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Note On: "Manufacturing Setup Cost Reduction And Quality Improvement For The Distribution Free Continuous-Review Inventory Model With A Service Level Constraint"*

Sushil Kumar Bhuiya†, Amal Sarkar‡

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Abstract

Sarkar et al. [6] recently proposed a distribution-free continuous-review inventory model with a service level constraint, effectively analyzing the benefits of setup cost reduction and quality improvement. However, they used an algorithm to obtain the optimal solution for the model under consideration. This note focuses on developing a closed-form optimal solution for that model. Additionally, a rigorous mathematical interrelationship between the parameters of the inventory system and the solutions is derived using Descartes' rule of signs.

1 Introduction

The continuous-review inventory model is one of the most important inventory systems, extensively studied in the literature and widely used in practice ([2], [5], [8]). In a recent study, Sarkar et al. [6] extended the distribution-free continuous-review inventory model with a service level constraint, originally developed by Moon et al. [3], to include the perspectives of manufacturing setup cost reduction and quality improvement. However, closed-form solutions for inventory systems are highly preferred by managers and practitioners due to their ease of use and the ability to provide clear managerial insights. From this perspective, Tajbakhsh [9] derived closed-form expressions for the order quantity and reorder point for Moon et al.'s [3] inventory model. Moon et al. [4] later extended Tajbakhsh model by considering variable lead times and a negative exponential crashing cost. They have obtained the closed-form solutions for the optimal order quantity, reorder point, and lead time. In this note, we consider the inventory system studied by Sarkar et al. [6] and derive closed-form expressions for the order quantity, reorder point, setup cost, and the probability of the production process going out of control. Moreover, we explore the mathematical relationships between the parameters and solutions of the inventory system using Descartes' rule of signs.

2 A Brief Review of the Work of Sarkar et al.

In this section, we will briefly revisit the work of Sarkar et al. [6] to facilitate the analysis. We use the same notations and assumptions employed in the analysis of Sarkar et al.'s [6] model. The annual expected total cost function is derived as follows.

$$\begin{aligned} EAC(Q,r,\phi,A) &= \frac{AD}{Q} + h\left(\frac{Q}{2} + r - \mu\right) + \alpha(Y - b\ln\phi - B\ln A) \\ &\quad + \frac{mDQ\phi}{2} \text{ for } 0 < A \le A_0 \text{ as well as } 0 < \phi \le \phi_0. \end{aligned}$$

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†Department of Mathematics, Krishnath College (Murshidabad University), Berhampore, West Bengal 742101, India (Corresponding Author)

‡Department of Economics, Krishnath College (Murshidabad University), Berhampore, West Bengal 742101, India

The expected cost function has been optimized by incorporating a fill rate. The fill rate is the fraction of demand fulfilled directly from the self. The fill rate is denoted by β and defined as follows.

$$\begin{aligned}\beta &= \frac{\text{expected demand satisfied per replenishment cycle}}{\text{expected demand per replenishment cycle}}, \\ &= 1 - \frac{E(X-r)^+}{Q} \Rightarrow E(X-r)^+ = (1-\beta)Q.\end{aligned}$$

The model under consideration assumes that the exact distributional information regarding lead time demand is unknown. Only, the first two moments of the lead-time demand are given to the decision-maker. Let us consider that $\mathcal{F}$ is the class of all probability distribution with mean μ and variance σ^2. Therefore, the objective is to find the worst possible distribution function $F \in \mathcal{F}$ for the decision variable and then minimize the annual cost function over the decision space. Mathematically, the model can be written as the following optimization problem.

$$(\mathrm{P}_1) \quad \begin{cases} \min\limits_{Q_p, r, \phi, A} \left[\max\limits_{F \in \mathcal{F}} EAC(Q, r, \phi, A) \right] \\ \text{such that } E(X-r)^+ = (1-\beta)Q, \\ A \leq A_0, \\ \phi \leq \phi_0, \\ Q, r, A, \phi > 0. \end{cases}$$

The above problem has been solved using the min-max distribution-free procedure developed by Scarf [7] and further improved by Gallego and Moon [1]. They proposed the following well-known lemma, which has been extensively used to address the min-max distribution-free inventory problem.

Lemma 2.1 **([1])** *Let $\mathcal{F}$ be the class of all cumulative distribution functions with mean μ and variance σ^2. Let X be a random variable of $\mathcal{F}$. Then*

$$E(X-r)^+ \leq \frac{\sqrt{\sigma^2 - (r-\mu)^2} - (r-\mu)}{2}.$$

Moreover, the above inequality is sharp for every r.

