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On The Ideal Kuratowski Convergence Of Nested Sequences Of Sets*

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Abstract

In this work, we show the equivalence of ideal Kuratowski convergence and Kuratowski convergence for nested sequences of sets. We also give some results regarding $\mathcal{I}$-nested sequences of sets and $\mathcal{I}$-Kuratowski limit sets.

1 Introduction

The inadequacy of pointwise limits in solving problems in multivalued analysis has led to the emergence of a new theory of convergence. Painleve first introduced the concepts of the outer and inner limit of a sequence of sets in 1902. On the other hand, Kuratowski [12] defined the condition that these two limits are equal to each other as Kuratowski convergence. Afterwards, many types of set convergence entered the literature. One of them is Hausdorff convergence, which is expressed using the concept of distance between two sets. Another is the Wijsman convergence [18, 19], which corresponds to the pointwise convergence of the distance functions. Recently, Apreutesei [3] proved that Wijsman convergence and Hausdorff convergence are equivalent to each other for monotone sequences of compact sets. These convergence concepts are enriched by using different theories. Some of these theories are the statistical convergence given by Fast [6] and Steinhaus [16] and ideal convergence given by Kostyrko et al. [11] (see also [4, 7, 8, 13, 14]). In this sense, Nuray and Rhoades [15] defined st-Kuratowski, st-Hausdorff and st-Wijsman convergence of a sequence of sets. While the concept of $\mathcal{I}$-Wijsman convergence was given by Kişi and Nuray [10], the concepts of $\mathcal{I}$-Kuratowski and $\mathcal{I}$-Hausdorff convergence were given by Talo and Sever [17]. In recent times, Khan et al. [9] extended the concept of $\mathcal{I}$-Kuratowski convergence given in metric space to intuitionistic fuzzy metric spaces. Albayrak [1] showed that $\mathcal{I}$-Wijsman limit and the $\mathcal{I}$-Hausdorff limit of nested sequence of sets are equivalent to each other for every admissible ideal. Furthermore, Albayrak et al. [2] examined set-theoretic operators preserving $\mathcal{I}$-Hausdorff convergence.

2 Preliminaries

Throughout this paper, (X, ρ) denotes a metric space. We denote the family of all nonempty closed subsets and the family of all nonempty compact subsets of X by $\mathrm{Cl}\,(X)$ and $\mathcal{K}\,(X)$, respectively. Let $\mathrm{cl}\,(A)$ be the closure of a set A.

The *distance* $d(x, A)$ from a point $x \in X$ to a set $A \subseteq X$ is defined as

$$d(x, A) = \inf_{y \in A} \rho(x, y)$$

(see [18]).

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The *open ball* with centre $a \in X$ and radius $r > 0$ is the set

$$\mathrm{B}\,(a; r) = \{x \in X : \rho(a, x) < r\}.$$

The *closed ball* with centre $a \in X$ and radius $r > 0$ is the set

$$\mathrm{B}\,[a; r] = \{x \in X : \rho(a, x) \leq r\}.$$

Now, we recall some basic concepts related to the ideals and the filters (see [5, 20]).
A family $\mathcal{I} \subseteq \mathcal{P}\,(\mathbb{N})$ is called an *ideal* on $\mathbb{N}$, if it satisfies the following conditions:

1. $\emptyset \in \mathcal{I}$,
2. $I, J \in \mathcal{I} \Longrightarrow I \cup J \in \mathcal{I}$,
3. $I \in \mathcal{I}$ and $J \subseteq I \Longrightarrow J \in \mathcal{I}$.

A family $\mathcal{F} \subseteq \mathcal{P}\,(\mathbb{N})$ is called a *filter* on $\mathbb{N}$, if it satisfies the following conditions:

1. $\emptyset \notin \mathcal{F}$,
2. $M, N \in \mathcal{F} \Longrightarrow M \cap N \in \mathcal{F}$,
3. $M \in \mathcal{F}$ and $M \subseteq N \Longrightarrow N \in \mathcal{F}$.

