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Solutions To The Heat Equation With Discontinuous Damping Coefficient*

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Abstract

In this note, we prove the existence of the solution to an initial-boundary value problem posed for damped heat equation, containing the Heaviside function with time dependent term as a damping coefficient. As a consequence of the Heaviside function, the problem breaks into two associated initial boundary value problems (IBVPs) and an unknown common boundary term appears in this process. We solve both the IBVPs in terms of Green's functions and show the existence and uniqueness of the derivative of the unknown function along positive t-axis with the help of Volterra's integral equation of the first kind.

1 Introduction

This paper deals with the initial-boundary value problem to the heat equation

$$v_t = v_{xx} - \frac{H(x)}{(1+t)}\,v, \quad -l \le x \le l, \quad t > 0, \tag{1}$$

subject to the initial data

$$v(x,0) = \cos\left(\frac{m\pi}{l}x\right), \quad -l \le x \le l, \tag{2}$$

and boundary conditions

$$v_x(-l,t) = 0, \quad t > 0, \tag{3}$$

$$v_x(l,t) = 0, \quad t > 0, \tag{4}$$

where H is the Heaviside function of x. The linear heat equation (1) plays a key role in studying the properties of solutions for viscous Burgers equation with source terms [6]. We show the existence of the solution to the initial-boundary value problem (1)–(4). For which, we split the problem (1)–(4) into two associated initial-boundary value problems and then form an Abel's integral equation of first kind. Existence of solution to the Abel's integral equation leads to conclude the existence of the solution to (1)—(4). Nonhomogeneous viscous Burgers equations have applications in various fields of science [1, 3, 15]. To analyze the existence, uniqueness and large time asymptotics to the solutions of the inhomogeneous viscous Burgers equation [6]

$$u_t + uu_x - u_{xx} = \frac{2\delta(x)}{1+t},$$

one generally applies Hopf-Cole transformation

$$u(x,t) = -\frac{v_x}{v},$$

which in turn produces the corresponding linear heat equation (1).

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Kreiss and Kreiss [14] studied the forced Burgers equation

$$u_t + uu_x - \epsilon u_{xx} = f(x), \quad 0 \le x \le 1, \ t > 0, \tag{5}$$

subjected to the initial and boundary data and showed that the solution of (5) converges to the unique solution of its steady state equation. The rate of convergence depends upon the boundary conditions. Ablowitz and Lillo [1] found exact solutions for the Cauchy problem of the forced Burgers equation when the force term is a product of some function of time variable t with the Dirac delta function. Petrovskii [16] discovered a technique for providing exact solutions to the Cauchy problem of non-linear Burgers equation. They applied it to the cases when force term was stationary and transient respectively. To study generalized solutions, we refer to [10, 9, 11, 12]. Chung, Kim and Slemrod [4] dealt with

$$\begin{aligned} u_t + uu_x - u_{xx} &= \delta, \quad x \in \mathbb{R}, \ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{6}$$

where $u_0(x) \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. Using Cole-Hopf transformation, they transformed it into heat conduction equation with a term containing Heaviside function

$$v_t - v_{xx} = -\frac{1}{2} H(x) v, \quad x \in \mathbb{R}, \ t > 0. \tag{7}$$

They discussed the existence, uniqueness and large time behavior of the solutions to the initial value problem posed in (7). Later, they studied the existence and uniqueness of the weak solution to the equation (6), derived by inverse Cole-Hopf transformation. They found the L^p-norm convergence of the weak solutions of (7) to its steady state solution on compact sets. The analysis of equation (7) showed that rather than second-order derivative terms, first-order derivative terms determine the compactness of solution trajectory. The initial value problem

$$\begin{cases} u_t + uu_x - \mu u_{xx} = 4x, \quad x \in (-\infty, \infty), \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in (-\infty, \infty), \end{cases} \tag{8}$$

for $\mu > 0$, was explicitly solved by Ding, Jiu and He [5]. Here, $u_0(x) \in L^1_{loc}(\mathbb{R})$ and $\int^x u_0(\xi) d\xi = o(x^2)$ as $|x| \to \infty$. From Cole-Hopf transformation, equation (8) converts into heat conduction equation

$$v_t - v_{xx} = -x^2 v, \quad x \in \mathbb{R}, \ t > 0.$$

Its solution is described using Fourier-Hermite series. Later they also discussed the asymptotic behavior of equation (8). Rao and Yadav [18] studied equation (8) for the case when right hand side of (8) is kx, where $k > 0$ and $\mu = 1$. Also, assumed that the initial data u_0 in the initial value problem (8)

$$\exp\left(\frac{-1}{2} \int^x u_0(\xi) d\xi\right) \in L^2\left(\mathbb{R}, \ \mathrm{e}^{\frac{-x^2}{2}}\right).$$