In connection to the fill rate β, let us define the safety stock as $\omega_\beta = r - \mu$, and we get

$$\frac{\sqrt{\sigma^2 + \omega_\beta}}{2} = (1-\beta)Q \Rightarrow \omega_\beta = \frac{\sigma^2}{4(1-\beta)Q} - (1-\beta)Q. \tag{1}$$

Plugging in the value of ω_β in the cost function and letting Q as Q_β, the expected annual cost function can be calculated as follows.

$$\begin{aligned} EACW(Q_\beta, A, \phi) \quad &= \quad \frac{AD}{Q_\beta} + hQ_\beta\left(\beta - \frac{1}{2}\right) + \alpha(Y - b\ln\phi - B\ln A) \\ &\quad + \frac{h\sigma^2}{4(1-\beta)Q_\beta} + \frac{mDQ_\beta\phi}{2}. \end{aligned} \tag{2}$$

Therefore, the optimization problem (P_1) is converted to the following minimization problem.

$$(\mathrm{P}_2) \quad \begin{cases} \min \ EACW(Q_\beta, A, \phi) \\ \text{such that} \quad A \leq A_0, \\ \phi \leq \phi_0, \\ Q_\beta, A, \phi > 0. \end{cases}$$

3 Analysis of Closed Form Solution of the Model

In this section, we derive and analyze the closed-form analytic solution to the optimization problem (P_2). Two inequality constraints of the problem (P_2) can be converted to equality constraints by adding two slack variables s_1^2 and s_2^2. Therefore, the Lagrange function is written as

$$\begin{aligned}\mathfrak{L}(Q_\beta, A, \phi, \lambda_1, \lambda_2) &= \frac{AD}{Q_\beta} + hQ_\beta\left(\beta - \frac{1}{2}\right) + \alpha(Y - b\ln\phi - B\ln A)\\ &\quad + \frac{h\sigma^2}{4(1-\beta)Q_\beta} + \frac{mDQ_\beta\phi}{2} + \lambda_1(A - A_0 + s_1^2) + \lambda_2(\phi - \phi_0 + s_2^2).\end{aligned}$$

Now, Karush-Kuhn-Tucker (KKT) necessary conditions for stationary points of the problem (P_2) are given by

$$\frac{\partial\mathfrak{L}}{\partial Q_\beta} = -\frac{AD}{Q_\beta^2} + h(\beta - \frac{1}{2}) - \frac{h\sigma^2}{4(1-\beta)Q_\beta^2} + \frac{mD\phi}{2} = 0, \tag{3}$$

$$\frac{\partial\mathfrak{L}}{\partial A} = \frac{D}{Q_\beta} - \frac{\alpha B}{A} + \lambda_1 = 0, \tag{4}$$

$$\frac{\partial\mathfrak{L}}{\partial \phi} = -\frac{\alpha b}{\phi} - \frac{mDQ_\beta}{2} + \lambda_2 = 0, \tag{5}$$

$$\begin{gathered} A - A_0 \leq 0, \quad \text{and} \quad \phi - \phi_0 \leq 0,\\ \lambda_1(A - A_0) = 0, \quad \text{and} \quad \lambda_2(\phi - \phi_0) = 0, \qquad (6)\\ \lambda_1, \lambda_2 \geq 0.\end{gathered}$$

Case-I: Let us consider $\lambda_1 = 0$ and $\lambda_2 = 0$. That is $A - A_0 \leq 0$ and $\phi - \phi_0 \leq 0$ are inactive constraints. From the optimality conditions (4) and (5), we get

$$A = \frac{\alpha B Q_\beta}{D}, \tag{7}$$

$$\phi = \frac{2\alpha b}{mDQ_\beta}. \tag{8}$$

Plugging in the values of A and ϕ in the optimality condition (3), we obtain the following quadratic equation for the order quantity Q_β,

$$h(2\beta - 1)Q_\beta^2 - 2\alpha(B - b)Q_\beta - \frac{h\sigma^2}{2(1-\beta)} = 0.$$

Theorem 3.1 *Let $\frac{1}{2} < \beta \leq 1$ and $B \geq b$. Then the order quantity is given by*

$$Q_\beta = \frac{\alpha(B-b) + \sqrt{\alpha^2(B-b)^2 + \frac{(2\beta-1)h^2\sigma^2}{2(1-\beta)}}}{h(2\beta - 1)}. \tag{9}$$