The ideal and the filter are dual concepts. If $\mathcal{I}$ is an ideal on $\mathbb{N}$ then the family $\mathcal{F}\,(\mathcal{I}) = \{\mathbb{N} \setminus I : I \in \mathcal{I}\}$ is a filter on $\mathbb{N}$. Conversely, if $\mathcal{F}$ is a filter on $\mathbb{N}$ then the family $\mathcal{I}\,(\mathcal{F}) = \{\mathbb{N} \setminus M : M \in \mathcal{F}\}$ is an ideal on $\mathbb{N}$. An ideal is called *proper* if $\mathbb{N} \notin \mathcal{I}$, and a proper ideal is called *admissible* if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. Obviously, an admissible ideal includes all finite subset of $\mathbb{N}$. The ideal $\mathcal{I}_{\text{fin}} = \{I \subseteq \mathbb{N} : I \text{ is finite}\}$ is the minimum admissible ideal according to the inclusion relation. Admissible ideals are expressed as: If $\mathcal{I}$ is a proper ideal and $\mathcal{I} \supseteq \mathcal{I}_{\text{fin}}$ then $\mathcal{I}$ is an admissible ideal.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A subset M of $\mathbb{N}$ is called $\mathcal{F}$-*stationary* if it has nonempty intersection with each member of the filter $\mathcal{F}\,(\mathcal{I})$. Denote the collection of all $\mathcal{F}$-stationary sets by $\mathcal{F}^*\,(\mathcal{I})$. In brief, for an $M \subseteq \mathbb{N}$,

$$M \in \mathcal{F}^*\,(\mathcal{I}) \Longleftrightarrow M \notin \mathcal{I}.$$

The following properties are provided:

- If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then
$$\mathcal{F}\,(\mathcal{I}_1) \subseteq \mathcal{F}\,(\mathcal{I}_2) \subseteq \mathcal{F}^*\,(\mathcal{I}_2) \subseteq \mathcal{F}^*\,(\mathcal{I}_1)\,.$$
Therefore, for every admissible ideal $\mathcal{I}$, we have
$$\mathcal{F}\,(\mathcal{I}_{\text{fin}}) \subseteq \mathcal{F}\,(\mathcal{I}) \subseteq \mathcal{F}^*\,(\mathcal{I}) \subseteq \mathcal{F}^*\,(\mathcal{I}_{\text{fin}})\,.$$
- If $\mathcal{I}$ a maximal ideal according to the inclusion relation, then $\mathcal{F}\,(\mathcal{I})$ is a maximal filter (i.e., an ultrafilter) and $\mathcal{F}\,(\mathcal{I}) = \mathcal{F}^*\,(\mathcal{I})$.

Definition 1 *Let (X, ρ) be a metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and $x_0 \in X$. Let $\mathcal{I}$ be any ideal on $\mathbb{N}$. If for every $\varepsilon > 0$*

$$\{n \in \mathbb{N} : \rho\,(x_n, x_0) \geq \varepsilon\} \in \mathcal{I},$$

then (x_n) is said to be $\mathcal{I}$-convergent to x_0. Then we write $\mathcal{I} - \lim x_n = x_0$ ([11]).

We will use the following notations for next definitions. The first notations were used in the literature. But since these sets have filter and $\mathcal{F}$-stationary counterparts, we will use the second notations.

$$\begin{aligned}\mathcal{N} &= \mathcal{F}(\mathcal{I}_{\text{fin}}) = \{M \subseteq \mathbb{N} : \mathbb{N} \setminus M \text{ is finite}\},\\ \mathcal{N}^{\#} &= \mathcal{F}^{*}(\mathcal{I}_{\text{fin}}) = \{M \subseteq \mathbb{N} : M \text{ is infinite}\},\\ \mathcal{N}_{\mathcal{I}} &= \mathcal{F}(\mathcal{I}) = \{M \subseteq \mathbb{N} : \mathbb{N} \setminus M \in \mathcal{I}\},\\ \mathcal{N}_{\mathcal{I}}^{\#} &= \mathcal{F}^{*}(\mathcal{I}) = \{M \subseteq \mathbb{N} : M \notin \mathcal{I}\}.\end{aligned}$$

Definition 2 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X and A be a nonempty subset of X. Then*

$$\limsup_{n\to\infty} A_n := \{x \in X : \forall\varepsilon > 0, \exists M \in \mathcal{F}^{*}(\mathcal{I}_{\text{fin}}) \text{ s.t. } A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset \text{ for } \forall n \in M\}$$

and

$$\liminf_{n\to\infty} A_n := \{x \in X : \forall\varepsilon > 0, \exists M \in \mathcal{F}(\mathcal{I}_{\text{fin}}) \text{ s.t. } A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset \text{ for } \forall n \in M\}$$

are called the outer limit and the inner limit of the sequence (A_n), respectively. The sequence $(A_n)_{n\in\mathbb{N}}$ is said to be Kuratowski convergent to the set A if

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = A.$$

In this case, we write K-$\lim A_n = A$ or $A_n \xrightarrow{K} A$ ([12]).

$\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$ are always closed subsets of X, and thus $A \in \mathrm{Cl}(X)$ if there is a nonempty set A such that $A_n \xrightarrow{K} A$.