But then the investigation of equation (8) was done via Kloosterziel's approach [13] which leads to the self-similar solutions and large time behavior of the concerned equation. Further, the asymptotic behavior of the solution to forced Burgers equation is determined via inverse Hopf-Cole transformation. Damped heat equation, containing a damping coefficient, given by

$$v_t - v_{xx} = -\frac{kx^2}{4(2\beta t + 1)^2} v, \quad x \in \mathbb{R}, \quad t > 0, \tag{9}$$

is considered by Rao and Yadav [19] while studying large time asymptotics for inhomogeneous viscous Burgers equations. They studied (9) subject to the initial functions from the class of $L^2(\mathbb{R}, \mathrm{e}^{\beta x^2/2})$. The solutions to the Cauchy problem for (9) are obtained in series form and asymptotic behavior of them is discussed. We refer to [20] for asymptotic analysis of solutions to generalized Burgers equation on half-line. Gianni and Hulshof [7] examined an initial-boundary value problem, containing a discontinuous term

$$\begin{cases} u_t = u_{xx} + H(u), \quad 0 \le x \le 1, \quad t \in (0,T]\,, \\ u(0,t) = -1, \quad t \in (0,T], \\ u(1,t) = 1, \quad t \in (0,T], \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{cases} \tag{10}$$

where H is the Heaviside furnction. They showed the existence, regularity and uniqueness of the solutions to (10). For the case, when initial data either touches zero from below or identically zero, the solutions are well-behaved. Engu, Sahoo and Berke [6] investigated initial value problem of the following equation

$$v_t = v_{xx} - \frac{H(x)}{(1+t)}v, \quad x \in \mathbb{R},\ t > 0,$$

$$v(x,0) = v_0(x), \quad x \in \mathbb{R},$$

where $v_0(x) \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R})$. The problem is solved by splitting the domain into two parts by introducing a common boundary condition. With the use of Abel's integral equation, the existence of the common boundary term is determined. As t approaches ∞, the solutions of the corresponding IBVPs converge to a non-zero real constant on the compact sets uniformly.

This paper is organized as follows. Section 2 deals with the existence of the solutions to the damped heat equation (1) subject to the class of initial functions and Neumann type boundary data (2)–(4). Section 3 gives the conclusion and future scope of the paper.

2 Damped Heat Equation with Heaviside Function

In this section, we study an initial-boundary value problem to the damped heat equation, given by

$$v_t = v_{xx} - \frac{H(x)}{(1+t)}\,v, \qquad -l \le x \le l, \quad t > 0, \tag{11a}$$

$$v(x,0) = \cos\left(\frac{m\pi}{l}x\right), \qquad -l \le x \le l, \tag{11b}$$

$$v_x(-l,t) = 0, \qquad t > 0, \tag{11c}$$

$$v_x(l,t) = 0, \qquad t > 0, \tag{11d}$$

where $H(x)$ is Heaviside function and $m \in \mathbb{N}$. For which, we divide the domain of the problem (11a)–(11d) into left side domain $\{-l \le x \le 0; t > 0\}$ and right side domain $\{0 \le x \le l; t > 0\}$ due to presence of the Heaviside function. During this process, we temporarily introduce a continuously differentiable function $\phi(t)$, derivative of the common boundary along positive t-axis, in the emerging associate initial-boundary value problems. Later we show the existence and uniqueness of this continuously differentiable function $\phi(t)$.

