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Local Linear Dependence And Local Multiplicity For Two Linear Operators*

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Abstract

Linear dependence is a very important and fundamental concept in linear algebra, operator theory, and all related fields. When linear maps $T_1, \ldots, T_n : X \to Y$ are linearly dependent, they are necessarily "locally linearly dependent", which means $T_1x, \ldots, T_nx$ are linearly dependent for all $x \in X$. For two linear operators $T, S : X \to Y$, T is said a local multiple to S, whenever T is a point-wise multiple of S, i.e., there is a scalar-valued function $c : X \to \mathbb{F}$ such that $Tx = c(x)Sx$ holds for all $x \in X$. If T is a local multiple to S, then T and S are locally linearly dependent, whereas, as we will show, the reverse of this claim is not necessarily true. In this paper, we characterize two local multiple operators. We show T is a local multiple to S, if and only if T is a scalar multiple to S. As a consequence, we apply our results to bounded linear operators defined on inner product spaces.

1 Introduction

For a vector space X, a linear operator $T : X \to X$ is said to be algebraic if there exists a non-trivial polynomial p such that $p(T) = 0$. Kaplansky ([7]) has proved that for a complex vector space X the linear operator $T : X \to X$ is algebraic if and only if for every $x \in X$, there exists a positive integer n such that $x, Tx, \ldots, T^nx$ are linearly dependent. For a vector space X over an algebraically closed field F, Cater ([4]) proved the linear operator $T : X \to X$ is algebraic if and only if

$$\sup_{x \in X} \dim \operatorname{span} \{x, Tx, T^2x, T^3x, \ldots\} < \infty. \tag{1}$$

In other words, by (1), T is algebraic if and only if there is $N \in \mathbb{N}$ such that for all $x \in X$ we have $\dim \operatorname{span}\{x, Tx, T^2x, T^3x, \ldots\} \leq N$.

Both of the equivalent conditions obtained by Kaplansky and Cater include a concept called *local linear dependence*.

Linear operators $T_1, \ldots, T_n : X \to Y$ are said locally linearly dependent if $T_1x, \ldots, T_nx$ are linearly dependent for every $x \in X$. Local linear dependence for linear operators $T_1, \ldots, T_n$ is obviously equivalent to for every $x \in X$ there is a nonzero $S \in \operatorname{span}\{T_1, \ldots, T_n\}$ such that $Sx = 0$. The structure of n-tuples of locally linearly dependent has applications in ring theory, derivations, and reflexivity of operator spaces (see, for examples [1, 2, 3, 5, 8]).

In what follows, we will discuss the concept of local multiple for two linear operators defined on a vector space over the field $\mathbb{F}$, where $\mathbb{F}$ denotes the real or complex numbers. In Section 2, we show for two linear operators $S, T : X \to Y$, S is local multiple to T, if and only if S is an scalar multiple to T. Meanwhile, we show the local linear dependence is not equivalent to linear dependence.

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2 Local Multiplicity and Local Linear Dependence Operators

First, we define local multiplicity and local linear dependence for linear operators. In the following, we will restrict to linear operators between vector spaces over the field $\mathbb{F}$ of real or complex numbers.

Definition 1 ([6, 9, 10, 12]) *For $n \in \mathbb{N}$ the linear operators $T_1, \dots, T_n : X \to Y$ are locally linearly dependent, if $T_1x, \dots, T_nx$ are linearly dependent for every $x \in X$.*

Linear dependence is stronger than local linear dependence. Moreover, $T_1, \dots, T_n$ are locally linearly dependent operators if and only if there are functions $c_1, \dots, c_n : X \to \mathbb{F}$ such that $\big(c_1(x), \dots, c_n(x)\big) \neq 0$ and that $c_1(x)T_1x + \cdots + c_n(x)T_nx = 0$, holds for all $x \in X$.

Local multiplicity is a special case of local linear dependence that is defined as follows:

Definition 2 *Let X and Y be two vector spaces over the field $\mathbb{F}$ and $T, S : X \to Y$ are two linear maps. Then T is said to be locally multiple to S if there is a function $c : X \to \mathbb{F}$ such that $Tx = c(x)Sx$ holds for all $x \in X$.*

For a linear map $f : X \to Y$, $\ker(f)$ denotes the set of all vectors $x \in X$ with $f(x) = 0$. The following theorem is a basic tool about functionals and will be used in the following results.

Theorem 1 ([11, Lemma 3.1]) *Let X be a linear space and $f, g_1, \dots, g_n$ be linear functionals on X. If $\cap_{i=1}^n \ker(g_i) \subseteq \ker(f)$, then f is a linear combination of $g_1, \dots, g_n$.*

We note that the previous theorem is not true for arbitrary linear maps. For example, if $C(\mathbb{R})$ denotes the set of all continuous real functions on $\mathbb{R}$, and $T, S : C(\mathbb{R}) \to \mathbb{C}(\mathbb{R})$ are defined by $Tf := f$ (the identity operator) and $Sf := g$, where g is defined by $g(x) := f(x + 1)$, for all $f \in C(\mathbb{R})$ and $x \in \mathbb{R}$, then it is clear that both T and S are one-to-one. So, we have $\ker(T) = \{0\} \subseteq \ker(S) = \{0\}$ while S is not a multiple of T.