Proof. Let

$$f(Q_\beta) = h(2\beta - 1)Q_\beta^2 - 2\alpha(B - b)Q_\beta - \frac{h\sigma^2}{2(1-\beta)}$$

be a quadratic polynomial with real coefficient. The polynomial $f(Q_\beta)$ has one sign change between the first and second terms. Therefore, $f(Q_\beta)$ has exactly one positive root as per Descartes' rule of signs. The two roots of $f(Q_\beta) = 0$ are given by

$$Q_\beta = \frac{\alpha(B-b) \pm \sqrt{\alpha^2(B-b)^2 + \frac{(2\beta-1)h^2\sigma^2}{2(1-\beta)}}}{h(2\beta - 1)}.$$

However, if possible

$$Q_\beta = \frac{\alpha(B-b) - \sqrt{\alpha^2(B-b)^2 + \frac{(2\beta-1)h^2\sigma^2}{2(1-\beta)}}}{h(2\beta-1)} > 0,$$

which gives $h^2(2\beta-1)\sigma^2 < 0$. Consequently, we get $\beta < \frac{1}{2}$. This is a contradiction, and the proof is complete. ∎

Remark 1 *For $\frac{1}{2} < \beta \leq 1$ and $B \geq b$, the hessian matrix of the objective function $EACW(Q_\beta, A, \phi)$ of the problem* (P_2) *is positive definite [6]. Moreover, all the constraints are linear. Therefore, The KKT conditions are necessary as well as sufficient for the optimal point of the optimization problem* (P_2)*.*

Therefore, the closed-form expression for the safety stock ω_β, set up cost A, probability of out of control ϕ and order quantity (Q_p) are obtained in equations (1), (7), (8), and (9), respectively are the optimal solution of the model.

Remark 2 *We shall find order quantity quantity Q_β for different context of β, B, and b. For each case, ω_β, A and ϕ can be obtained by putting the value Q_β in (1), (7), and (8), respectively.*

Theorem 3.2 *Let $\frac{1}{2} < \beta \leq 1$ and $B < b$. Then the order quantity is given by*

$$Q_\beta = \frac{-\alpha(b-B) + \sqrt{\alpha^2(b-B)^2 + \frac{(2\beta-1)h^2\sigma^2}{2(1-\beta)}}}{h(2\beta-1)}. \tag{10}$$

Proof. The polynomial

$$f(Q_\beta) = h(2\beta-1)Q_\beta^2 + 2\alpha(b-B)Q_\beta - \frac{h\sigma^2}{2(1-\beta)}$$

has one sign change between second and third terms. By applying Descartes' rule of signs, we can say that $f(Q_\beta) = 0$ has exactly one positive root, and the positive root is obtained by

$$Q_\beta = \frac{-\alpha(b-B) + \sqrt{\alpha^2(b-B)^2 + \frac{(2\beta-1)h^2\sigma^2}{2(1-\beta)}}}{h(2\beta-1)}.$$

∎

Theorem 3.3 *If $\frac{1}{2} < \beta \leq 1$ and $B < b$, then KKT point (A, ϕ, Q_p) of problem* (P_2) *obtained in (7), (8), and (10) will be the optimal point if $Bh\sigma^2 > 2AD(1-\beta)(b-B)$.*

Proof. The determinant of all principal minors of the Hessian matrix for the objective function $EACW(Q_\beta, A, \phi)$ of the problem (P_2) are given by [6]

$$\det H_{11} = \frac{2AD}{Q_\beta^3} + \frac{h\sigma^2}{2(1-\beta)Q_\beta^3} > 0,$$

$$\det H_{22} = \frac{\alpha B}{A^2 Q_\beta^3}\left[AD + \frac{h\sigma^2}{2(1-\beta)}\right] > 0,$$

$$\det H_{33} = \frac{\alpha^2 Bb}{A^2\phi^2 Q_\beta^3}\left[\frac{h\sigma^2}{2(1-\beta)} - AD\left(\frac{b}{B} - 1\right)\right].$$

Now, $\det H_{33} > 0$ if $Bh\sigma^2 > 2AD(1-\beta)(b-B)$. Moreover, constraints of optimization problem (P_2) are linear. This completes the proof. ∎

Theorem 3.4 *If $0 \leq \beta < \frac{1}{2}$ and $B \geq b$, then the inventory system has no feasible solution.*

Proof. The polynomial

$$f(Q_\beta) = -h(1-2\beta)Q_\beta^2 - 2\alpha(B-b)Q_\beta - \frac{h\sigma^2}{2(1-\beta)}$$

has no sign change. Therefore, the polynomial has no positive root by Descartes' rule of signs. Thus, the inventory system has no optimal solution. ■