Let us write the initial boundary value problem (11a)–(11d) in the left side domain $\{-l \le x \le 0\,;\, t > 0\}$ as

$$L_t = L_{xx}, \qquad -l \le x \le 0, \quad t > 0, \tag{12a}$$

$$L(x,0) = \cos\left(\frac{m\pi}{l}x\right), \qquad -l \le x \le 0, \tag{12b}$$

$$L_x(-l,t) = 0, \qquad t > 0, \tag{12c}$$

$$L_x(0,t) = \phi(t), \qquad t > 0. \tag{12d}$$

Solving (12a)–(12d), we get

$$L(x,t) = \int_0^l \cos\left(\frac{m\pi}{l}\xi\right) G(-x,\xi,t)\,d\xi + \int_0^t \phi(\tau)\,G(-x,0,t-\tau)\,d\tau, \tag{13}$$

where G is the modified Green's function given by

$$G(x,\xi,t) = \sum_{n=0}^{\infty} \frac{y_n(x)y_n(\xi)}{\|y_n\|^2} \mathrm{e}^{-\lambda_n t}, \quad \|y_n\|^2 = \int_{-l}^{0} y_n^2(x)\,dx. \tag{14}$$

In equation (14), λ_n are eigenvalues and y_n are corresponding eigenvectors of the following Sturm-Liouville problem

$$\begin{cases} y'' + \lambda y = 0, & -l < x < 0, \\ y'(-l) = 0, \\ y'(0) = 0. \end{cases} \tag{15}$$

Solving the boundary value problem (15), one obtains the eigenvalues and corresponding eigenvectors as follows;

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \text{ and } y_n(x) = \cos\left(\frac{n\pi}{l}x\right).$$

Hence, modified Green's function (14) reduces to

$$G(x,\xi,t) = \frac{1}{l} + \sum_{n=1}^{\infty} \frac{2}{l} \cos(\frac{n\pi}{l}x)\, \cos(\frac{n\pi}{l}\xi)\, \mathrm{e}^{-(\frac{n\pi}{l})^2 t}. \tag{16}$$

Let $R(x,t)$ be the solution of the associate initial-boundary value problem for (11a)–(11d) in the right side domain $\{0 \le x \le l; t > 0\}$. Then,

$$R_t = R_{xx} - \frac{R}{1+t}, \qquad 0 \le x \le l, \quad t > 0, \tag{17a}$$

$$R(x,0) = \cos\left(\frac{m\pi}{l}x\right), \qquad 0 \le x \le l, \tag{17b}$$

$$R_x(0,t) = \phi(t), \qquad t > 0, \tag{17c}$$

$$R_x(l,t) = 0, \qquad t > 0. \tag{17d}$$

Introducing

$$R(x,t) = \frac{W(x,t)}{(1+t)},$$

into (17a)–(17d) gives rise to

$$W_t = W_{xx}, \qquad 0 \le x \le l, \quad t > 0, \tag{18a}$$

$$W(x,0) = \cos\left(\frac{m\pi}{l}x\right), \qquad 0 \le x \le l, \tag{18b}$$

$$W_x(0,t) = (1+t)\phi(t), \qquad t > 0, \tag{18c}$$

$$W_x(l,t) = 0, \qquad t > 0. \tag{18d}$$

Solving (18a)–(18d) for $W(x,t)$, we have

$$W(x,t) = \int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\, \tilde{G}(x,\xi,t)\,d\xi - \int_0^t (1+\tau)\phi(\tau)\,\tilde{G}(x,0,t-\tau)\,d\tau,$$

where modified Green's function $\tilde{G}$ is written as

$$\tilde{G}\,(x,\xi,t) = \sum_{n=0}^{\infty} \frac{\tilde{y}_n(x)\tilde{y}_n(\xi)}{\|\tilde{y}_n\|^2} \mathrm{e}^{-\tilde{\lambda}_n t}, \quad \|\tilde{y}_n\|^2 = \int_0^l \tilde{y}_n^2(x)\,dx. \tag{19}$$

Here $\tilde{\lambda}_n$ are eigenvalues and $\tilde{y}_n$ are corresponding eigenvectors of the ordinary differential equation;

$$\tilde{y}''(x) + \tilde{\lambda}\tilde{y}(x) = 0, \text{ with } \tilde{y}'(0) = 0 \text{ and } \tilde{y}'(l) = 0.$$

Hence, we obtain

$$\tilde{\lambda}_n = \left(\frac{n\pi}{l}\right)^2 \text{ and } \tilde{y}_n(x) = \cos\left(\frac{n\pi}{l}x\right).$$

