infty} f(u_n) = 0$, there exists $m \in \mathbb{N}$ such that $f(u_m) \le \frac{\alpha\beta}{3+\alpha\beta} \cdot f(x)$. Similarly, for each $x' \in Y$, since $g(x') > 0$ and $\lim_{n\to\infty} g(v_n) = 0$, there exists $m' \in \mathbb{N}$ such that $g(v_m) \le \frac{\alpha\beta}{3+\alpha\beta} \cdot g(x')$. We put $STx = u_m$ and $TSx' = v_{m'}$. Then it is obvious that

$$f(STx) \le \frac{\alpha\beta}{3+\alpha\beta} \cdot f(x) \text{ and } STx \in \{u_n : n \in \mathbb{N}\} \text{ for all } x \in X,$$

and

$$g(TSx') \le \frac{\alpha\beta}{3+\alpha\beta} \cdot g(x') \text{ and } TSx' \in \{v_n : n \in \mathbb{N}\} \text{ for all } x' \in Y.$$

Then $STx \ne x$ for all $x \in X$ and $TSx' \ne x'$ for all $x' \in Y$. Since $ST(X) \subset \{u_n : n \in \mathbb{N}\}$ and $TS(Y) \subset \{v_n : n \in \mathbb{N}\}$, the coniditoin (4) in Theorem 4 holds. Also, it is not difficult to prove the coniditoin (5) in Theorem 4. Let us prove (13) and (14). Fix $x, y \in X$ with $\beta \cdot d(x, STx) \le d(x, y)$. If $f(y) > 2f(x)$, we have

$$\begin{aligned} d(Tx, Ty) \le g(Tx) + g(Ty) &= f(STx) + f(STy) \\ &\le \frac{\alpha\beta}{3+\alpha\beta}(f(x) + f(y)) \\ &\le \frac{\alpha}{3}(f(x) + f(y)) \\ &\le \frac{\alpha}{3}(f(x) + f(y)) + \frac{2\alpha}{3}(f(y) - 2f(x)) \\ &= \alpha(f(y) - f(x)) \le \alpha d(x, y). \end{aligned}$$

In the other case, where $f(y) \le 2f(x)$, we have

$$d(x, y) \ge \beta d(x, STx) \ge \beta(f(x) - f(STx)) \ge \beta(1 - \frac{\alpha\beta}{3+\alpha\beta})f(x) = \frac{3\beta}{3+\alpha\beta}f(x),$$

and hence

$$\begin{aligned} d(Tx, Ty) \le g(Tx) + g(Ty) &= f(STx) + f(STy) \\ &\le \frac{\alpha\beta}{3+\alpha\beta}(f(x) + f(y)) \\ &\le \frac{3\alpha\beta}{3+\alpha\beta}f(x) \\ &\le \alpha d(x, y). \end{aligned}$$

Thus we have shown (13). Similarly, we can show that (14) satisfied. Hence $T, S \in N_{\alpha,\beta}$. By (**III**), T and S have best proximity points $u \in X$ and $v \in Y$ respectively, such that $Tu = v$, $Sv = u$, and hence $STu = u$, $TSv = v$, which yields a contradiction. Hence X and Y are complete. This completes the proof. ■

4 Conclusion

We have formulated the best proximity point theorem for non-self mappings in a Suzuki-type framework, which extends Theorem 2, refer to the examples presented in Section 2 for further clarification. In addition, it has been demonstrated that our findings also define the completeness of the metric space, as stated in Theorem 4. Our theorems provide a pathway to new and improved proximity point results and their applications.

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