Definition 3 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X and A be a nonempty subset of X. Then*

$$\mathcal{I}\text{-}\limsup_{n\to\infty} A_n := \{x \in X : \forall\varepsilon > 0, \exists M \in \mathcal{F}^{*}(\mathcal{I}) \text{ s.t. } A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset \text{ for } \forall n \in M\}$$

and

$$\mathcal{I}\text{-}\liminf_{n\to\infty} A_n := \{x \in X : \forall\varepsilon > 0, \exists M \in \mathcal{F}(\mathcal{I}) \text{ s.t. } A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset \text{ for } \forall n \in M\}$$

are called the $\mathcal{I}$-outer limit and the $\mathcal{I}$-inner limit of the sequence (A_n), respectively. The sequence $(A_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-Kuratowski convergent to the set A if

$$\mathcal{I}\text{-}\limsup_{n\to\infty} A_n = \mathcal{I}\text{-}\liminf_{n\to\infty} A_n = A.$$

In this case, we write $\mathcal{I}$-K-$\lim A_n = A$ or $A_n \xrightarrow{\mathcal{I}\text{-}K} A$ ([17]). Here $\mathcal{I}$-$\limsup_{n\to\infty} A_n$, $\mathcal{I}$-$\liminf_{n\to\infty} A_n$ and A are closed subsets of X.

Lemma 1 ([17]) *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X and $A \in \mathrm{Cl}(X)$. Let $\mathcal{I}$ be an admissible ideal. Then we have*

$$\liminf_{n\to\infty} A_n \subseteq \mathcal{I}\text{-}\liminf_{n\to\infty} A_n \subseteq \mathcal{I}\text{-}\limsup_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$$

and

$$K\text{-}\lim A_n = A \Longrightarrow \mathcal{I}\text{-}K\text{-}\lim A_n = A.$$

Hausdorff distance of two sets is defined in three different ways: For nonempty sets $A, B \subseteq X$,

1. $$H(A,B) = \max\{h(A,B), h(B,A)\}$$

   where $h(A,B) = \sup_{a\in A} d(a,B)$ is the excess of A over B.

2.
$$H(A,B)=\inf\{\varepsilon>0 : A\subseteq B^{\varepsilon} \text{ and } B\subseteq A^{\varepsilon}\}$$
where $A^{\varepsilon}=\bigcup_{a\in A}\{x\in X:\rho(a,x)<\varepsilon\}=\{x\in X:d(x,A)<\varepsilon\}$ is the ε-enlargement of A.

3.
$$H(A,B)=\sup_{x\in X}|d(x,A)-d(x,B)|.$$

Definition 4 *A sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X is said to be Hausdorff convergent to a set $A\subseteq X$ if*
$$\lim_{n\to\infty}H(A_n,A)=0.$$
In this case, we write $A_n\xrightarrow{H}A$ ([18]).

Definition 5 *We say that the sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X is $\mathcal{I}$-Hausdorff convergent to the set A if*
$$\mathcal{I}\text{-}\lim H(A_n,A)=0,$$
i.e., for every $\varepsilon>0$, we have
$$\{n\in\mathbb{N}:H(A_n,A)\geq\varepsilon\}\in\mathcal{I}.$$
In this case, we write $\mathcal{I}$-H-$\lim A_n=A$ or $A_n\xrightarrow{\mathcal{I}\text{-}H}A$ ([17]).

Definition 6 *We say that a sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X is Wijsman convergent to a set $A\subseteq X$ if*
$$\lim_{n\to\infty}d(x,A_n)=d(x,A) \text{ for each } x\in X.$$
In this case, we write $A_n\xrightarrow{W}A$ ([18, 19]).

Definition 7 *We say that a sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X is $\mathcal{I}$-Wijsman convergent to a set $A\subseteq X$ if*
$$\mathcal{I}\text{-}\lim d(x,A_n)=d(x,A) \text{ for each } x\in X,$$
i.e., for each $x\in X$ we have
$$\{n\in\mathbb{N}:|d(x,A_n)-d(x,A)|\geq\varepsilon\}\in\mathcal{I} \text{ for every } \varepsilon>0.$$
In this case, we write $\mathcal{I}$-$W-\lim A_n=A$ or $A_n\xrightarrow{\mathcal{I}\text{-}W}A$ ([10]).

For definitions of statistical Wijsman convergence and statistical Hausdorff convergence, the reader can refer to [15].

From the results obtained in [3], we can give the following lemma.