In the following theorem, we show the two notions of local multiplicity and multiplicity are equivalent.

Theorem 2 *Let $T, S : X \to Y$ be two linear maps. If T is a local multiple to S, then T is a multiple to S.*

Proof. Suppose that $Tx = c(x)Sx$, holds for all $x \in X$. We consider the following three cases:

Case I. $\dim \operatorname{Im}(S) = 0$; Then it is obvious that $S = T = 0$.

Case II. $\dim \operatorname{Im}(S) = 1$; Then we have from using Theorem 1 that $T = cS$, for some constant $c \in \mathbb{F}$.

Case III. $\dim \operatorname{Im}(S) \geq 2$; Suppose that $x, y \notin \ker(S)$. If Sx and Sy are linearly independent, then we have

$$\begin{aligned} c(x)Sx + c(y)Sy &= Tx + Ty \\ &= T(x + y) \\ &= c(x + y)S(x + y) \\ &= c(x + y)Sx + c(x + y)Sy. \end{aligned}$$

Therefore, $(c(x + y) - c(x))Sx + (c(x + y) - c(y))Sy = 0$, which follows from the linear independence of Sx and Sy that $c(x+y) = c(x)$ and $c(x+y) = c(y)$. Thus $c(x) = c(y)$. Now, if Sx and Sy are linearly dependent, then

$$\dim \operatorname{span}\{Sx, Sy\} = 1 < 2 \leq \dim \operatorname{Im}(S). \tag{2}$$

Therefore, there is $z \in X$ such that the two sets $\{Sx, Sz\}$ and $\{Sy, Sz\}$ are independent. So, from using *Case II* we have $c(x) = c(y)$. Thus $c(x) = c(y)$ holds for all $x, y \notin \ker(S)$, which shows the function $c : X \to \mathbb{F}$ is constant on $X \setminus \ker(S)$, with the constant value λ. Therefore, $T = \lambda S$ which completes the proof. ∎

Example 1 *Suppose that H is a Hilbert space, $0 \neq x_0 \in H$, and $\theta_1, \theta_2 \in H^*$, where H^* denotes the dual of H. Let $T_1, T_2 : H \to H$ are both bounded linear operators which are defined as $T_1x = \theta_1(x)x_0$ and $T_2x = \theta_2(x)x_0$, for all $x \in H$. Then we have*

(i) *T_1 and T_2 are locally linearly dependent ((i) follows from the equality $\{T_1x, T_2x\} = \{\theta_1(x)x_0, \theta_2(x)x_0\}$).*

(ii) *T_1 is a local multiple to T_2 if and only if T_1 is multiple to T_2 (using Theorem 2).*

(iii) *T_1^* and T_2^* are not necessarily locally linearly dependent (To show (iii), using Riesz representation theorem, there are $x_1, x_2 \in H$ such that $T_1x = \langle x, x_1\rangle x_0$ and $T_2x = \langle x, x_2\rangle x_0$. Therefore, $T_1^*x = \langle x, x_0\rangle x_1$ and $T_2^*x = \langle x, x_0\rangle x_2$, that shows T_1^* and T_2^* are not necessarily locally linearly dependent operators).*

Theorem 3 *Suppose $T, S : X \to Y$ are linear operators. Then the following conditions are equivalent.*

(i) T and S are locally linearly dependent and $\ker(T) \subseteq \ker(S)$;

(ii) S is a multiple of T; and

(iii) S is a local multiple to T.

Proof. (i) $\Rightarrow$ (ii): There are two functions $c_1, c_2 : X \to \mathbb{F}$ such that

$$(c_1(x), c_2(x)) \neq 0 \;\text{ and }\; c_1(x)Tx + c_2(x)Sx = 0 \;\text{ for all } x \in X. \tag{3}$$

For any given $x \in X$ we consider the following two cases:

Case 1. $Sx \neq 0$. Then $c_2(x) \neq 0$, otherwise it follows from (3) that $c_1(x) \neq 0$ and $c_1(x)Tx = 0$. So $Tx = 0$ which contradicts the assumption that $\ker(T) \subseteq \ker(S)$. Therefore, $c_2(x) \neq 0$ which implies from (3) that

$$Sx = \frac{-c_1(x)}{c_2(x)}Tx \;\text{ for all } x \in X \text{ with } Sx \neq 0. \tag{4}$$

Case 2. $Sx = 0$. Then it is clear that

$$Sx = 0Tx \;\text{ for all } x \in X \text{ with } Sx = 0. \tag{5}$$

From using (4) and (5) it follows that S is a local multiple of T. Thus, by Theorem 2, S is a multiple of T.
(ii) $\Rightarrow$ (i): It is obvious.
(ii) $\Leftrightarrow$ (iii): Follows from Theorem 2. ∎

The following result is a direct consequence of Theorem 3.