By substituting values of $\tilde{\lambda}_n$ and $\tilde{y}_n$, modified Green's function (19) becomes

$$\tilde{G}(x,\xi,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos(\frac{n\pi}{l}x)\, \cos(\frac{n\pi}{l}\xi)\, \mathrm{e}^{-\left(\frac{n\pi}{l}\right)^2 t}. \tag{20}$$

Thus, the solution of IBVP (17a)–(17d) is

$$R(x,t) = \frac{1}{(1+t)}\left[\int_0^l \cos\left(\frac{m\pi}{l}\xi\right) \tilde{G}(x,\xi,t)\, d\xi - \int_0^t (1+\tau)\, \phi(\tau)\, \tilde{G}(x,0,t-\tau)\, d\tau\right],$$

where $\tilde{G}(x,\xi,t)$ is given in (20). From (20) and (16), observe that $\tilde{G}(x,\xi,t) = G(x,\xi,t)$. Therefore,

$$R(x,t) = \frac{1}{(1+t)}\left[\int_0^l \cos\left(\frac{m\pi}{l}\xi\right) G(x,\xi,t)\, d\xi - \int_0^t (1+\tau)\phi(\tau)\, G(x,0,t-\tau)\, d\tau\right], \tag{21}$$

where $G(x,\xi,t)$ is given in (16). Since the solution of the initial-boundary value problem (11a)–(11d) must be continuous, we impose the condition

$$\lim_{x\to 0} L(x,t) = \lim_{x\to 0} R(x,t), \qquad \text{for all } t > 0.$$

Then, from (13) and (21), we get

$$\begin{aligned} &\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\, G(0,\xi,t)\, d\xi + \lim_{x\to 0}\int_0^t \phi(\tau)\, G(-x,0,t-\tau)\, d\tau \\ =\ & \frac{1}{(1+t)}\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\, G(0,\xi,t)\, d\xi - \lim_{x\to 0}\int_0^t \left(\frac{1+\tau}{1+t}\right) \phi(\tau)\, G(x,0,t-\tau)\, d\tau. \end{aligned}$$

Since $\frac{1+\tau}{1+t} \le 1$ and if we assume $\phi(\tau)G(x,0,t-\tau)$ is bounded then from dominated convergence theorem, we get

$$\begin{aligned} &\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\, G(0,\xi,t)\, d\xi + \int_0^t \phi(\tau)\, G(0,0,t-\tau)\, d\tau \\ =\ & \frac{1}{(1+t)}\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\, G(0,\xi,t)\, d\xi - \int_0^t \left(\frac{1+\tau}{1+t}\right) \phi(\tau)\, G(0,0,t-\tau)\, d\tau. \end{aligned} \tag{22}$$

Re-arranging the terms in (22) leads to

$$\left(\frac{-t}{1+t}\right)\int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\, G(0,\xi,t)\, d\xi = \int_0^t \left[\left(\frac{1+\tau}{1+t}\right)+1\right] \phi(\tau)\, G(0,0,t-\tau)\, d\tau. \tag{23}$$

Note that

$$\begin{aligned} \int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\, G(0,\xi,t)\, d\xi &= \int_0^l \cos\left(\frac{m\pi}{l}\xi\right)\left[\frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{l}\xi\right) \mathrm{e}^{-\left(\frac{n\pi}{l}\right)^2 t}\right] d\,\xi \\ &= \mathrm{e}^{-\left(\frac{m\pi}{l}\right)^2 t}, \end{aligned}$$

where the orthogonality of the sequence $\{\cos\left(\frac{n\pi\, x}{l}\right) : n \in \mathbb{N}\}$ is used. Hence, equation (23) reduces to

$$\left(\frac{-t}{1+t}\right) \mathrm{e}^{-\left(\frac{m\pi}{l}\right)^2 t} = \int_0^t \left[\left(\frac{1+\tau}{1+t}+1\right)\left(\frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \mathrm{e}^{-\left(\frac{n\pi}{l}\right)^2 (t-\tau)}\right)\right] \phi(\tau) d\tau. \tag{24}$$