Lemma 2 ([3]) *Let $(A_n)_{n\in\mathbb{N}}$ be a nested sequence where $A_n\in\mathcal{K}(X)$ for every $n\in\mathbb{N}$.*

1. *If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence and $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}A_n\right)\in\mathcal{K}(X)$, then*
$$K\text{-}\lim A_n=W\text{-}\lim A_n=H\text{-}\lim A_n=\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}A_n\right).$$

2. *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence, then*
$$K\text{-}\lim A_n=W\text{-}\lim A_n=H\text{-}\lim A_n=\bigcap_{n\in\mathbb{N}}A_n.$$

In [1], the following result was obtained.

Lemma 3 ([1]) *Let $(A_n)_{n\in\mathbb{N}}$ be a nested sequence where $A_n \in \mathcal{K}(X)$ for every $n \in \mathbb{N}$.*

1. *If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence and $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \in \mathcal{K}(X)$, then*

$$\mathcal{I}\text{-}W\text{-}\lim A_n = \mathcal{I}\text{-}H\text{-}\lim A_n = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right)$$

for every admissible ideal $\mathcal{I}$ on $\mathbb{N}$.

2. *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence then*

$$\mathcal{I}\text{-}W\text{-}\lim A_n = \mathcal{I}\text{-}H\text{-}\lim A_n = \bigcap_{n\in\mathbb{N}} A_n$$

for every admissible ideal $\mathcal{I}$ on $\mathbb{N}$.

The condition $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \in \mathcal{K}(X)$ cannot be removed in the hypothesis of Lemmas 2 and 3. Otherwise the given equations may not be satisfied as seen in the example below.

Example 1 *Let's consider the sequence $(A_n)_{n\in\mathbb{N}}$ defined by $A_n = [0,n]$ for every $n \in \mathbb{N}$ in $\mathbb{R}$. We have $A_n \in \mathcal{K}(X)$ for every $n \in \mathbb{N}$, but $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = [0,\infty) \notin \mathcal{K}(X)$. Even though $K-\lim A_n = W-\lim A_n = [0,\infty)$, the sequence $(A_n)_{n\in\mathbb{N}}$ is not Hausdorff convergent.*

3 Main Results

First, we start by defining the concept of $\mathcal{I}$-nested sequence. The results are given our results for nested sequences and $\mathcal{I}$-nested sequences of sets. We show that Kuratowski convergence and $\mathcal{I}$-Kuratowski convergence are equivalent for nested sequences. Finally, we give some results regarding $\mathcal{I}$-Kuratowski limit sets.

Definition 8 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of X and $\mathcal{I}$ be any ideal on $\mathbb{N}$. We say that the sequence $(A_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$-monotone increasing if there exist a set*

$$M = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}(\mathcal{I})$$

such that $A_{n_k} \subseteq A_{n_{k+1}}$ for every $k \in \mathbb{N}$. Similarly, $(A_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-monotone decreasing if there exist a set

$$M = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}(\mathcal{I})$$

such that $A_{n_{k+1}} \subseteq A_{n_k}$ for every $k \in \mathbb{N}$. If $(A_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$-monotone increasing or $\mathcal{I}$-monotone decreasing, then we say that $(A_n)_{n\in\mathbb{N}}$ is an $\mathcal{I}$-nested sequence.

Theorem 1 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of non-empty subsets of X and $\mathcal{I}$ be any ideal on $\mathbb{N}$. If a sequence $(A_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$-nested then there exist a nested sequence $(B_n)_{n\in\mathbb{N}}$ and a sequence $(C_n)_{n\in\mathbb{N}}$ such that $A_n \cup B_n = C_n \cup B_n$ for every $n \in \mathbb{N}$ and $\{n \in \mathbb{N} : C_n \neq \emptyset\} \in \mathcal{I}$.*

Proof. The proof is given for $\mathcal{I}$-monotone increasing sequences. Another case is similar.

Let $(A_n)_{n\in\mathbb{N}}$ be $\mathcal{I}$-monotone increasing. Then there is a set

$$M = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}(\mathcal{I})$$

such that $(A_n)_{n\in M}$ is monotone increasing, i.e., $A_{n_k} \subseteq A_{n_{k+1}}$ for every $k \in \mathbb{N}$. Let's define the sets B_n and C_n for each $n \in \mathbb{N}$ as follows:

$$B_n = \begin{cases} A_n & \text{if } n \in M, \\ A_{n_1} & \text{if } n < n_1, \\ A_{n_{k+1}} & \text{if } n_k < n < n_{k+1}, \end{cases} \quad (k \in \mathbb{N}) \ \text{ and } \ C_n = A_n \setminus B_n.$$

