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On The Number Of Zeros Of A Polynomial In A Certain Region*

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Abstract

The polynomial $P(z)=\sum_{j=0}^{n}a_jz^j$, where the coefficients satisfy $a_j\geq a_{j-1}$, $a_0\geq 0$, $j=1,2,3,...,n$ and $a_n\neq 0$, is the focus of the classical Enestrom-Kakeya [3] result, which states that all the zeros of $P(z)$ lie within the disk $|z|\leq 1$. This result has been extended by Shah and Liman [4], who introduced additional conditions on the coefficients and proved that if $\sum_{j=0}^{n}|a_j|<|a_n|$, then all the zeros of $P(z)$ also lie within the disk $|z|\leq 1$. The main aim of the paper is to extend this result further by introducing new conditions on the coefficients. These generalizations provide more specific information about the exact number of zeros of a polynomial within certain regions, offering a deeper understanding of the distribution of zeros.

1 Introduction

The study of the zeros of polynomials is a long-standing topic in the analytical theory of polynomials, with significant applications in both mathematics and other fields. This area of research has evolved over time, with contributions from notable figures such as Fourier, Gauss, Cauchy, and Laguerre. The problem of finding the zeros of a polynomial is central to the theory of equations, and the techniques for solving it are complex, involving both algebraic and analytical methods. To address these complexities, mathematicians have developed methods to identify regions that contain the zeros of polynomials with real and complex coefficients. A key classical result in this area is due to Cauchy [1], which is well-known as the Cauchy's classical result, which provides bounds for the zeros of polynomials.

Theorem 1 *If $P(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0$ is a polynomial of degree n with complex coefficients and $M=\max\limits_{0\leq j\leq n-1}|\frac{a_j}{a_n}|$, then all the zeros of $P(z)$ lie in the disk $|z|<1+M$.*

There are many similar results pertaining to the location of zeros of polynomials, a review of literature can be found in [3, 4, 8, 11]. Another elegant result concerning the distribution of zeros of polynomials with restricted coefficients was established by Enestrom and Kakeya [3] which is known as the Enestrom-Kakeya theorem.

Theorem 2 *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that $a_n\geq a_{n-1}\geq ...\geq a_1\geq a_0>0$. Then all the zeros of $P(z)$ lie in $|z|\leq 1$.*

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However, this result is limited in scope due to the restrictive nature of the hypothesis. Consequently, mathematicians have relaxed the conditions on the coefficients, leading to numerous Eneström-Kakeya-type results, along with their generalizations and extensions. In this context, Shah and Liman [4] established a significant result concerning the distribution of zeros of a polynomial within a specified region.

Theorem 3 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a complex polynomial of degree n such that $\sum_{j=0}^{n-1} |a_j| \leq |a_n|$, then $P(z)$ has all its zeros in $|z| \leq 1$.*

In 2017, Gulzar and Bashir [8] obtained an extension of Theorem 3 by weakening the hypothesis on the coefficients. Infact, they proved:

Theorem 4 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying $\sum_{j=0}^{n} |a_j - a_{j-1}| \leq |a_n|$, $a_{-1} = 0$. Then all the zeros of $P(z)$ lie in $|z| \leq 1$.*

The distribution of zeros of polynomials has been extensively studied, and much literature exists on this subject [5, 7, 9]. However, there is relatively little work focused on determining the exact number of zeros of a polynomial in a given region [6, 10]. The aim of this paper is to address this gap by deriving results that specify the exact number of zeros within a particular region, under certain conditions on the polynomial coefficients.

In this paper, we generalize Theorem 3 and Theorem 4, leading to new results that provide precise information about the number of zeros of a polynomial in a specific region. To prove these results, we utilize the following fundamental tool, known as the Rouche's theorem [2].