Note that (24) is in the form of the Volterra's integral equation of 1st kind;

$$f(t) = \int_0^t K(t,\tau)\phi(\tau)d\tau, \tag{25}$$

where f and kernel K are given by

$$f(t) = \left(\frac{-t}{1+t}\right) \mathrm{e}^{-(\frac{m\pi}{l})^2 t}, \tag{26}$$

and

$$K(t,\tau) = \left(\frac{1+\tau}{1+t} + 1\right)\left(\frac{1}{l} + \sum_{n=1}^{\infty} \frac{2}{l} \mathrm{e}^{-(\frac{n\pi}{l})^2 (t-\tau)}\right). \tag{27}$$

Theorem 1 *There exists a unique, continuously differentiable solution ϕ for the Volterra's integral equation of the first kind* (25)–(27)*.*

Proof. Re-write the integral equation (25)–(27) in terms of Abel's integral equation of first kind;

$$\frac{1}{\sqrt{\pi}} \int_0^t \frac{\tilde{K}(t,\tau)}{\sqrt{t-\tau}} \phi(\tau) d\tau = F(t),$$

where

$$\tilde{K}(t,\tau) = \sqrt{\frac{\pi}{8}} \sqrt{t-\tau} \left(\frac{1+\tau}{1+t} + 1\right)\left(\frac{1}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \mathrm{e}^{-(\frac{n\pi}{l})^2 (t-\tau)}\right), \tag{28}$$

and

$$F(t) = \frac{1}{\sqrt{8}} f(t) = \frac{1}{\sqrt{8}} \left(\frac{-t}{1+t}\right) \mathrm{e}^{-(\frac{m\pi}{l})^2 t}. \tag{29}$$

Note that $F(0) = 0$. Let us consider the domain $T = \{(t,\tau) \in \mathbb{R}^2 : 0 < \tau < t < \infty\}$. The re-arrangement of kernel (28) yields

$$\begin{aligned} \tilde{K}(t,\tau) &= \frac{\sqrt{\pi}}{2\sqrt{2}} \sqrt{t-\tau} \left(\frac{1+\tau}{1+t} + 1\right)\left(\frac{1}{l} + \frac{2}{l}\left(\sum_{n=0}^{\infty} \mathrm{e}^{-(\frac{n\pi}{l})^2 (t-\tau)} - 1\right)\right) \\ &= \frac{1}{2}\left(\frac{1+\tau}{1+t} + 1\right)\left[\sqrt{\frac{2}{\pi}\left(\frac{\pi}{l}\right)^2 (t-\tau)} \sum_{n=0}^{\infty} \mathrm{e}^{-(\frac{n\pi}{l})^2 (t-\tau)} - \frac{1}{l}\sqrt{\frac{\pi}{2}(t-\tau)}\right]. \end{aligned}$$

It is known [21] that $\sqrt{\frac{2\Theta}{\pi}} \sum_{n=0}^{\infty} \mathrm{e}^{-n^2\Theta} \sim 1$ as $\Theta \to 0$. Hence, we get

$$\tilde{K}(t,\tau) \to 1, \text{ as } \tau \to t.$$

To show $\tilde{K}(t,\tau)$ is continuously differentiable, we represent $\tilde{K}(t,\tau)$ in terms of *Jacobi 3-Theta function*;

$$\tilde{K}(t,\tau) = \frac{1}{l}\sqrt{\frac{\pi}{8}} \sqrt{t-\tau} \left(\frac{1+\tau}{1+t} + 1\right) \vartheta_3\left(0, \mathrm{e}^{-(\frac{\pi}{l})^2 (t-\tau)}\right),$$

where $\vartheta_3(z,q)$ is the *Jacobi 3-Theta function* [2, 17] defined by

$$\vartheta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \text{ for } z,\, q \in \mathbb{C} \text{ and } |q| < 1.$$

Jacobi 3-Theta function shows that kernel $\tilde{K}(t,\tau)$ is twice continuously differentiable on the domain T. Further, consider a function ΩF defined by

$$(\Omega F)(t) := \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{F(\tau)}{\sqrt{t-\tau}} d\tau.$$

Applying integration by parts gives

$$(\Omega F)(t) = \frac{2}{\sqrt{\pi}} \frac{d}{dt} \int_0^t F'(\tau)\sqrt{t-\tau} d\tau = \frac{1}{\sqrt{\pi}} \int_0^t \frac{F'(\tau)}{\sqrt{t-\tau}} d\tau. \tag{30}$$