Then the sequence $(B_n)_{n\in\mathbb{N}}$ is monotone increasing. Also, it's easy to see that $A_n \cup B_n = C_n \cup B_n$ for every $n \in \mathbb{N}$. Finally, for each $n \in M$ we have $C_n = A_n \setminus B_n = A_n \setminus A_n = \emptyset$. Hence we get

$$\{n \in \mathbb{N} : C_n \neq \emptyset\} \subseteq \mathbb{N} \setminus M \in \mathcal{I} \ \text{ and } \ \{n \in \mathbb{N} : C_n \neq \emptyset\} \in \mathcal{I}.$$

■

Theorem 2 *Let $(A_n)_{n\in\mathbb{N}}$ be a nested sequence of closed subsets of X, $A \in \mathrm{Cl}(X)$ and $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. Then we have:*

$$A_n \xrightarrow{K} A \Longleftrightarrow A_n \xrightarrow{\mathcal{I}\text{-}K} A. \tag{1}$$

Proof. $(\Longrightarrow)$: It was given in Lemma 1.
$(\Longleftarrow)$: Assume that $A_n \xrightarrow{\mathcal{I}\text{-}K} A$, that is,

$$\mathcal{I}\text{-}\liminf_{n\to\infty} A_n = \mathcal{I}\text{-}\limsup_{n\to\infty} A_n = A. \tag{2}$$

From Lemma 1, we can write

$$\liminf_{n\to\infty} A_n \subseteq A \subseteq \limsup_{n\to\infty} A_n. \tag{3}$$

We will continue the proof in two parts according to whether the sequence $(A_n)_{n\in\mathbb{N}}$ is increasing or decreasing.

(1) Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence such that $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$.

Firstly, we show that $A_n \subseteq A$ for every $n \in \mathbb{N}$. Let's fix $n \in \mathbb{N}$ and let $u \in A_n$. Since (A_n) is increasing, we have $u \in A_m$ for every $m \geq n$. Then, for every $m \geq n$ and every $\varepsilon > 0$

$$A_m \cap \mathrm{B}(u; \varepsilon) \neq \emptyset.$$

Since

$$\mathbb{N} \setminus \{1, 2, ..., n-1\} \in \mathcal{F}(\mathcal{I}_{\text{fin}}) \subseteq \mathcal{F}(\mathcal{I}),$$

we get $u \in \mathcal{I}\text{-}\liminf_{n\to\infty} A_n = A$. So we get $A_n \subseteq A$ for every $n \in \mathbb{N}$.

Let $x \in \limsup_{n\to\infty} A_n$ and $\varepsilon > 0$. Then there is an infinite set $M(x, \varepsilon) \in \mathbb{N}$ such that $A_n \cap \mathrm{B}(x; \varepsilon) \neq \emptyset$ for every $n \in M(x, \varepsilon)$. Since $A_n \subseteq A$, we get

$$A \cap \mathrm{B}(x; \varepsilon) \neq \emptyset$$

for each $\varepsilon > 0$. From the closedness of A, we have $x \in A$. Therefore we get

$$\limsup_{n\to\infty} A_n = A.$$

Now we show that $\liminf_{n\to\infty} A_n = A$. Let $x \in A$ and $\varepsilon > 0$. Since $x \in \mathcal{I}\text{-}\liminf_{n\to\infty} A_n$, there is a set $N(x, \varepsilon) \in \mathcal{F}(\mathcal{I})$ such that $A_n \cap \mathrm{B}(x; \varepsilon) \neq \emptyset$ for every $n \in N(x, \varepsilon)$. Let $n_0 = n_0(x, \varepsilon) := \min N(x, \varepsilon)$. Since (A_n) is increasing, for every $n \geq n_0$ we get $A_{n_0} \subseteq A_n$ and so

$$\emptyset \neq A_{n_0} \cap \mathrm{B}(x; \varepsilon) \subseteq A_n \cap \mathrm{B}(x; \varepsilon).$$

Hence, for each $\varepsilon > 0$ there is an $n_0(x, \varepsilon) \in \mathbb{N}$ such that

$$A_n \cap \mathrm{B}(x; \varepsilon) \neq \emptyset \text{ for every } n \geq n_0.$$

So we get $x \in \liminf_{n\to\infty} A_n$. This implies that $A \subseteq \liminf_{n\to\infty} A_n$. From (3), we have $\liminf_{n\to\infty} A_n = A$.

Consequently, we obtain $A_n \xrightarrow{K} A$.

(2) Let $(A_n)_{n\in\mathbb{N}}$ be a decreasing sequence such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$.