Rouche's Theorem *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) \pm g(z)$ have the same number of zeros inside C.*

2 Main Results

Theorem 5 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for $0 < t \leq 1$ and $1 \leq k \leq n$ with $\sum_{j=0,\, j\neq k}^{n} |a_j| \leq t^k |a_k|$, $a_k \neq 0$, then $P(z)$ has exactly k zeros in $|z| < t$.*

Proof. Consider the polynomial

$$g(z) = \frac{1}{a_k} \sum_{j=0,\, j\neq k}^{n} a_j z^j = \frac{1}{a_k}(a_0 + a_1 z + a_2 z^2 + ... + a_{k-1} z^{k-1} + a_{k+1} z^{k+1} + ... + a_n z^n).$$

Now on $|z| = t$, we have

$$|g(z)| = \frac{1}{|a_k|} \left| \sum_{j=0,\, j\neq k}^{n} a_j z^j \right| \leq \frac{1}{|a_k|} \sum_{j=0,\, j\neq k}^{n} |a_j||z|^j = \frac{1}{|a_k|} \sum_{j=0,\, j\neq k}^{n} |a_j| t^j.$$

Since $0 < t \leq 1$, we see that

$$|g(z)| \leq \frac{1}{|a_k|} \sum_{j=0,\, j\neq k}^{n} |a_j|.$$

Using the condition $\sum_{j=0,\ j\neq k}^{n} |a_j| \leq t^k |a_k|,\ 1 \leq k \leq n$, we obtain

$$|g(z)| \leq t^k \quad \text{or} \quad |g(z)| \leq |z|^k, \ \ |z| = t,$$

where $1 \leq k \leq n$. By applying Rouche's theorem, it follows that z^k and $g(z) + z^k$ have the same number of zeros in $|z| < t$. Since $g(z) + z^k = P(z)$, z^k and $P(z)$ have the same number of zeros in $|z| < t$. But z^k has exactly k zeros in $|z| < t$, therefore it follows that $P(z)$ also has exactly k zeros in $|z| < t$. This completes the proof of Theorem 5. ■

Example 1 *Consider a polynomial* $P(z) = z^5 - 5z^2 - 0.25z + 2.5$ *of degree* 5. *For* $t = 1$ *and* $k = 2$, *since* $P(z)$ *satisfies the conditions of Theorem 5, it follows that* $P(z)$ *has exactly* 2 *zeros viz;* $z = 0.707$ *and* -0.707 *(approx.) in* $|z| < 1$.

Example 2 *Consider a polynomial* $P(z) = 12z^5 - 26z^3 + 6z^2 + 4z - 1$ *of degree* 5. *For* $t = 1$ *and* $k = 3$, *since* $P(z)$ *satisfies the conditions of Theorem 5, it follows that* $P(z)$ *has exactly* 3 *zeros viz;* $z = 0.258,\ 0.408$ *and* -0.408 *(approx.) in* $|z| < 1$.

Remark 1 *By taking* $k = n,\ t = 1$ *in Theorem 5, we obtain Theorem 3.*

The following result immediately follows if we put $k = n$ in Theorem 5.

Corollary 1 *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that* $\sum_{j=0}^{n-1} |a_j| \leq t^n |a_n|$, $0 < t \leq 1$, *then* $P(z)$ *has all its zeros in* $|z| < t$.

Example 3 *Consider a polynomial* $P(z) = 2z^6 + 0.4z^5 + 0.3z^4 - 0.2z^3 - 0.1z^2 + 0.01z + 0.02$ *of degree* 6. *By applying Theorem 3, the zeros of* $P(z)$ *lie in* $|z| \leq 1$, *while by applying Corollary 1 for* $t = 0.9$, *it follows that all the zeros of* $P(z)$ *lie in* $|z| < 0.9$.

Theorem 6 *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that for* $0 < t \leq 1$ *and* $1 \leq k \leq n$ *with* $\sum_{j=0,\ j\neq k}^{n} |ta_j - a_{j-1}| \leq t^k |a_k|,\ a_k \neq 0, (a_{-1} = 0)$ *and* $|ta_k - a_{k-1}| \geq |a_k|$, *then* $P(z)$ *has exactly* k *zeros in* $|z| < t$.