Corollary 1 *If $T, S : X \to Y$ are locally linearly dependent operators with $\ker(T) = \ker(S)$, then S is a non-zero multiple of T.*

Example 2 *Suppose that $T_1, T_2, T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by $T_1 = \mathrm{id}$, $T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $T_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then it is clear that, however, $\{T_1, T_2, T_3\}$ is independent, but it is a locally linearly dependent set. In fact if we set $c_1(x) = x_1^2$, $c_2(x) = -x_1x_2$, and $c_3(x) = x_2^2 - x_1^2$, for each $0 \neq x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^3$ and we set $c_1(0) = c_2(0) = c_3(0) = 1$, then $(c_1(x), c_2(x), c_3(x)) \neq 0$. Furthermore, $c_1(x)T_1x + c_2(x)T_2x + c_3(x)T_3x = 0$.*

As the previous example shows, in the general case, local dependence does not necessarily follow dependence. The next theorem can be proved easily and is an extension of the previous example to a more general state.

Theorem 4 *Suppose that $m, n \in \mathbb{N}$, X and Y are two vector spaces and $m = \dim Y < n$. Then every linear maps $T_1, \ldots, T_n : X \to Y$ are locally linearly dependent. In particular, every two linear functionals θ_1, θ_2 on X are locally linearly dependent.*

Theorem 5 *Let X and Y be two vector spaces, $Y \neq 0$, and $\dim X \geq n > 1$. Then there are linear maps $T_1, \ldots, T_n : X \to Y$ such that $\{T_1, \ldots, T_n\}$ is locally linearly dependent but not dependent, i.e., $\{T_1, \ldots, T_n\}$ is independent.*

Proof. Because $Y \neq 0$, it implies that there is $y_0 \in Y$ with $y_0 \neq 0$. On the other hand, from using the assumption $\dim X \geq n > 1$, there are $\mathrm{e}_1, \ldots, \mathrm{e}_n \in X$ such that $\mathcal{B}_0 = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\}$ is independent. If $\mathcal{B} \subseteq X$ is an extension of $\mathcal{B}_0$ to some algebraic basis of X, then for each $x \in X$, there are unique $x_0 = \alpha_1 \mathrm{e}_1 + \cdots + \alpha_n \mathrm{e}_n \in \operatorname{span}(\mathcal{B}_0)$ and $x_1 \in \operatorname{span}(\mathcal{B} \setminus \mathcal{B}_0)$ such that $x = x_0 + x_1$. If we set $T_k x = \alpha_k y_0$ for all $1 \leq k \leq n$ and $x \in X$, then it is clear that $\{T_1, \ldots, T_n\}$ is independent. Moreover, because

$$\dim \operatorname{span}\{T_1 x, \ldots, T_n x\} \leq \dim \operatorname{span}\{y_0\} = 1 < n,$$

it follows that $\{T_1, \ldots, T_n\}$ is locally linearly dependent. ■

The next corollary is an application of Theorem 3 for operators on inner product spaces.

Corollary 2 *Let X be a linear space and Y an inner product space over the field $\mathbb{F}$. If $T, S : X \to Y$ are linear maps which satisfy $|\langle Tx, y\rangle| \leq |\langle Sx, y\rangle|$, for all $x \in X$ and $y \in Y$, then T is a multiple of S.*

Proof. First, we show T and S are locally linearly dependent. Because, otherwise, there is $x \in X$ such that $\{Tx, Sx\}$ is an independent subset of Y. Now we consider $y = Tx - \frac{\langle Tx, Sx\rangle}{\|Sx\|^2} Sx$. Therefore, we have

$$\langle Tx, y\rangle = \langle Tx, Tx\rangle - \frac{\overline{\langle Tx, Sx\rangle}}{\|Sx\|^2}\langle Tx, Sx\rangle = \|Tx\|^2 - \frac{|\langle Tx, Sx\rangle|^2}{\|Sx\|^2} \tag{6}$$

Because $\{Tx, Sx\}$ is independent, the equality is not satisfies in the Cauchy-Schwarz inequality $|\langle Tx, Sx\rangle| \leq \|Tx\| \, \|Sx\|$. Therefore, from (6) and using the assumption we obtain

$$0 < |\langle Tx, y\rangle| \leq |\langle Sx, y\rangle| = |\langle Sx, Tx\rangle - \langle Sx, Tx\rangle| = 0,$$

which contradicts. This contradiction shows T and S are locally linearly dependent. On the other hand, it is clear that $\ker(S) \subseteq \ker(T)$. Thus from Theorem 3, T is a scalar multiple of S. ■

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