Firstly, we show that $A \subseteq A_n$ for every $n \in \mathbb{N}$. Let's fix $n \in \mathbb{N}$ and let $u \in A$. Since $A = \mathcal{I}\text{-}\liminf_{n\to\infty} A_n$, for every $\varepsilon > 0$ there is an $M(u,\varepsilon) \in \mathcal{F}(\mathcal{I})$ such that

$$A_m \cap \mathrm{B}(u;\varepsilon) \neq \emptyset \text{ for every } m \in M(u,\varepsilon).$$

Since $M(u,\varepsilon)$'s are infinite sets, for each $\varepsilon > 0$ we can choose an $m_\varepsilon \in M(u,\varepsilon)$ such that $m_\varepsilon \geq n$. Since (A_n) is decreasing, we have $A_{m_\varepsilon} \subseteq A_n$ and

$$\emptyset \neq A_{m_\varepsilon} \cap \mathrm{B}(u;\varepsilon) \subseteq A_n \cap \mathrm{B}(u;\varepsilon).$$

Thus we have $A_n \cap \mathrm{B}(u;\varepsilon) \neq \emptyset$ for every $\varepsilon > 0$. From the closedness of A_n, we get $u \in A_n$. So we get $A_n \subseteq A$ for every $n \in \mathbb{N}$.

Let $x \in A$ and $\varepsilon > 0$. For every $n \in \mathbb{N}$ we have $x \in A_n$ and so $A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$. Thus, we get $x \in \liminf_{n\to\infty} A_n$ and so

$$\liminf_{n\to\infty} A_n = A.$$

Now we show that $\limsup_{n\to\infty} A_n = A$. Let $x \in \limsup_{n\to\infty} A_n$ and $\varepsilon > 0$. Then there is an

$$N(x,\varepsilon) \in \mathcal{F}^*(\mathcal{I}_{\text{fin}})$$

such that $A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$ for every $n \in N(x,\varepsilon)$. Fix $n \in \mathbb{N}$.

Hence there is an $n_\varepsilon \in N(x,\varepsilon)$ such that $n_\varepsilon \geq n$. Since (A_n) is decreasing, we have $A_{n_\varepsilon} \subseteq A_n$ and

$$\emptyset \neq A_{n_\varepsilon} \cap \mathrm{B}(x;\varepsilon) \subseteq A_n \cap \mathrm{B}(x;\varepsilon).$$

Then we get $A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$ for every $n \in \mathbb{N}$. Thus we obtain $x \in \mathcal{I}\text{-}\limsup_{n\to\infty} A_n = A$. This implies that $\limsup_{n\to\infty} A_n = A$.

Consequently, we obtain $A_n \xrightarrow{K} A$. ■

Combining the Theorem 2, Lemma 2 and Lemma 3, we can give the following corollary.

Corollary 1 *Let $(A_n)_{n\in\mathbb{N}}$ be a nested sequence where $A_n \in \mathcal{K}(X)$ for every $n \in \mathbb{N}$.*

1. *If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence and $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \in \mathcal{K}(X)$, then*

$$\mathcal{I}\text{-}K\text{-}\lim A_n = \mathcal{I}\text{-}W\text{-}\lim A_n = \mathcal{I}\text{-}H\text{-}\lim A_n = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right)$$

for every admissible ideal $\mathcal{I}$ on $\mathbb{N}$.

2. *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence, then*

$$\mathcal{I}\text{-}K\text{-}\lim A_n = \mathcal{I}\text{-}W\text{-}\lim A_n = \mathcal{I}\text{-}H\text{-}\lim A_n = \bigcap_{n\in\mathbb{N}} A_n$$

for every admissible ideal $\mathcal{I}$ on $\mathbb{N}$.

Definition 9 *A set A is said to be an ideal Kuratowski limit set (briefly, $\mathcal{I}$-K-limit set) of a sequence $(A_n)_{n\in\mathbb{N}}$ if there exists an*

$$M = \{n_1 < n_2 < \ldots < n_k < \ldots\} \in \mathcal{F}^*(\mathcal{I})$$

such that $K\text{-}\lim_{k\to\infty} A_{n_k} = A$. We will denote by $\mathcal{I}\text{-}K\text{-}\mathrm{LIM}\, A_n$ the collection of all $\mathcal{I}$-K-limit sets of the sequence (A_n).

Lemma 4 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X and $A \in \mathrm{Cl}(X)$. Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If A is an $\mathcal{I}$-K-limit set of a sequence $(A_n)_{n\in\mathbb{N}}$, then*

$$\mathcal{I}\text{-}\liminf_{n\to\infty} A_n \subseteq A \subseteq \mathcal{I}\text{-}\limsup_{n\to\infty} A_n.$$

Proof. Let's assume that A is an $\mathcal{I}$-K-limit set of a sequence $(A_n)_{n\in\mathbb{N}}$. In this case, there exists an

$$M = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}^*(\mathcal{I})$$

such that $K\text{-}\lim_{k\to\infty} A_{n_k} = A$.