From equation (29), F' is written as

$$F'(t) = \frac{-1}{\sqrt{8}} \left[-\left(\frac{m\pi}{l}\right)^2 \frac{t}{1+t} + \frac{1}{1+t} - \frac{t}{(1+t)^2} \right] \mathrm{e}^{-(\frac{m\pi}{l})^2 t}. \tag{31}$$

Applying inequality $t \le 1+t \le (1+t)^2$, we get

$$\left| \sqrt{t} F'(t) \right| \le C, \quad \forall\, t > 0,$$

where C is a constant. Therefore, taking modulus on both sides of equation (30) gives

$$|(\Omega F)(t)| \le \frac{1}{\sqrt{\pi}} \int_0^t \frac{|F'(\tau)|}{\sqrt{t-\tau}} d\tau \le \frac{C}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\tau}\sqrt{t-\tau}} d\tau = C\sqrt{\pi}, \quad \forall\ t > 0.$$

Hence, $(\Omega F)(t)$ is bounded and continuous for all $t > 0$. Now, differentiating ΩF in (30) and using $F'(0) = -\frac{1}{\sqrt{8}}$, we get

$$\frac{d}{dt}(\Omega F)(t) = \frac{1}{\sqrt{8\pi t}} - \frac{1}{\sqrt{\pi}} \int_0^t \frac{F''(\tau)}{\sqrt{t-\tau}}\, d\tau.$$

In view of (31), we get

$$F''(t) = -\left(\frac{m\pi}{l}\right)^2 F'(t) + \frac{\mathrm{e}^{-(\frac{m\pi}{l})^2 t}}{\sqrt{8}} \left[\frac{\left(\frac{m\pi}{l}\right)^2 + 1}{(t+1)^2} + \frac{1-t}{(1+t)^3} \right],$$

which implies

$$\left| \sqrt{t} F''(t) \right| \le C', \qquad \forall\ t > 0,$$

where C' is a constant. Hence,

$$\left| \frac{1}{\sqrt{\pi}} \int_0^t \frac{F''(\tau)}{\sqrt{t-\tau}} d\tau \right| \le \frac{C'}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\tau}\sqrt{t-\tau}} d\tau = C'\sqrt{\pi},\ \forall\ t > 0.$$

Thus, ΩF is continuously differentiable function. Therefore, by [Gorenflo and Vessella [8], Theorem 5.1.4], there exists a unique, continuously differentiable solution ϕ to the Volterra's integral equation (25)–(27). ■

Using above theorem, the solution of damped heat equation (11a)–(11d) is given by

$$v(x,t) = \begin{cases} L(x,t), & -l \le x \le 0, \quad t > 0, \\ R(x,t), & 0 \le x \le l, \quad t > 0, \end{cases}$$

where $L(x,t)$ and $R(x,t)$ are given in (13) and (21) respectively.

Remark 1 *Using the results from [8], we obtain bounds for ϕ;*

$$\|\phi(t)\|_{C^1(0,\infty)} \le \triangle(\sigma_2)\, \|(\Omega F)(t)\|_{C^1(0,\infty)},$$

where $\sigma_2 = \|\tilde{K}\|_{C^2(0,\infty)}$ and $\triangle(\sigma_2)$ is a constant depends on σ_2.

3 Conclusions and Future Scope

We studied an initial-boundary value problem of heat equation (11a)–(11d), containing a source term involving the Heaviside function. We divided the domain into two parts and considered one common boundary term for both the associated IBVPs. Assuming the limits of the solutions of associated IBVPs are equal as $x \to 0$, we obtained Volterra's integral equation of the first kind. We proved the existence and uniqueness of the solution of Volterra's integral equation by converting it into Abel's integral equation of the first kind. Anti-derivative of the solution of Volterra's integral equation is the required unknown function along the positive t-axis. This in turn helped to show the existence of the continuous solution of the problem (11a)–(11d) giving the explicit representation of the solutions.

We hope that our study paves the way to obtain the explicit representation of the solutions for more general initial-boundary data to (11a). Applying inverse Cole-Hopf transformation, one can investigate the large time asymptotics to the inhomogeneous viscous Burgers equation namely,

$$u_t + uu_x - u_{xx} = \frac{2\ \delta(x)}{1+t},$$

subject to the concerned initial and boundary conditions, where δ is a Dirac-delta function of x.

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