Theorem 3.5 *If $0 \le \beta < \frac{1}{2}$, $B < b$, and $2\alpha^2(1-\beta)(b-B)^2 \ge h^2\sigma^2(1-2\beta)$, then the order quantity is given by*

$$Q_\beta = \frac{2\alpha(b-B) \mp \sqrt{\alpha^2(b-B)^2 - \frac{h^2\sigma^2(1-2\beta)}{2(1-\beta)}}}{h(1-2\beta)}. \tag{11}$$

Proof. The polynomial

$$f(Q_\beta) = -h(1-2\beta)Q_\beta^2 + 2\alpha(b-B)Q_\beta - \frac{h\sigma^2}{2(1-\beta)}$$

has two sign changes and $f(-Q_\beta)$ has no sign change. Therefore, the polynomial has either two positive real roots or two complex roots by Descartes' rule of signs. The discriminant is non negative since $2\alpha^2(1-\beta)(b-B)^2 \ge h^2\sigma^2(1-2\beta)$. Thus, the order quantity is given by

$$Q_\beta = \frac{2\alpha(b-B) \mp \sqrt{\alpha^2(b-B)^2 - \frac{h^2\sigma^2(1-2\beta)}{2(1-\beta)}}}{h(1-2\beta)}.$$

■

Remark 3 *As per Theorem 3.3, the order quantities calculated in (11) will be optimal if the corresponding KKT point satisfies the condition $Bh\sigma^2 > 2AD(1-\beta)(b-B)$. The decision maker will take the suitable order quantity for which cost is minimum if the problem has two optimal solutions. Moreover, if $2\alpha^2(1-\beta)(b-B)^2 < h^2\sigma^2(1-2\beta)$ then the problem has no optimal solution for $0 \le \beta < \frac{1}{2}$ and $B < b$.*

Theorem 3.6 *If $\beta = \frac{1}{2}$ and $B < b$, then the order quantity is obtained as*

$$Q_\beta = \frac{h\sigma^2}{2\alpha(b-B)}. \tag{12}$$

Proof. The equation

$$h(2\beta-1)Q_\beta^2 - 2\alpha(B-b)Q_\beta - \frac{h\sigma^2}{2(1-\beta)} = 0$$

is reduced to $2\alpha(b-B)Q_\beta = h\sigma^2$ for $\beta = \frac{1}{2}$. Thus, $Q_\beta = \frac{h\sigma^2}{2\alpha(b-B)} > 0$. ■

Remark 4 *For $B < b$, the order quantities calculated in (12) will be optimal if the corresponding KKT point satisfies the condition $Bh\sigma^2 > 2AD(1-\beta)(b-B)$ as per Theorem 3.3. On the other hand, the inventory system has no feasible solution if $\beta = \frac{1}{2}$ and $B \ge b$.*

Case-II: Let us consider $\lambda_1 = 0$ and $\lambda_2 > 0$. That is $A - A_0 \le 0$ is an inactive constraint and $\phi - \phi_0 \le 0$ is the active constraint. From the optimality condition (4) and complementary slackness condition (6), we get

$$A = \frac{\alpha B Q_\beta}{D}, \tag{13}$$

$$\phi = \phi_0. \tag{14}$$

Plugging in the values of A and ϕ in the optimality condition (3), we obtain the following quadratic equation for the order quantity Q_β,

$$[h(2\beta-1) + mD\phi_0]Q_\beta^2 - 2\alpha BQ_\beta - \frac{h\sigma^2}{2(1-\beta)} = 0.$$

Theorem 3.7 *For any $0 \leq \beta \leq 1$ with $[h(2\beta-1)+mD\phi_0] > 0$, the order quantity is given by*

$$Q_\beta = \frac{\alpha B + \sqrt{\alpha^2 B^2 + \frac{h\sigma^2[h(2\beta-1)+mD\phi_0]}{2(1-\beta)}}}{[h(2\beta-1)+mD\phi_0]}. \tag{15}$$

Proof. The proof is similar as the proof of Theorem 3.1. ■

Remark 5 *For any $0 \leq \beta \leq 1$ with $[h(2\beta-1)+mD\phi_0] > 0$, the KKT point (A, ϕ, Q_β) obtained in (13), (14), and (15) is optimal solution of (P_2) if the KKT point satisfy the condition*

$$(1-\beta)\{4\alpha^2 BbQ_\beta - m^2D^2\phi^2Q_\beta^3\} + 2h\sigma^2 > 0.$$

This result can be proved using the same argument of the Theorem 3.3.