- Firstly, we show that $A \subseteq \mathcal{I}\text{-}\limsup_{n\to\infty} A_n$. Let $x \in A$. Take $\varepsilon > 0$. Then there exits a $k_0 \in \mathbb{N}$ such that $A_{n_k} \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$ for every $k \geq k_0$. Let

$$N(x,\varepsilon) := M \setminus \{n_1, n_2, ..., n_{k_0}\} \in \mathcal{F}^*(\mathcal{I}).$$

  So we get

$$A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$$

  for every $n \in N$. Hence we obtain $x \in \mathcal{I}\text{-}\limsup_{n\to\infty} A_n$.

- Now, we show that $\mathcal{I}\text{-}\liminf_{n\to\infty} A_n \subseteq A$. Let $x \in \mathcal{I}\text{-}\liminf_{n\to\infty} A_n$. Take $\varepsilon > 0$. Then there exists an $N(x,\varepsilon) \in \mathcal{F}(\mathcal{I})$ such that $A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$ for every $n \in N$. Since $M \cap N \in \mathcal{F}^*(\mathcal{I})$, the set $M \cap N$ is an infinite set. Therefore, we get

$$A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$$

  for every $n \in M \cap N$. That is, we have $A_{n_k} \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$ for infinitely many k. Hence we obtain $x \in \limsup_{k\to\infty} A_{n_k} = A$.

■

Theorem 3 *Let $(A_n)_{n\in\mathbb{N}}$ be any sequence where $A_n \in \mathcal{K}(X)$ for every $n \in \mathbb{N}$. Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If $\mathcal{I}$-K-LIM $A_n = \{B_1, B_2, ..., B_m\}$ is finite such that $(A_n)_{n\in M_i} \xrightarrow{K} B_i$ for each $i \in \{1,2,...,m\}$ where $M_i \in \mathcal{F}^*(\mathcal{I})$ $(i \in \{1,2,...,m\})$ and $M = \bigcup_{i=1}^m M_i \in \mathcal{F}(\mathcal{I})$, then we get*

$$\mathcal{I}\text{-}\limsup_{n\to\infty} A_n = \bigcup_{i=1}^m B_i$$

and

$$\mathcal{I}\text{-}\liminf_{n\to\infty} A_n = \bigcap_{i=1}^m B_i.$$

Proof. From Lemma 4, it is clear that

$$\bigcup_{i=1}^m B_i \subseteq \mathcal{I}\text{-}\limsup_{n\to\infty} A_n$$

and

$$\mathcal{I}\text{-}\liminf_{n\to\infty} A_n \subseteq \bigcap_{i=1}^m B_i.$$

For the first equality, let's take an arbitrary $x \in \mathcal{I}\text{-}\limsup_{n\to\infty} A_n$. Assume that $x \notin \bigcup\limits_{i=1}^m B_i$. In this case, for each $i \in \{1,2,...,m\}$ there exists $\varepsilon_i > 0$ such that $A_n \cap \mathrm{B}(x;\varepsilon_i) = \emptyset$ for all but finitely many $n \in M_i$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_m\}$. Then we have

$$A_n \cap \mathrm{B}(x;\varepsilon) = \emptyset \text{ for all but finitely many } n \in M = \bigcup_{i=1}^m M_i. \tag{4}$$

From $x \in \mathcal{I}\text{-}\limsup_{n\to\infty} A_n$, there exists $N(x,\varepsilon) \in \mathcal{F}^*(\mathcal{I})$ such that

$$A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset \text{ for every } n \in N(x,\varepsilon). \tag{5}$$

We have $M \cap N(x,\varepsilon) \in \mathcal{F}^*(\mathcal{I})$ and therefore $M \cap N(x,\varepsilon)$ has infinite elements. Hence, statements (4) and (5) contradict each other. So, we get $x \in \bigcup_{i=1}^{m} B_i$.

For the second equality, let $x \in \bigcap_{i=1}^{m} B_i$ and $\varepsilon > 0$. Then for each $i \in \{1,2,...,m\}$ there is a finite set $L_i(x,\varepsilon)$ such that $A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$ for all $n \in M_i \setminus L_i$. Let $L(x,\varepsilon) = \bigcup_{i=1}^{m} L_i$ and

$$M^*(x,\varepsilon) = \bigcup_{i=1}^{m} M_i \setminus L = M \setminus L \in \mathcal{F}(\mathcal{I}).$$

Then we get $A_n \cap \mathrm{B}(x;\varepsilon) \neq \emptyset$ for all $n \in M^*$. Thus we get $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$. ∎