Proof. Consider the polynomial

$$\begin{aligned} F(z) &= (t - z)P(z) \\ &= -a_n z^{n+1} + (ta_n - a_{n-1})z^n + (ta_{n-1} - a_{n-2})z^{n-1} + ... + (ta_1 - a_0)z + ta_0. \end{aligned}$$

Let

$$\begin{aligned} g(z) &= \frac{1}{ta_k - a_{k-1}}\{-a_n z^{n+1} + (ta_n - a_{n-1})z^n + ... \\ &\quad +(ta_{k-1} - a_{k-2})z^{k-1} + (ta_{k+1} - a_k)z^{k+1} + ... + (ta_1 - a_0)z + ta_0\}. \end{aligned}$$

Now on $|z| = t$,

$$\begin{aligned} |g(z)| &\leq \frac{1}{|ta_k - a_{k-1}|}[|a_n||z|^{n+1} + |ta_n - a_{n-1}||z|^n + ... \\ &\quad +|ta_{k-1} - a_{k-2}||z|^{k-1} + |ta_{k+1} - a_k||z|^{k+1} + ... + |ta_1 - a_0||z| + |ta_0|] \\ &\leq \frac{1}{|a_k|} \sum_{j=0, j\neq k}^{n} |ta_j - a_{j-1}|t^j. \end{aligned}$$

Since $0 < t \leq 1$, we see that

$$|g(z)| \leq \frac{1}{|a_k|} \sum_{j=0,\ j\neq k}^{n} |ta_j - a_{j-1}|.$$

Using the condition $\sum_{j=0,\ j\neq k}^{n} |ta_j - a_{j-1}| \leq t^k |a_k|$, we get

$$|g(z)| \leq t^k \quad \text{and} \quad |g(z)| \leq |z|^k \text{ for } |z| = t,$$

where $1 \leq k \leq n$. Since $|ta_k - a_{k-1}| \geq 1$, we have $|z|^k \leq |ta_k - a_{k-1}||z|^k$ for $0 < t \leq 1$. This implies $|g(z)| \leq |ta_k - a_{k-1}||z|^k$ for $|z| = t$, for $0 < t \leq 1$.

Hence by applying Rouche's theorem, we see that $(ta_k - a_{k-1})z^k$ and $g(z) + (ta_k - a_{k-1})z^k$ have the same number of zeros in $|z| < t$ and since $g(z) + (ta_k - a_{k-1})z^k = F(z)$, it follows that $(ta_k - a_{k-1})z^k$ and $F(z)$ have the same number of zeros in $|z| < t$. But the zeros of $(ta_k - a_{k-1})z^k$ are same as the zeros of z^k and the zeros of $P(z)$ are also the zeros of $F(z)$, hence it follows that z^k and $P(z)$ have the same number of zeros in $|z| < t$. Now z^k has exactly k zeros in $|z| < t$, therefore it follows that $P(z)$ also has exactly k zeros in $|z| < t$.

This completes the proof of Theorem 6. ■

Example 4 *Consider a polynomial* $P(z) = 2z^4 + 1.8z^3 + 0.5z + 0.01$ *of degree* 4. *For* $t = \frac{15}{16}$ *and* $k = 3$ *in Theorem 6, we obtain for above* $P(z)$

$$\sum_{j=0,\ j\neq k}^{n} |ta_j - a_{j-1}| = 1.043125 \leq 1.4831543 = t^k |a_k|$$

and $|ta_k - a_{k-1}| = 1.6875 \geq 1$. *Hence,* $P(z)$ *satisfies the conditions of Theorem 6 for* $t = \frac{15}{16}$, $k = 3$ *and therefore, it follows that* $P(z)$ *has exactly* 3 *zeros in the disk* $|z| < \frac{15}{16}$, *viz;* $z \approx -0.02,\ 0.1110 + 0.4635i$ *and* $0.1110 - 0.4635i$ *while one zero* $(z \approx -1.1021)$ *lies outside* $|z| < \frac{15}{16}$.

3 Conclusion

The results obtained in this paper assist us to determine the exact number of zeros of polynomials in a certain region, thereby reducing the efforts of locating these zeros. These results have applications in many areas of scientific discipline such as Communication Theory, Coding Theory, Crytography, Control Theory etc. Since less literature is available with reference to the location of exact number of zeros of polynomials, there is a scope of further extending these results, thereby setting out an inspiration for mathematicians to undertake further research in this area.

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