Theorem 3.8 *For any $0 \leq \beta \leq 1$ with $[h(2\beta-1)+mD\phi_0] \leq 0$, the problem has no feasible solution.*

Proof. The proof is similar as the proof of Theorem 3.4. ■

Case-III: Let us consider $\lambda_1 > 0$ and $\lambda_2 = 0$. That is $A - A_0 \leq 0$ is an active constraint, and $\phi - \phi_0 \leq 0$ is the inactive constraint. From the complementary slackness condition (6) and optimality condition (5), we calculate

$$A = A_0,$$

$$\phi = \frac{2\alpha b}{mDQ_\beta}.$$

Plugging in the values of A and ϕ in the optimality condition (3), we obtain the following quadratic equation for the order quantity Q_β,

$$h(2\beta-1)Q_\beta^2 + 2\alpha bQ_\beta - \left[2A_0D + \frac{h\sigma^2}{2(1-\beta)}\right] = 0.$$

Theorem 3.9 *For $\frac{1}{2} < \beta \leq 1$, the order quantity is given by*

$$Q_\beta = \frac{-\alpha b + \sqrt{\alpha^2b^2 + h(2\beta-1)\left[2A_0D + \frac{h\sigma^2}{2(1-\beta)}\right]}}{h(2\beta-1)}. \tag{16}$$

Proof. The proof is same as the proof of the Theorem 3.2. ■

Theorem 3.10 *For $0 < \beta < \frac{1}{2}$ and $\alpha^2b^2 \geq h(1-2\beta)\left[2A_0D + \frac{h\sigma^2}{2(1-\beta)}\right]$, the order quantity is obtained as*

$$Q_\beta = \frac{\alpha b \mp \sqrt{\alpha^2b^2 - h(1-2\beta)\left[2A_0D + \frac{h\sigma^2}{2(1-\beta)}\right]}}{h(1-2\beta)}. \tag{17}$$

Proof. The proof is similar to the proof of the Theorem 3.5. ■

Theorem 3.11 *For $\beta = \frac{1}{2}$, the order quantity is calculated by*

$$Q_\beta = \frac{2A_0D + h\sigma^2}{2\alpha b}. \tag{18}$$

Proof. The proof is the same as the proof of the Theorem 3.6. ■

Remark 6 *For different contexts of β, the KKT points corresponding to order quantities obtained in (16), (17), and (18) will be optimal if (A, ϕ, Q_β) satisfy the conditions*

$$\alpha B Q_\beta [4AD(1-\beta) + h\sigma^2] \geq 2(1-\beta)A^2D^2 \quad \text{and} \quad 2(1-\beta)AD + h\sigma^2 \geq 2\alpha b(1-\beta)Q.$$

This can be proved by using the same type of argument of the Theorem 3.3.

Case-IV: Let us consider $\lambda_1 = 0$ and $\lambda_2 = 0$. That is $A - A_0 \leq 0$ and $\phi - \phi_0 \leq 0$ are inactive constraints. From the complementary slackness condition (6), we obtain

$$A = A_0, \tag{19}$$

$$\phi = \phi_0. \tag{20}$$

Plugging in the values of A and ϕ in the optimality condition (3), we calculate the following quadratic equation for the order quantity Q_β,

$$[h(2\beta - 1) + mD\phi_0]Q_\beta^2 - \left[2A_0D + \frac{h\sigma^2}{2(1-\beta)}\right] = 0.$$

Therefore, the optimal quantity is obtained as

$$Q_\beta = \sqrt{\frac{2A_0D + \frac{h\sigma^2}{2(1-\beta)}}{h(2\beta-1) + mD\phi_0}}. \tag{21}$$

Remark 7 *For any $0 \leq \beta \leq 1$ with $[h(2\beta - 1) + mD\phi_0] > 0$, the KKT point (A, ϕ, Q_β) calculated in (19), (20), and (21) is optimal if the KKT point satisfy the conditions*

$$\alpha B Q_\beta [4AD(1-\beta) + h\sigma^2] \geq 2(1-\beta)A^2D^2$$

and

$$2\alpha b[2(1-\beta)AD + h\sigma^2] \geq (1-\beta)m^2D^2\phi^2Q^3.$$

The proof of this is similar to the proof of Theorem 3.3. Moreover, For any $0 \leq \beta \leq 1$ the problem has no feasible solution if $[h(2\beta - 1) + mD\phi_0] \leq 0$.

4 Conclusion

In this note, we have derived closed-form expressions for the order quantity, reorder point, setup cost per cycle, and the probability of the production process going into an out-of-control state in a distribution-free continuous-review inventory model with a service level constraint, where the effectiveness of setup cost reduction and quality improvement has been studied. Therefore, the decision-maker needs not worry about the time complexity of any algorithm used to obtain the optimal solution for the inventory model. Moreover, Descartes' rule of signs has been incorporated to establish the interrelationships between the parameters and optimal solutions of the inventory system in various contexts.

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