Example 2 *The family $\mathcal{I}_{(a_n)} = \left\{I \subseteq \mathbb{N} : \sum_{n\in I} a_n < +\infty\right\}$ is an admissible ideal on $\mathbb{N}$ and is called a summable ideal, where $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers such that $\sum_{n\in\mathbb{N}} a_n = +\infty$. Let's take the sequence $(a_n)_{n\in\mathbb{N}} = (1/n)_{n\in\mathbb{N}}$ specifically. In this case, we have*

$$\mathcal{F}^*_{(1/n)}\left(\mathcal{I}_{(1/n)}\right) = \{M \subseteq \mathbb{N} : \sum_{n\in M} (1/n) = +\infty\}.$$

Now, on the Euclidean space $\mathbb{R}$ let's define a sequence $(A_n)_{n\in\mathbb{N}}$ by

$$A_n = \begin{cases} \left[-1+\dfrac{1}{2n}, 2+\dfrac{1}{2n}\right] & , \text{ if } n \text{ is odd,} \\ \left[-2-\dfrac{1}{n}, 1-\dfrac{1}{n}\right] & , \text{ if } n \text{ is even,} \end{cases}$$

for each $n \in \mathbb{N}$. Since $M_1 = \{2k-1 : k \in \mathbb{N}\}$ and $M_2 = \{2k : k \in \mathbb{N}\}$ belong to $\mathcal{F}^*_{(1/n)}\left(\mathcal{I}_{(1/n)}\right)$, it is easy to see that the sets $A = [-1,2]$ and $B = [-2,1]$ are $\mathcal{I}$-K-limit sets of the sequence $(A_n)_{n\in\mathbb{N}}$. Additionally, we get

$$\mathcal{I}\text{-}\limsup_{n\to\infty} A_n = A \cup B = [-2,2] \quad \textit{and} \quad \mathcal{I}\text{-}\liminf_{n\to\infty} A_n = A \cap B = [-1,1].$$

If the family $\mathcal{I}$-K-$\lim A_n$ is infinite, Theorem 3 may not be satisfied, as seen in the example below.

Example 3 *Let's consider the ideal $\mathcal{I}_{(1/n)} = \left\{I \subseteq \mathbb{N} : \sum_{n\in I} (1/n) < +\infty\right\}$ again. On the Euclidean space $\mathbb{R}$ let's define a sequence $(A_n)_{n\in\mathbb{N}}$ by*

$$A_n = \left\{\frac{1}{k}\right\} \text{ if } n \in M_k$$

for each $n \in \mathbb{N}$ where

$$M_k := 2^{k-1}(2\mathbb{N}-1) = \left\{2^{k-1}, 3\cdot 2^{k-1}, 5\cdot 2^{k-1}, ..., (2n-1)\,2^{k-1}, ...\right\}$$

for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, since $\sum_{n\in M_k} \frac{1}{n} = \frac{1}{2^{k-1}} \sum_{n\in\mathbb{N}} \frac{1}{2n-1} = +\infty$, we have $M_k \in \mathcal{F}^*_{(1/n)}\left(\mathcal{I}_{(1/n)}\right)$. Then we get $\mathcal{I}$-K-LIM $A_n = \left\{\left\{\frac{1}{k}\right\} : k \in \mathbb{N}\right\}$,

$$\mathcal{I}\text{-}\liminf_{n\to\infty} A_n = \bigcap_{A\in\mathcal{I}\text{-}K\text{-}\mathrm{LIM}\,A_n} A = \emptyset \quad \textit{and} \quad \bigcup_{A\in\mathcal{I}\text{-}K\text{-}\mathrm{LIM}\,A_n} A = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{k}, ...\right\}$$

but

$$\mathcal{I} - \limsup_{n\to\infty} A_n = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{k}, ...\right\} \cup \{0\}.$$

The following theorem is easily obtained from Lemma 2.

Theorem 4 *Let $A_n \in \mathcal{K}(X)$ for every $n \in \mathbb{N}$ and let $M = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}^*(\mathcal{I})$ where $\mathcal{I}$ is an admissible ideal.*

1. *If the subsequence $(A_{n_k})_{k\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ is monotone increasing and* $\operatorname{cl}\left(\bigcup_{k\in\mathbb{N}} A_{n_k}\right) \in \mathcal{K}(X)$. *Then the set* $\operatorname{cl}\left(\bigcup_{k\in\mathbb{N}} A_{n_k}\right)$ *is an $\mathcal{I}$-K-limit set of $(A_n)_{n\in\mathbb{N}}$.*
2. *If the subsequence $(A_{n_k})_{k\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ is monotone decreasing. Then the set $\bigcap_{k\in\mathbb{N}} A_{n_k}$ is an $\mathcal{I}$-K-limit set of $(A_n)_{n\in\mathbb{N}}$.*

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